MATHEMATICAL MODELLING OF COUETTE FLOW OF AN ELECTRICALLY CONDUCTING FLUID BOUNDED BY TWO PARALLEL POROUS PLATES

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A THESIS SUBMITTED TO THE POSTGRADUATE SCHOOL
FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA, NIGERIA
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF DOCTOR OF PHILOSOPHY (PhD)
IN MATHEMATICS

NOVEMBER, 2022

ABSTRACT

This thesis presents mathematical model for steady and unsteady Couette flow of an electrically conducting viscous incompressible fluid bounded by two parallel non-conducting porous plates incorporating species equation, temperature-dependent viscosity and thermal radiation. The partial differential equations governing the phenomenon were non-dimensionalized, using some dimensionless quantities. The conditions for the existence and uniqueness of solution of the model were established using Lipschitz continuity approach. The properties of solution were examined using upper and lower solution method and Kolodner and Pederson lemma. The dimensionless equations were transformed and considered in three forms: Transient state with time dependent pressure gradient; transient state with constant pressure gradient and steady state with constant pressure gradient. The equations for each case considered were solved using perturbation method and eigenfunction expansion technique and direct integration. The results obtained were presented graphically and discussed. From the results obtained, it was observed that the fluid concentration is at maximum value $\phi(y,t) = 2.5$ when $y = -0.5$ while the secondary velocity is at maximum value $w(y,t) = 8.0$ when $y = -0.5$. It was also observed that increase in Reynolds number and pressure gradient leads to enhancement in the velocity profiles while suction parameter, Hartman number and porosity parameter reduced velocity profiles. Also, radiation parameter enhanced the temperature profile while Reynolds number, suction parameter and Prandtl number reduced the temperature profile. Fluid flow is observed to attain maximum velocity $u(y) = 55$ when $y = 0$. Reynolds number, suction parameter, constant pressure gradient chemical reaction parameter and thermo diffusion parameter enhanced the concentration profile while radiation parameter and Eckert number reduced the concentration profile. The result from this research work is of importance to industries that produce domestic consumables like toothpaste and food industries in production of tomato paste and fruit juice.

TABLE OF CONTENTS

Content

Page

LIST OF FIGURES

LIST OF TABLES

CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

Mathematical Models are used to examine different phenomena with each model representing a definite schematization of the phenomenon taken into consideration. In modeling, the researcher is always restricted by a finite number of parameters called the governing factors within the limits of which the investigation is being carried out. In fluid dynamics, Couette flow is the laminar flow of a viscous fluid in the space between two parallel plates one of which is moving at a velocity relative to the other. The flow is driven by the virtue of viscous drag force acting on the fluid and the applied pressure gradient parallel to the plates. The study of magnetohydrodynamic (MHD) Couette flow with heat transfer of an electrically conducting fluid through two parallel plates known as Hartman flow is a classical problem that has many applications in MHD power generators, MHD pumps, aerodynamic heating, nuclear reactors and geothermal energy extractions. Fluid flow through porous media has several engineering and geophysical applications such as in the field of chemical engineering for filtration and purification processes, in agricultural engineering to study the underground water resources, in petroleum industry to study the movement of natural gas, oil and water through the oil channels and reservoirs while in astrophysics it is applied to study the stellar and solar structures (Makinde & Mhone, 2005)

Hartman *et al.* (1973) studied the influence of a transverse uniform magnetic field on the flow of a conducting fluid between two infinite, parallel, stationary and insulated plates. Afterwards, a lot of research work concerning the Hartman flow has been carried out under different physical conditions and flow geometries. In most cases, the Hall and ion slip terms were ignored in applying Ohm's law as they have negligible effects for small and moderate values of the magnetic field. In recent research works, the trend for the application of magnetohydrodynamics is towards a strong magnetic field resulting to a significant effect of electromagnetic force. Under these conditions, the Hall and ion slip are important and they also have a marked effect on the magnitude and direction of the current density and consequently on the magnetic force term.

Over the years, considerable interest has been observed on the effect of MHD in viscous, incompressible, non-Newtonian fluid flow with heat transfer. These interests on non-Newtonian fluids are owed to its important applications in various branches of science, engineering and technology, particularly in chemical and nuclear industries, material processing, geophysics and bio-engineering. In view of these applications, an extensive range of mathematical models have been developed to simulate the diverse hydrodynamic behavior of these non-Newtonian fluids. However, different non-Newtonian fluid models have been presented by researchers and solved using various types of analytical and computational schemes. The most important non-Newtonian fluid possessing a yield value is the Casson fluid, which has significant applications in polymer processing industries and biomechanics. Casson fluid is shear thinning liquid which has an infinite velocity at a zero rate of strain. Cassons constitute equation represents a nonlinear relationship between the rates of stress and strain and has been noticed to be accurately applicable in silicon suspensions and lithographic varnishes used for printing inks. Casson fluid when acted upon by pressure gradient and is subjected to a uniform magnetic field is a good approximation of some practical situations such as heat exchangers and flow meters and pipes (Muchin *et al.*, 2012). The basic set of equations that governs the flow of fluids are; the continuity equation (mass conservation), equation of motion (momentum), and energy equation. They are all based on the assumption that fluid flow is continuous.

i. Continuity equation: This equation is derived on the assumption that matter can neither be created nor destroyed but can be transformed from one form to another. It is expressed as:

$$\nabla . v = 0 \tag{1.1}$$

ii. Equation of motion (momentum): This is derived from applying Newton's second law of motion to the fluid motion together with the assumption that the fluid stress is the sum of a diffusing viscous term which is proportional to the velocity gradient plus a pressure term. It is expressed as:

$$\rho \frac{\partial \bar{q}}{\partial t} + \rho\left(\bar{q}\nabla\right)\vec{q} = \rho\vec{F} - \nabla P + \mu\nabla^2\vec{q} \tag{1.2}$$

where $\vec{F} = (\bar{J} \times \bar{B})$ for electrically conducting fluids in addition to other body forces.

iii. Energy equation: This is derived based on the assumption that the energy of the fluid is conserved during its motion and it is expressed as:

$$\rho C_p\left(\frac{\partial T}{\partial t} + \left(\bar{v}\nabla\right)T\right) = \mu\left(\nabla\bar{v}\right)^2 + k\nabla^2 T + \frac{J^2}{\sigma} \tag{1.3}$$

where S, V, $\vec{q}$, n, ρ, μ, $\vec{F}$, k, C_p, $\bar{V}$, T,$\bar{J}$, $\bar{B}$ are closed surface, fluid volume, fluid velocity, unit normal vector, fluid density, fluid viscosity, body force, thermal conductivity, specific heat capacity, velocity, temperature, current density and magnetic flux respectively and ∇ is the Laplacian operator.

1.2 Statement of the Research Problem

Fluids in which the shear stresses are not linearly proportional to the velocity gradient are characterized as non-Newtonian fluids and are of much interest among researchers. Among the non-Newtonian fluids, Casson fluid has attracted more attention of researchers due to its application in the field of metallurgy, food processing, drilling operations and bioengineering operations. Some more applications of Casson fluid can be seen in manufacturing of pharmaceutical products, paints, synthetic lubricants and biological fluids such as sewage jelly, tomato sauce, honey, soup and blood due to its contents such as plasma, fibrinogen and protein. Hence, there is need to see the effect of parameters involved on the concentration, temperature, primary and secondary velocities.

1.3 Aim and Objectives of the Study

The aim of this research is to carry out a study on mathematical model of transient Couette flow of an electrically conducting fluid bounded by two parallel porous plates.

The objectives of the present research work are to:

i. Formulate the mathematical model describing transient Couette flow of an electrically conducting fluid.
ii. Establish the criteria for the existence and uniqueness of solution of the model formulated using Lipschitz continuity approach.
iii. Examine the properties of the solutions of the modelusing method of upper and lower solution.
iv. Solve the model equations using perturbation method and eigenfunction expansion technique.

v. Provide the graphical representation of the system responses.

1.4 Justification of the Study

The importance of MHD Couette flow of Casson fluid through channels cannot be over emphasized, hence the need for this study to conduct research considering thermal radiation and chemical reaction in the presence of uniform suction and injection.

1.5 Scope and Limitations of the Study

This research work focuses on the formulation of the mathematical model and the analytic simulation of the MHD Casson fluid model. This research is therefore limited to the mathematical analysis of the problem.

1.6 Significance of the Study

The study of unsteady magnetohydrodynamics (MHD) Couette flow in the presence of transverse magnetic field has wide range of applications in many areas of science and engineering such as MHD pumps, MHD generators, MHD accelerators and MHD flow meters. Therefore, this study is of great significance for proper understanding of the working processes of these machines.

1.7 Definition of Terms

This section presents definition of terms used in the thesis

Brinkman Number$(\boldsymbol{Br})$: This is the ratio of heat generated by viscous dissipation to heat transported by molecular conduction (external heating). It is denoted by

$$Br = \frac{\mu u^2}{k\left(T_w - T_0\right)} \tag{1.4}$$

where μ is the dynamic viscosity, u is the flow velocity, k is the thermal conductivity, T_0 is the bulk temperature, T_w is the wall temperature.

Casson Fluid: This is a shear thinning fluid which is assumed to have an infinite viscosity at zero rates of shear, a yield stress below which no flow occurs and a zero viscosity at an infinite rate of shear.

Couette flow: This is defined as the laminar flow of a viscous fluid in the area between two boundaries one of which is moving relative to the other.

Definitions:

Definition 1: A smooth function $\underline{u}$ is said to be a lower solution of the problem

$$Lu = f\left(x,t,u\right)$$

where

$$L = \frac{\partial}{\partial t} + a\left(x,t\right)\frac{\partial^2}{\partial x^2} + b\left(x,t\right)\frac{\partial}{\partial x} + c\left(x,t\right)$$

If $\underline{u}$ satisfies

$$L\underline{u} \leq f\left(x,t,\underline{u}\right)$$

$$\underline{u}\left(x,0\right) \leq f\left(x\right), \quad \underline{u}\left(0,\mathrm{t}\right) \leq h_1\left(t\right), \quad \underline{u}\left(\mathrm{L},\mathrm{t}\right) \leq h_2\left(t\right)$$ (Olayiwola and Ayeni, 2011)

Definition 2: A smooth function $\bar{u}$ is said to be an upper solution of the problem

$$Lu = f\left(x,t,u\right)$$

Where

$$L = \frac{\partial}{\partial t} + a(x,t)\frac{\partial^2}{\partial x^2} + b(x,t)\frac{\partial}{\partial x} + c(x,t)$$

If $\bar{u}$ satisfies

$$L\bar{u} \geq f\left(x,t,\bar{u}\right)$$

$\bar{u}(x,0) \geq f(x), \quad \bar{u}(0,\mathrm{t}) \geq h_1(t), \quad \bar{u}(\mathrm{L},\mathrm{t}) \geq h_2(t)$ (Olayiwola and Ayeni, 2011)

Definition 3: A smooth function $\underline{u}$ is said to be a lower solution of the problem

$$Lu = f(x,u)$$

where

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)$$

If $\underline{u}$ satisfies

$$L\underline{u} \geq f(x,\underline{u})$$

$\underline{u}(0) \leq h_1, \quad \underline{u}(\mathrm{L}) \leq h_2$ (Olayiwola and Ayeni 2011)

Definition 4: A smooth function $\bar{u}$ is said to be an upper solution of the problem

$$Lu = f(x,u)$$

Where

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)$$

If $\bar{u}$ satisfies

$$L\bar{u} \le f\left(x, \bar{u}\right)$$

$\bar{u}(0) \ge h_1, \quad \bar{u}(\mathrm{L}) \ge h_2$ (Olayiwola and Ayeni, 2011)

Eckert Number (Ec): This provides a measure of the kinetic energy of the flow relative to the enthalpy difference across the thermal boundary layer. It is also used to characterize heat dissipation in high speed flows for which viscous dissipation is significant.

$$Ec = \frac{u^2}{c_p \Delta T} \tag{1.5}$$

Grashof Number (Gr): This is a dimensionless number which approximates the ratio of the buoyancy to viscous force acting on a fluid. It is given by

$$Gr_L = \frac{g\beta(T_s - T_\infty)L^3}{v^2} \tag{1.6}$$

Hartmann Number (Ha): This is the ratio of electromagnetic force to the viscous force, defined by

$$Ha = BL\sqrt{\frac{\sigma}{\mu}} \tag{1.7}$$

where B is the magnetic field, L the characteristic length, σ the electrical conductivity and μ dynamic viscosity.

Incompressible fluids: These are fluids that do not change the volume of its container due to external pressure.

Joule dissipation: This is the process by which the passage of an electric current through a conductor releases heat. The amount of heat released is proportional to the square of the current.

Laminar flow: Laminar flow occurs when fluid flows at low velocity, in parallel layers with no disruption between the layers.

Lipschitz condition: A real valued function $f : R \to R$ is called Lipschitz continuous if there exist a real constant $K \geq 0$ such that for all $x_1 \; and \; x_2 \; in \; X$,

$$\left| f\left(x_1\right) - f\left(x_2\right) \right| \leq K \left| x_1 - x_2 \right|. \tag{1.8}$$

Any such K is referred to as a Lipschitz constant for the function f.

Magnetohydrodynamics (MHD): This is a branch of science which deals with the dynamics of conducting fluids moving in an electromagnetic field.

Prandtl Number (Pr): this is a ratio of momentum diffusivity to thermal diffusivity.

Poiseuille Flow: This is defined as the laminar fluid flow of a viscous fluid in the area bounded by two stationary boundaries where the flow is induced by pressure gradient.

Reynold Number (Re): This is the ratio of inertial forces to viscous forces and it is a convenient parameter for predicting if a flow condition will be laminar or turbulent. When the viscous forces are dominant, then the flow is laminar while when the inertia forces dominate then the flow is turbulent.

Schmidt Number (Sc): This is a dimensionless number defined as the ratio of the shear component of diffusivity viscosity/density to the diffusivity for mass transfer. It physically relates the relative thickness of the hydrodynamic layer and mass transfer boundary layer.

Viscosity: This is the measure of a fluid's resistance to gradual deformation by shear or tensile stress.

Viscous dissipation: This is defined as an irreversible process where kinetic energy of the moving fluid is converted into thermal energy.

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Review of Previous Works

The study of unsteady magnetohydrodynamic (MHD) Couette flow in the presence of transverse magnetic field has various and wide applications in many areas of science and engineering such as MHD pumps, MHD generators, MHD accelerators and MHD flow meters. Katagiri (1962) studied unsteady MHD Couette flow of a viscous, incompressible and electrically conducting fluid in the presence of uniform transverse magnetic field. The fluid flow through the channel was assumed to be induced by the impulsive movement of one of the plates of the channel. He deduced that increase in magnetic field brings about increase in skin friction while it retards the velocity of the fluid. In recent years, the study of Couette flow in rotating systems enhances the interest in researchers due to its applications in secular variation of earth’s magnetic field, the internal rotation rate of the sun, the structure of rotating magnetic stars, rotating hydromagnetic generators, vortex type MHD power generators and other centrifugal machines. Taking these facts into cognizance, Hazem (2009) studied the ion slip effect on unsteady Couette flow with heat transfer under exponential decaying pressure gradient. Taiwo and Jha (2018) studied the transient pressure driven flow in an annulus partially filled with porous material. They obtained the exact solution of the governing equations using Laplace transform technique and deduced that as Darcy number increases the permeability of the porous region increases. Ajibade and Bichi (2019) investigated the variable fluid properties and thermal radiation effects on natural convection Couette flow through a vertical porous channel using the Adomian decomposition method (ADM) and maintained that both fluid velocity and its temperature within the channel were observed to increase with growing thermal radiation and decreases with increase in thermal conduction of the fluid.Yusuf *et al.*(2018) examined the boundary layer flow of a nanofluid in an inclined wavy wall with convective boundary condition. They observed fluid flow back at the wavy wall. Alsabery *et al.* (2017) studied using finite difference method, the natural convection flow of a nanofluid in an inclined square enclosure partially filled with a porous medium and they deduced that heat transfer is considerably affected by the porous layer increment. Aiyesimi *et al.* (2015) analytically investigated the convective boundary layer flow of a nanofluid past a stretching sheet with radiation. They solved the governing equations using the Adomain decomposition method (ADM). They observed that both thermal and concentration Grashof numbers enhance the velocity, temperature and concentration profiles of the fluid. Laila and Marwat (2021) examined the nanofliud flow in a converging and diverging channel of rectangular heated walls. They dedeuced that both the temperature and concentration profiles are enhanced with increase in thermophoretic forces. Recently, Jiya *et al.* (2015) studied using the Adomain decomposition method the solutions of a boundary layer flow past a stretching plate with heat transfer, viscous dissipation and Grashof number. They observed that ADM provides highly precise numerical solution for non-linear differential equations.

Chutia *et al.* (2017) numerically studied the solution of unsteady hydromagnetic Couette flow in a rotating system bounded by two porous plates with Hall effects. The governing equations were solved using the finite difference method. Jana *et al.* (2012)

investigated Couette flow through a porous medium in a rotating system and observed that a thin boundary layer which increases in thickness as porosity parameter increases is formed near the moving plate. In another related work, Seth *et al.* (2011) studied using Laplace transform technique, the effects of rotation and magnetic field on unsteady Couette flow in a porous channel. They observed that magnetic field retards the fluid flow in both primary and secondary flow directions. Seth *et al.* (2010) studied the unsteady hydromagnetic Couette flow within porous plates in a rotating system. They observed that suction has a retarding influence on both the primary and secondary flow where as injection and time have accelerating influence on the flow velocities. Casson fluid as an example of non-Newtonian fluid is a shear thinning liquid with an infinite viscosity at a zero rate of strain. It is an important fluid in mechanics due to its practical applications such as in silicon suspension and suspensions of bentonic in water. Pramanik (2014) focused on Casson fluid flow and heat transfer past an exponentially porous stretching surface in the presence of thermal radiation. Afikuzzaman *et al.* (2015) have investigated an unsteady MHD Casson fluid flow through a parallel plate with hall current using an explicit finite difference technique. In another related research, hydrodynamic impulsive lid driven flow and heat transfer of a Casson fluid was studied by Attia & Sayed-Ahmed (2006).

Sayed-Ahmed *et al.* (2011) considered the time dependent pressure gradient effect on unsteady MHD Couette flow and heat transfer of Casson fluid. The two components of the momentum equation are given by:

$$\rho\frac{\partial u}{\partial t}+\rho v_0\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)-\frac{\sigma B_0^2}{1+m^2}\left[u+mw\right] \tag{2.1}$$

$$\rho\frac{\partial w}{\partial t}+\rho v_0\frac{\partial w}{\partial y}=\frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right)-\frac{\sigma B_0^2}{1+m^2}\left[w+mu\right] \tag{2.2}$$

The energy equation in dimensional form is given as

$$\rho c \frac{\partial T}{\partial t} + \rho c_p v_0 \frac{\partial T}{\partial y} = k \frac{\partial^2 T}{\partial y^2} + \mu \left[\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] + \frac{\sigma B_0^2}{1+m^2} \left[u^2 + w^2 \right] \tag{2.3}$$

Subject to

$$\left. \begin{array}{l} u(y,0)=0, u(-h,t)=0, u(h,t)=U_0 \\ w(y,0)=0, w(-h,t)=0, w(h,t)=0 \\ T(y,0)=T_1, T(-h,t)=T_1, T(h,t)=T_2 \end{array} \right\} \tag{2.4}$$

where apparent viscosity is given by

$$\mu = \left(K_c + \left(\frac{\tau_0}{\sqrt{\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2}} \right)^{1/2} \right)^2 \tag{2.5}$$

where

ρ is fluid density,

μ is apparent of viscosity of the fluid,

K_c^2 is Casson's coefficient of viscosity,

τ_0 is yield stress,

σ is electric conductivity,

β is Hall factor,

B_i is ion slip parameter,

$m = \sigma\beta B_0$ is Hall parameter,

c_p is specific heat capacity of the fluid,

k is thermal conductivity of the fluid,

u is primary velocity,

w is secondary velocity,

T fluid temperature.

Bhattacharyya *et al.* (2013) studied analytically the solution for magnetohydrodynamic boundary layer flow of Casson fluid over a stretching/shrinking sheet with wall mass transfer. The unsteady boundary layer flow of a Casson fluid due to an impulsively started moving plate was considered by Mustafa *et al.* (2011). Recently, Mukhopadhyay *et al.* (2011) investigated the steady boundary layer flow and heat transfer over a porous moving plate in the presence of thermal radiation. Makinde and Mhone (2005) studied the heat transfer to MHD flow in a channel filled with porous medium.

Sharada and Shankar (2016) investigated steady three-dimensional Casson fluid over an exponentially stretching surface in the presence of Lorentz force. They consider the effect of heat generation and mixed convection. Their model equations were transformed from partial differential equations to set of ordinary differential equations using similarity transformations and the transformed equations were solved by applying Keller Box method. The effects of magnetic parameter, mixed convection parameter, heat source/sink, casson parameter and ratio parameter were investigated on the velocity and temperature profiles graphically.

Pushapalata *et al.* (2016) investigated the unsteady free convective flow of a casson fluid bounded by a moving vertical plane in a rotating system. The governing equations of the flow were solved analytically using perturbation technique. The effects of various parameters such as casson, magnetic field, thermal diffusion, chemical reaction and thermal radiations on velocity, temperature and concentration profiles were discussed.

Kushpala *et al.* (2017) analyzed the effects of cross diffusion on casson fluid over an unsteady stretching surface with boundary effects. The governing equations were solved numerically using Runge-Kutta fourth order along with shooting technique.

Maleque (2016) investigated an exothermic/endothermic binarychemical reaction on unsteady MHD non-Newtonian casson fluid flow with heat and mass transfer past a flat porous plate. Considering the effects of casson parameter on velocity profile for cooling and heating plate, the exothermic/endothermic chemical reaction rate and Arrhenius energy on the concentration, the governing equations were solved numerically by adopting implicit Runge-Kutta and shooting method using the Nachtsheim-Swigert iteration technique.

Vedavathi *et al.* (2016) examined the chemical reaction, radiation and dufour effects on casson MHD flow over a vertical plate with heat source/sink and the problem was solved numerically using perturbation technique.

Gireesha *et al.* (2016) examined similarity solution to the problem of two-dimensional boundary layer flow, heat and mass transfer of non-newtonian Casson fluid over a porous stretching surface. The governing equations were transformed into self-similar nonlinear ordinary differential equations and solved numerically by an efficient Runge-Kutta fourth-fifth order method.

Kirughashankar *et al.* (2016) investigated Casson fliud flow and heat transfer over an unsteady porous stretching surface and obtain analytical expression for axial velocity and temperature field of the fluid.

Hussanan *et al.* (2016) examined the effects of Newtonian heating and inclined magnetic field on two-dimensional flow of a casson fluid over a stretching sheet. The governing partial differential equations were transformed into nonlinear ordinary differential equation by using similarity transformation and the solution of the coupled nonlinear equations obtained using analytical technique.

In most of these investigations, it was observed that the effects of Hall current are not taken into account. Hall effects results in a development of an individual potential difference between opposite surfaces of conductors for which a current is induced perpendicular to both the electric and magnetic field. Hall current has many applications such as in MHD power generators, nuclear power reactors, underground energy systems and in several areas of astrophysical and geophysical interests. Keeping these facts in view, Balamurugan *et al.* (2015) considered an unsteady MHD free convective flow past a moving vertical plate with time dependent suction and chemical reaction in a slip flow regime. The slip flow conditions for the velocity, jump in temperature and jump in concentration are taken into account in the boundary conditions.

Murthy (2020) made numerical assessment on magnetohydrodynamic flow of Casson fluid over a deformable porous layer with slip conditions. They observed that the liquid velocity and solid displacements are found are rotted for higher estimations of magnetic parameter and contrary nature was observed for the impact of Casson Parameter.

Nagaraju *et al.* (2020) investigated the radiation and chemical reaction effects on MHD Casson fluid flow of a porous medium with suction and injection. They found that velocity decreases while temperature and concentration increases when magnetic field and permeability parameter increases.

Gireesha and Roja (2020) made a second law analysis of MHD natural convection slip flow of Casson fluid through an inclined microchannel. They observed an increasing behaviour of heat transfer rate for enhancement values of Eckert number and heat source ratio parameter. Also, drag force are retarded with higher estimation of Reynolds number.

Samrat *et al.* (2021) studied the buoyancy effect on magnetohydodynamics flow of Casson fluid with Brownian movement and thermophoresis. They observed that the magnetic field signifies additional conflicting force to fluid motion and dissipation impacts to accelerate the temperature.

Opanuga *et al.* (2020) examined the impact of Hall current on entropy generation of radiative MHD mixed convection Casson fluid. Muhammed *et al.* (2021) examined the Couette flow of viscoelastic dusty fluid in a rotating frame along with heat transfer. Goud and Malga (2020) investigated the effect of heat source on an unsteady MHD free convection flow of a Casson fluid past a vertical Oscillating plate in porous medium using finite element analysis. Mohammed *et al.* (2015) investigated the simulation of heat and mass transfer in the flow of incompressible viscous fluid past an infinite vertical plate

In another related work, Ghosh and Pop (2004) investigated Hall effects on MHD plasma Couette flow in a rotating environment. Hayat *et al.* (2004) studied the Hall effects on the unsteady hydromagnetic oscillatory flow of a second grade fluid. Recently, Das *et al.* (2017) analyzed Hall Effects on Unsteady MHD Reactive Flow Through a Porous Channel with Convective Heating at the Arrhenius Reaction Rate. The Hall effects on MHD Couette flow in a rotating system with arbitrary magnetic field was considered by (Ghosh 2002). Research on Hall effects on MHD couette flow in a channel partially filled with porous medium in a rotating system was carried out by (Chauhan & Agrawal 2012). Seth *et al.* (2009; 2012) studied the Hall effects on oscillatory hydromagnetic Couette flow in a rotating system and Hall current and rotation effects on unsteady hydromagnetic Couette flow within a porous channel respectively. The effect of porosity on unsteady Couette flow with heat transfer in the presence of uniform suction and injection was numerically analyzed by Hazem (2009).

Venkateswarlu and Padma (2015) analyzed unsteady MHD free convective heat and mass transfer in a boundary layer flow past a vertical peameable plate with thermal radiation and chemical reaction. Chamkha and Ahmed (2012) examined unsteady MHD heat and mass transfer by mixed convection flow in the forward stagnation region of a rotating sphere at different wall conditions. The effects of thermal radiation and magnetic field on unsteady mixed convection flow and heat transfer over a stretching in the presence of internal heat generation/absorption was studied by Elbashbeshy and Aldawody (2011). Dulal and Bulal(2010) investigated the buoyancy and chemical reaction effects on MHD mixed convection heat and mass transfer in a porous medium with thermal radiation and ohmic heating. Mohammed *et al.* (2015) analyzed radiation and mass transfer effects on MHD oscillatory flow in a channel filled with porous medium in the presence of chemical reaction.

2.2 Eigenfunction Expansion Method

The approach of eigenfunction is closely associated to the Fourier's method, which is commonly known as the method of separation of variables that is intended in sorting out a specific solution to differential equations. In using this approach, our primary preoccupation is geared towards the peculiar function being solutions of an eigenvalue problem. The technique of separation of variables was put forth by d'Alembert (1749).

However, in the 18th century, the same approach was engaged by Euler, Bernolli and Lagrange in tackling the question of oscillation of string. This method requires that both partial differential equation (PDE) and boundary conditions be homogenous. We also transform non-homogenous boundary conditions into homogenous one before using the general formulation. The eigenfunction expansion method is easiest to apply to diffusion problemsin space dimension. Eigenfunction expansion approach was used by Ibrahim*et al.* (2017).

2.3 Summary of the Literature Review and Gap to Fill

In reviewing the above literature, several works have been carried out on transient and steady Couette flow. Some authors considered time dependent pressure gradient without considering thermal radiation. Others concentrated on heat source and ignored chemical reaction. Most authors simulated their models numerically due to the complexity of the equations.

However, this research work seeks to consider an analytical solution of transient and steady Couette flow of an electrically conducting incompressible fluid bounded by two parallel non-conducting porous plates incorporating the following:

(i) Thermal radiation

(ii) Chemical reaction

(iii) Constant and time dependent pressure gradient

(iv) Temperature dependent viscosity.

CHAPTER THREE

3.0 MATERIALS AND METHODS

3.1 Mathematical Formulation

Following Sayed-Ahmed *et al.* (2011), we consider the unsteady flow of a viscous, incompressible, non-conducting fluid through a channel with chemical reaction, thermal radiation constant and variable pressure gradient in the presence of magnetic field. The flow is assumed to be laminar, incompressible and flows between two infinite horizontal plates located at $y = \pm h$ which extends from $x = -\infty$ to ∞ and from $z = -\infty$ to ∞ as shown in Figure 1.

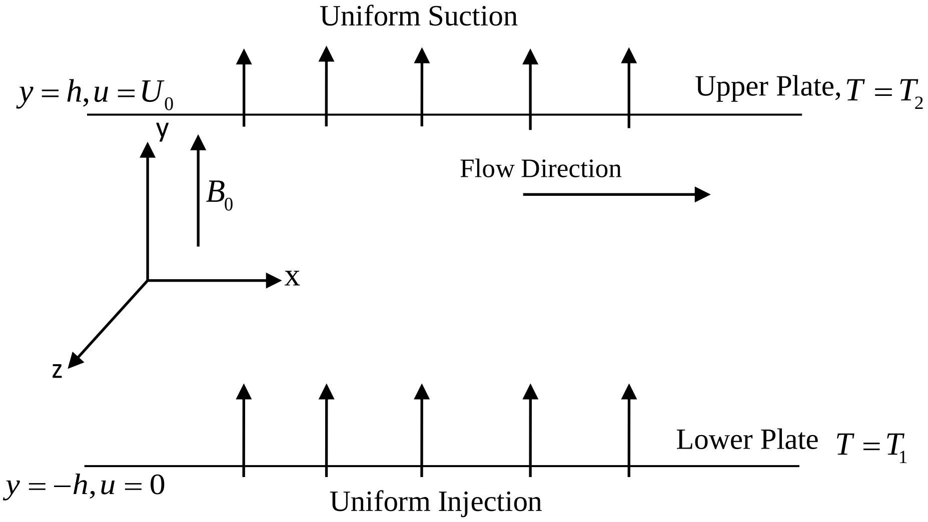

Figure 3.1: Schematic diagram of the problem

It is also assumed that:

(i) Axial conduction in the fluid through the surfaces are negligible.

(ii) The fluid is optically thin with relatively low density.

(iii) There is small electrical conductivity and electromagnetic force in the region of flow.

(iv) No heat generation and constant thermo-physical properties.

The upper plate is suddenly set into motion and moves with a uniform velocity U_0 while the lower plate is kept stationary. The upper and lower plates are kept at two constant temperatures T_2 and T_1 respectively with $T_2 > T_1$. The fluid flows between the two plates under the influence of an exponential decaying with time dependent pressure gradient in the x-direction which is a generalization of a constant pressure gradient. A uniform suction from above and injection from below with constant velocity v_0 which are all applied at $t = 0$. The system is subjected to a uniform magnetic field B_0 in the positive y-direction and is assumed undisturbed as the induced magnetic field is neglected by assuming a small magnetic Reynolds number. The Hall effect is taken into consideration and consequently a z-component of the velocity is expected to arise. Thus the fluid velocity vector is given by:

$$v = ui + v_0 j + wk \tag{3.1}$$

The fluid motion starts from rest at $t = 0$, and the no-slip condition of the plates implies that the fluid velocity has no z-component at $y = \pm h$. The initial temperature of the fluid is assumed to be equal to T_1. The flow of the fluid is governed by the momentum equation-

$$\rho \frac{Dv}{Dt} = \nabla \cdot (\mu \nabla v) - \nabla p + J \times B_0 \tag{3.2}$$

where ρ is the density of the fluid and μ is the apparent viscosity of the model which is assumed to be a function of temperature only i.e, $\mu = \mu(T)$.

Introducing a Chapman-Rubesin viscosity law, with $w = 1$as shown in Olayiwola (2016) and using the condition at the lower plate, gives

$$\mu = \frac{c\mu_1 T}{T_1} \tag{3.3}$$

where μ_1 is the Casson coefficient of viscosity, c is a constant. If the Hall term is retained, the current density J is given by

$$J = \sigma\left(v \times B_0 - \beta(J \times B_0) + \frac{\beta Bi}{B_0}(J \times B_0) \times B_0 \right) \tag{3.4}$$

where σ is the electric conductivity of the fluid, Bi is the ion slip parameter and β is the Hall factor (Sutton and Sherman (1965)). Equation (3.4) above may be solved in J to yield

$$J \times B_0 = -\frac{\sigma B_0^2}{\left((1 + BiBe)^2 + Be^2\right)}\left(\left((1 + BiBe)u + Bew\right)i + \left((1 + BiBe)w - Beu\right)k\right) \tag{3.5}$$

where $Be = \sigma\beta B_0$ is the Hall parameter.

Following Sayed-Ahmed *et al.* (2011), the governing momentum equations in terms of equation (3.5), in dimensional form are as follows:

$$\rho\frac{\partial u}{\partial t}+\rho v_0\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)-\frac{\sigma B_0^2}{\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)u+Bew\right)-$$
$$\mu\frac{u}{k}+g\beta_T\left(T-T_1\right)+g\beta_C\left(C-C_1\right) \tag{3.6}$$

$$\rho\frac{\partial w}{\partial t}+\rho v_0\frac{\partial w}{\partial y}=\frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right)-\frac{\sigma B_0^2}{\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)w-Beu\right)-\mu\frac{w}{k} \tag{3.7}$$

The energy equation from Sayed-Ahmed *et al.* (2011) in dimensional form is given by

$$\left.\begin{array}{l}\rho c_P\dfrac{\partial T}{\partial t}+\rho c_P v_0\dfrac{\partial T}{\partial y}=\dfrac{c_p}{\Pr}\dfrac{\partial}{\partial y}\left(\mu\dfrac{\partial T}{\partial y}\right)+\mu\left(\left(\dfrac{\partial u}{\partial y}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2\right)+\\ \dfrac{\sigma B_0^2}{\left((1+BiBe)^2+Be^2\right)}\left(u^2+w^2\right)-\dfrac{1}{\rho c_P}\dfrac{\partial q}{\partial y}\end{array}\right\} \tag{3.8}$$

The concentration equation in dimensional form is given as:

$$\rho\frac{\partial C}{\partial t}+\rho v_0\frac{\partial C}{\partial y}=\frac{1}{Sc}\frac{\partial}{\partial y}\left(\mu\frac{\partial C}{\partial y}\right)+D\frac{\partial^2 T}{\partial y^2}-\eta\left(C-C_1\right) \tag{3.9}$$

Subject to the initial and boundary conditions;

$$\left.\begin{array}{lll}u(y,0)=0, & u(-h,t)=0, & u(h,t)=U_0\\ w(y,0)=\dfrac{U_0 y}{h}\left(1-\dfrac{y}{h}\right), & w(-h,t)=0, & w(h,t)=0\\ T(y,0)=0, & T(-h,t)=T_1, & T(h,t)=T_2\\ C(y,0)=0, & C(-h,t)=C_1, & C(h,t)=C_2\end{array}\right\} \tag{3.10}$$

where apparent viscosity μ is represented by equation (3.3) as:

$$\mu=\frac{c\mu_1 T}{T_1}$$

where

ρ is the density,

μ apparent viscosity of the fluid,

y is distance,

σ is electrical conductivity,

β is Hall factor,

Bi is ion slip parameter,

p is pressure

v_0 is constant suction/injection velocity,

U_0 is upper plate velocity,

$Be = \sigma\beta B_0$ is Hall parameter,

k is porous media permeability coefficient,

c_p is specific heat capacity,

u is primary velocity in x direction,

w is secondary velocity in y direction,

h is distance between the plates,

θ is dimensionless fluid temperature,

Gr_c is solutal Grashof number,

Gr_θ is thermal Grashof number,

β_T is the coefficient of volume expansion due to temperature,

β_C is the coefficient of volumetric expansion due to concentration,

T is fluid temperature,

T_1 is lower plate temperature,

T_2 is upper plate temperature,

C is fluid concentration,

C_1 is lower plate concentration,

C_2 is upper plate concentration,

D is thermal diffusivity,

η is chemical reaction constant,

t is time,

q is radiative heat flux,

B_0 is uniform magnetic field,

α_* is coefficient of mean radiation absorption.

3.2 Methods of Solution

3.2.1 Non-dimensionlization

To write the governing dimensional equations (3.6)-(3.9) with their corresponding boundary conditions (3.10) in non-dimensional form, we use the following dimensionless variables:

$$\left.\bar{u}=\frac{u}{U_0},\ \bar{w}=\frac{w}{U_0},\ \bar{y}=\frac{y}{h},\ \bar{x}=\frac{x}{h},\ \bar{t}=\frac{tU_0}{h},\ \theta=\frac{T-T_1}{T_2-T_1},\ \phi=\frac{C-C_1}{C_2-C_1},\ \bar{p}=\frac{p}{\rho U_0^2}\right\} \tag{3.11}$$

Then,

$$\bar{u}=\frac{u}{U_0}\Rightarrow u=U_0\bar{u},\qquad \partial u=U_0\partial\bar{u},\qquad \partial^2 u=U_0\partial^2\bar{u} \tag{3.12}$$

$$\bar{w}=\frac{w}{U_0}\Rightarrow w=U_0\bar{w},\qquad \partial w=U_0\partial\bar{w},\qquad \partial^2 w=U_0\partial^2\bar{w} \tag{3.13}$$

$$\left.\begin{aligned}&\theta=\frac{T-T_1}{T_2-T_1}\Rightarrow\ T-T_1=\left(T_2-T_1\right)\theta\Rightarrow T=\left(T_2-T_1\right)\theta+T_1,\\&\partial T=\left(T_2-T_1\right)\partial\theta,\qquad \partial^2 T=\left(T_2-T_1\right)\partial^2\theta\end{aligned}\right\} \tag{3.14}$$

$$\left.\begin{aligned}&\phi=\frac{C-C_1}{C_2-C_1}\Rightarrow C-C_1=\left(C_2-C_1\right)\phi\Rightarrow C=\left(C_2-C_1\right)\phi+C_1,\\&\partial C=\left(C_2-C_1\right)\partial\phi,\qquad \partial^2 C=\left(C_2-C_1\right)\partial^2\phi\end{aligned}\right\} \tag{3.15}$$

$$\bar{p}=\frac{p}{\rho U_0^2}\Rightarrow\ p=\rho U_0^2\,\bar{p}\Rightarrow\partial p=\rho U_0^2\partial\bar{p} \tag{3.16}$$

$$\bar{x}=\frac{x}{h}\Rightarrow x=h\bar{x}\Rightarrow\partial x=h\partial\bar{x} \tag{3.17}$$

Substituting the above transformations into (3.6) and multiplying through by $\dfrac{h}{\rho U_0^2}$

$$\left.\begin{array}{l}\left(\frac{h}{\rho U_0^2}\cdot\frac{\rho U_0^2}{h}\right)\frac{\partial \bar{u}}{\partial \bar{t}}+\left(\frac{h}{\rho U_0^2}\cdot\frac{\rho U_0 v_0}{h}\right)\frac{\partial \bar{u}}{\partial \bar{y}}=-\left(\frac{h}{\rho U_0^2}\cdot\frac{\rho U_0^2}{h}\right)\cdot\frac{\partial \bar{p}}{\partial \bar{x}}+\\ \left(\frac{h}{\rho U_0^2}\cdot\frac{U_0}{h^2}\right)\frac{\partial}{\partial \bar{y}}\left(\frac{c\mu_1}{T_1}\left(\left(T_2-T_1\right)\theta+T_1\right)\frac{\partial \bar{u}}{\partial \bar{y}}\right)-\\ \left(\frac{h}{\rho U_0^2}\right)\frac{\sigma B_0^2 U_0}{\left(\left(1+BiBe\right)^2+Be^2\right)}\left(\left(1+BiBe\right)\bar{u}+Be\bar{w}\right)-\left(\frac{h}{\rho U_0^2}\right)\frac{c\mu_1}{T_1}\left(\left(T_2-T_1\right)\theta+T_1\right)\frac{U_0\bar{u}}{k}+\\ \left(\frac{h}{\rho U_0^2}\cdot g\beta_T\left(\left(T_2-T_1\right)\theta+T_1\right)-T_1\right)+\left(\frac{h}{\rho U_0^2}\cdot g\beta_C\left(\left(C_2-C_1\right)\phi+C_1\right)-C_1\right)\end{array}\right\} \quad (3.18)$$

$$\left.\begin{array}{l}\frac{\partial \bar{u}}{\partial \bar{t}}+\frac{v_0}{U_0}\frac{\partial \bar{u}}{\partial \bar{y}}=-\frac{\partial \bar{p}}{\partial \bar{x}}+\frac{c\mu_1}{\rho U_0 h}\frac{\partial}{\partial \bar{y}}\left(\left(\frac{\left(T_2-T_1\right)}{T_1}\theta+1\right)\frac{\partial \bar{u}}{\partial \bar{y}}\right)-\\ \frac{\sigma B_0^2 h}{\rho U_0\left(\left(1+BiBe\right)^2+Be^2\right)}\left(\left(1+BiBe\right)\bar{u}+Be\bar{w}\right)-\\ \frac{hc\mu_1}{\rho U_0 k}\left(\left(\frac{\left(T_2-T_1\right)}{T_1}\theta+1\right)\bar{u}\right)+\frac{g\beta_T\left(T_2-T_1\right)h}{\rho U_0^2}\theta+\frac{g\beta_C\left(C_2-C_1\right)h}{\rho U_0^2}\phi\end{array}\right\} \quad (3.19)$$

Dropping bar, we have

$$\left.\begin{array}{l}\frac{\partial u}{\partial t}+S\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{c}{\mathrm{Re}}\frac{\partial}{\partial y}\left(\left(\alpha\theta+1\right)\frac{\partial u}{\partial y}\right)-\frac{Ha^2}{\left(\mathrm{Re}\left(1+BiBe\right)^2+Be^2\right)}\left(\left(1+BiBe\right)u+Bew\right)-\\ \frac{cP}{\mathrm{Re}}\left(\left(\alpha\theta+1\right)u\right)+Gr_\theta\theta+Gr_\phi\phi\end{array}\right\}$$

(3.20)

where

$S=\frac{v_0}{U_0}$ =Suction parameter,

$\mathrm{Re}=\frac{\rho U_0 h}{\mu_1}$ =Reynold's Number,

$Ha^2=\frac{\sigma B_0^2 h^2}{\mu_1}$ =Hartman Number,

$\alpha = \frac{(T_2 - T_1)}{T_1}$ =Coefficient of viscosity,

$P = \frac{1}{P^*}$, $P^* = \frac{k}{h^2}$ =Porosity parameter,

$Gr_\theta = \frac{hg\beta_T (T_2 - T_1)}{\rho U_0^2}$ =Thermal Grashof number,

$Gr_\phi = \frac{hg\beta_c (C_2 - C_1)}{\rho U_0^2}$ =Solutal Grashof number.

To write the second momentum equation (3.7) in its dimensionless form, we multiply equation (3.7) by $\frac{h}{\rho U_0^2}$

$$\left.\begin{array}{l}\left(\frac{h}{\rho U_0^2}\cdot\frac{\rho U_0^2}{h}\right)\frac{\partial \overline{w}}{\partial \overline{t}} + \left(\frac{h}{\rho U_0^2}\cdot\frac{\rho U_0 v_0}{h}\right)\frac{\partial \overline{w}}{\partial \overline{y}} = \left(\frac{h}{\rho U_0^2}\cdot\frac{U_0}{h^2}\right)\frac{\partial}{\partial \overline{y}}\left(\frac{c\mu_1}{T_1}\left((T_2 - T_1)\theta + T_1\right)\frac{\partial \overline{w}}{\partial \overline{y}}\right) \\ -\frac{h}{\rho U_0^2}\frac{c\mu_1}{T_1}\left((T_2 - T_1)\theta + T_1\right)\frac{U_0 \overline{w}}{k} - \left(\frac{h}{\rho U_0^2}\right)\frac{\sigma B_0^2 U_0}{\left((1+BiBe)^2 + Be^2\right)}\left((1+BiBe)\overline{w} - Be\overline{u}\right)\end{array}\right\} \tag{3.21}$$

Simplifying, we obtain,

$$\left.\begin{array}{l}\frac{\partial \overline{w}}{\partial \overline{t}} + \frac{v_0}{U_0}\frac{\partial \overline{w}}{\partial \overline{y}} = \frac{c\mu_1}{\rho U_0 h}\frac{\partial}{\partial \overline{y}}\left(\left(\frac{(T_2 - T_1)}{T_1}\theta + 1\right)\frac{\partial \overline{w}}{\partial \overline{y}}\right) - \\ \frac{\sigma B_0^2 h}{\rho U_0\left((1+BiBe)^2 + Be^2\right)}\left((1+BiBe)\overline{w} - Be\overline{u}\right) - \frac{hc\mu_1}{\rho U_0 k}\left(\left(\frac{(T_2 - T_1)}{T_1}\theta + 1\right)\overline{w}\right)\end{array}\right\} \tag{3.22}$$

Dropping bar, we have

$$\left.\begin{aligned}&\frac{\partial w}{\partial t}+S\frac{\partial w}{\partial y}=\frac{c}{\mathrm{Re}}\frac{\partial}{\partial y}\left((\alpha\theta+1)\frac{\partial w}{\partial y}\right)-\frac{Ha^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)w-Beu\right)-\\&\frac{cP}{\mathrm{Re}}\left((\alpha\theta+1)w\right)\end{aligned}\right\}\quad(3.23)$$

To write the energy equation (3.8) in its dimensionless form, we introduce the Rosseland approximation radiative heat flux vector by

$$\frac{\partial q}{\partial y}=4\alpha^2\left(T-T_1\right)\text{(Venkateswarlu and Padma 2015).}\qquad(3.24)$$

We rewrite equation (3.8) as

$$\left.\begin{aligned}&\rho c_p\frac{U_0}{h}(T_2-T_1)\frac{\partial\theta}{\partial t}+\rho c_p v_0(T_2-T_1)\frac{\partial\theta}{h\partial\bar{y}}=\frac{c_p}{\mathrm{Pr}\,h}\frac{\partial}{\partial\bar{y}}\left(\frac{c\mu_1}{T_1}\left((T_2-T_1)\theta+T_1\right)\frac{(T_2-T_1)}{h}\frac{\partial\theta}{\partial y}\right)+\\&\frac{c\mu_1}{T_1}\left((T_2-T_1)\theta+T_1\right)\left(\left(\frac{U_0}{h}\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2+\left(\frac{U_0}{h}\frac{\partial\bar{w}}{\partial\bar{y}}\right)^2\right)+\frac{\sigma B_0^2}{\left((1+BiBe)^2+Be^2\right)}\left[U_0^2\bar{u}^2+U_0^2\bar{w}^2\right]\\&-\frac{4\alpha_*^2(T_2-T_1)\theta}{h\rho c_p}\end{aligned}\right\}$$

(3.25)

Where

α_* is the mean radiation absorption parameter

Multiplying the above equation (3.25) by $\dfrac{h}{\rho c_p(T_2-T_1)U_0}$, we get

$$\left.\begin{aligned}&\left(\frac{h}{\rho c_p\left(T_2-T_1\right)U_0}\right)\rho c_p\left(T_2-T_1\right)\frac{\partial\theta}{\partial t}\cdot\frac{U_0}{h}+\rho c_p v_0\left(T_2-T_1\right)\left(\frac{h}{\rho c_p\left(T_2-T_1\right)U_0}\right)\frac{\partial\theta}{h\partial\bar{y}}=\\&\left(\frac{h}{\rho c_p\left(T_2-T_1\right)U_0}\right)\frac{c_p}{\Pr h}\frac{\partial}{\partial\bar{y}}\left[\frac{c\mu_1}{T_1}\left(\left(T_2-T_1\right)\theta+T_1\right)\frac{\left(T_2-T_1\right)}{h}\frac{\partial\theta}{\partial\bar{y}}\right]+\\&\left(\frac{h}{\rho c_p\left(T_2-T_1\right)U_0}\right)\frac{c\mu_1}{T_1}\left(\left(T_2-T_1\right)\theta+T_1\right)\left(\left(\frac{v_0}{h}\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2+\left(\frac{v_0}{h}\frac{\partial\bar{w}}{\partial\bar{y}}\right)^2\right)+\\&\left(\frac{h}{\rho c_p\left(T_2-T_1\right)U_0}\right)\frac{\sigma B_0^2}{\left(\left(1+BiBe\right)^2+Be^2\right)}\left[U_0^2\bar{u}^2+U_0^2\bar{w}^2\right]-\\&\left(\frac{h}{\rho c_p\left(T_2-T_1\right)U_0}\right)\frac{4\alpha_*^2\left(T_2-T_1\right)\theta}{h\rho c_p}\end{aligned}\right\}\quad(3.26)$$

Simplifying, we obtain

$$\left.\begin{aligned}&\frac{\partial\theta}{\partial\bar{t}}+\frac{v_0}{U_0}\frac{\partial\theta}{\partial\bar{y}}=\frac{c\mu_1}{\rho c_p U_0 h}\frac{c_p}{\Pr}\frac{\partial}{\partial\bar{y}}\left[\left(\frac{T_2-T_1}{T_1}\theta+1\right)\frac{\partial\theta}{\partial\bar{y}}\right]+\\&\frac{c\mu_1 U_0}{\rho c_p\left(T_2-T_1\right)h}\left(\frac{T_2-T_1}{T_1}\theta+1\right)\left[\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2+\left(\frac{\partial\bar{w}}{\partial\bar{y}}\right)^2\right]\\&+\frac{hU_0}{\rho c_p\left(T_2-T_1\right)}\frac{\sigma B_0^2}{\left((1+BiBe)^2+Be^2\right)}\left[\bar{u}^2+\bar{w}^2\right]-\frac{4\alpha_*^2 h}{\rho^2 c_p^2 U_0}\theta\end{aligned}\right\}\qquad(3.27)$$

Dropping bar for convenience yields

$$\left.\begin{aligned}&\frac{\partial\theta}{\partial t}+S\frac{\partial\theta}{\partial y}=\frac{c}{\mathrm{Re}\Pr}\frac{\partial}{\partial y}\left(\left(\alpha\theta+1\right)\frac{\partial\theta}{\partial y}\right)+\frac{cEc}{\mathrm{Re}}\left(\alpha\theta+1\right)\left(\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right)+\\&\frac{EcHa^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left(u^2+w^2\right)-Ra^2\theta\end{aligned}\right\}\qquad(3.28)$$

where

$Ra^2=\dfrac{4\alpha_*^2 h}{\left(\rho c_p\right)^2 U_0}$ =Radiation parameter,

$Ec = \dfrac{U_0^2}{c_p\left(T_2 - T_1\right)}$ =Eckert number.

To write the species equation in its dimensionless form, we rewrite equation (3.9) as

$$\left.\begin{aligned}&\frac{\rho\left(C_2-C_1\right)U_0}{h}\frac{\partial\phi}{\partial \bar{t}}+\frac{\rho v_0\left(C_2-C_1\right)}{h}\frac{\partial\phi}{\partial \bar{\bar{y}}}=\frac{1}{Sch}\frac{\partial}{\partial \bar{\bar{y}}}\left(\frac{c\mu_1}{T_1}\left(\left(T_2-T_1\right)\theta+T_1\right)\frac{\left(C_2-C_1\right)}{h}\frac{\partial\phi}{\partial \bar{\bar{y}}}\right)+\\&D\left(\frac{\left(T_2-T_1\right)}{h^2}\frac{\partial^2\theta}{\partial \bar{y}^2}\right)-\eta\left(\left(C_2-C_1\right)\phi\right)\end{aligned}\right\} \tag{3.29}$$

Multiplying through by $\dfrac{h}{\rho\left(C_2-C_1\right)U_0}$ gives

$$\left.\begin{aligned}&\left(\frac{h}{\rho\left(C_2-C_1\right)U_0}\right)\frac{\rho\left(C_2-C_1\right)U_0}{h}\frac{\partial\phi}{\partial \bar{t}}+\left(\frac{h}{\rho\left(C_2-C_1\right)U_0}\right)\frac{\rho v_0\left(C_2-C_1\right)}{h}\frac{\partial\phi}{\partial \bar{\bar{y}}}=\\&\left(\frac{h}{\rho\left(C_2-C_1\right)U_0}\right)\frac{1}{Sch}\frac{\partial}{\partial \bar{\bar{y}}}\left(\frac{c\mu_1}{T_1}\left(\left(T_2-T_1\right)\theta+T_1\right)\frac{\left(C_2-C_1\right)}{h}\frac{\partial\phi}{\partial \bar{\bar{y}}}\right)+\\&D\left(\frac{h}{\rho\left(C_2-C_1\right)U_0}\right)\left(\frac{\left(T_2-T_1\right)}{h^2}\frac{\partial^2\theta}{\partial \bar{y}^2}\right)-\left(\frac{h}{\rho\left(C_2-C_1\right)U_0}\right)\eta\left(\left(C_2-C_1\right)\phi\right)\end{aligned}\right\} \tag{3.30}$$

Simplfying,

$$\frac{\partial\phi}{\partial \bar{t}}+\frac{v_0}{U_0}\frac{\partial\phi}{\partial \bar{y}}=\frac{c\mu_1}{Sc\rho U_0 h}\frac{\partial}{\partial \bar{y}}\left[\left(\frac{T_2-T_1}{T_1}\theta+1\right)\frac{\partial\phi}{\partial \bar{\bar{y}}}\right]+\left(\frac{D\left(T_2-T_1\right)}{\rho U_0\left(C_2-C_1\right)h}\right)\frac{\partial^2\theta}{\partial \bar{y}^2}-\frac{h\eta}{\rho U_0}\phi \tag{3.31}$$

Dropping bar and simplifying further, we get

$$\frac{\partial\phi}{\partial t}+S\frac{\partial\phi}{\partial y}=\frac{c}{Sc\,\mathrm{Re}}\frac{\partial}{\partial y}\left(\left(\alpha\theta+1\right)\frac{\partial\phi}{\partial y}\right)+T_D\frac{\partial^2\theta}{\partial y^2}-K_r\phi \tag{3.32}$$

where

$Sc = \dfrac{U_0 h}{D}$ = Schimidt Number,

$$T_D = \left(\frac{D\left(T_2 - T_1\right)}{\rho U_0 \left(C_2 - C_1\right) h} \right) = \text{Thermo diffusion parameter,}$$

$$K_r = \frac{h\eta}{\rho U_0} = \text{Chemical reaction parameter.}$$

Next is to write the initial and boundary conditions in dimensionless form. To do this, we substitute the dimensionless variables into equation (3.10) and we have

$$\left.\begin{array}{l}
u(y,0) = U_0 \bar{u}(\bar{y},0) = 0, \quad u(-1,t) = U_0 \bar{u}(-1,t) = 0, \quad u(h,t) = U_0 \bar{u}(1,t) = U_0 \\
i.e \\
\bar{u}(\bar{y},0) = 0, \quad \bar{u}(-1,t) = 0, \quad \bar{u}(1,t) = 1 \\
w(y,0) = U_0 \bar{w}(\bar{y},0) = \frac{U_0 y}{h}(1-y) = U_0 \bar{y}(1-\bar{y}) \quad w(-h,t) = U_0 \bar{w}(-1,t) = 0, \quad w(h,t) = U_0 \bar{w}(1,t) = 0 \\
i.e \\
w(y,0) = \bar{y}(1-\bar{y}), \quad w(-1,t) = 0, \quad w(1,t) = 0 \\
T(y,0) = (T_2 - T_1)\theta(\bar{y},0) + T_1 = 0, \quad T(-h,t) = (T_2 - T_1)\theta(-1,t) + T_1 = T_1, \quad T(h,t) = (T_2 - T_1)\theta(1,t) + T_1 = T_2 \\
i.e \\
\theta(\bar{y},0) = -\frac{T_1}{T_2 - T_1} = \frac{T_1}{T_1 - T_2}, \quad \theta(-1,t) = 0, \quad \theta(1,t) = \frac{T_2 - T_1}{T_2 - T_1} = 1 \\
C(y,0) = (C_2 - C_1)\phi(\bar{y},0) + C_1 = 0, \quad C(-h,t) = (C_2 - C_1)\phi(-1,t) + C_1 = C_1, \quad C(h,t) = (C_2 - C_1)\phi(1,t) + C_1 = C_2 \\
i.e \\
\phi(\bar{y},0) = -\frac{C_1}{C_2 - C_1} = \frac{C_1}{C_1 - C_2}, \phi(-1,t) = 0, \quad \phi(1,t) = \frac{C_2 - C_1}{C_2 - C_1} = 1
\end{array}\right\} \tag{3.33}$$

The corresponding initial and boundary conditions in dimensionless form become

$$\left.\begin{array}{lll}
u(y,0) = 0, & u(-1,t) = 0, & u(1,t) = 1 \\
w(y,0) = y(1-y), & w(-1,t) = 0, & w(1,t) = 0 \\
\theta(y,0) = d_1, & \theta(-1,t) = 0, & \theta(1,t) = 1 \\
\phi(y,0) = d_2, & \phi(-1,t) = 0, & \phi(1,t) = 1
\end{array}\right\}, \tag{3.34}$$

where

$$d_1 = \frac{T_1}{T_1 - T_2}, \; d_2 = \frac{C_1}{C_1 - C_2}$$

Therefore, the dimensionless equations together with their initial and boundary conditions are given by

$$\frac{\partial u}{\partial t}+S\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{c}{\mathrm{Re}}\frac{\partial}{\partial y}\left((1+\alpha\theta)\frac{\partial u}{\partial y}\right)-\frac{Ha^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)u+Bew\right)-$$
$$\frac{cP}{\mathrm{Re}}(1+\alpha\theta)u+Gr_\theta\theta+Gr_\phi\phi$$

$$\frac{\partial w}{\partial t}+S\frac{\partial w}{\partial y}=\frac{c}{\mathrm{Re}}\frac{\partial}{\partial y}\left((1+\alpha\theta)\frac{\partial w}{\partial y}\right)-\frac{Ha^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)w-Beu\right)-$$
$$\frac{cP}{\mathrm{Re}}(1+\alpha\theta)w$$

$$\frac{\partial \theta}{\partial t}+S\frac{\partial \theta}{\partial y}=\frac{c}{\mathrm{Re}\,\mathrm{Pr}}\frac{\partial}{\partial y}\left((1+\alpha\theta)\frac{\partial \theta}{\partial y}\right)+\frac{cEc}{\mathrm{Re}}(1+\alpha\theta)\left(\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right)+$$
$$\frac{EcHa^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left(u^2+w^2\right)-Ra^2\theta$$

$$\frac{\partial \phi}{\partial t}+S\frac{\partial \phi}{\partial y}=\frac{c}{Sc\,\mathrm{Re}}\frac{\partial}{\partial y}\left((1+\alpha\theta)\frac{\partial \phi}{\partial y}\right)+T_D\frac{\partial^2\theta}{\partial y^2}-K_r\phi$$

Subject to the initial and boundary conditions

$$\left.\begin{array}{lll} u(y,0)=0, & u(-1,t)=0, & u(1,t)=1 \\ w(y,0)=y(1-y), & w(-1,t)=0, & w(1,t)=0 \\ \theta(y,0)=d_1, & \theta(-1,t)=0, & \theta(1,t)=1 \\ \phi(y,0)=d_2, & \phi(-1,t)=0, & \phi(1,t)=1 \end{array}\right\}$$

3.2.2 Transformation

Since the boundary conditions are from -1 to 1, we transform the problem into a half-plane problem for singularity.

Let

$$z=\frac{y+1}{2} \tag{3.35}$$

$$\left.\frac{\partial}{\partial y}=\frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial y}, \qquad \frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial z^2}\left(\frac{\partial z}{\partial y}\right)^2\right\} \tag{3.36}$$

Using equation (3.36), the dimensionless equations together with their initial and boundary conditions are transformed as follows:

First momentum equation (3.20) is transformed thus;

$$\left.\begin{array}{l}\frac{\partial u}{\partial t}+S\left(\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{c}{\mathrm{Re}}\left(\frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial y}\right)\left((\alpha\theta+1)\left(\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)\right)- \\ \frac{Ha^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)u+Bew\right)-\frac{cP}{\mathrm{Re}}(\alpha\theta+1)u+Gr_\theta\theta+Gr_\phi\phi\end{array}\right\} \tag{3.37}$$

i.e

$$\left.\begin{array}{l}\frac{\partial u}{\partial t}+S\left(\frac{\partial u}{\partial z}\cdot\frac{1}{2}\right)=-\frac{\partial p}{\partial x}+\frac{c}{\mathrm{Re}}\left(\frac{\partial}{\partial z}\cdot\frac{1}{2}\right)\left((\alpha\theta+1)\left(\frac{\partial u}{\partial z}\cdot\frac{1}{2}\right)\right)- \\ \frac{Ha^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)u+Bew\right)-\frac{cP}{\mathrm{Re}}(\alpha\theta+1)u+Gr_\theta\theta+Gr_\phi\phi\end{array}\right\} \tag{3.38}$$

Simplifying gives

$$\left.\begin{array}{l}\frac{\partial u}{\partial t}+\frac{S}{2}\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}+\frac{c}{4\mathrm{Re}}\frac{\partial}{\partial z}\left((\alpha\theta+1)\frac{\partial u}{\partial z}\right)-\frac{Ha^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)u+Bew\right)- \\ \frac{cP}{\mathrm{Re}}(\alpha\theta+1)u+Gr_\theta\theta+Gr_\phi\phi\end{array}\right\} \tag{3.39}$$

Second momentum equation (3.23) is transformed to

$$\left.\begin{array}{l}\frac{\partial w}{\partial t}+S\left(\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial y}\right)=\frac{c}{\mathrm{Re}}\left(\frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial y}\right)\left((\alpha\theta+1)\left(\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial y}\right)\right)-\\ \frac{Ha^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)w-Beu\right)-\frac{Pc}{\mathrm{Re}}(\alpha\theta+1)w\end{array}\right\} \qquad (3.40)$$

Simplifying using equation (3.36) gives

$$\left.\begin{array}{l}\frac{\partial w}{\partial t}+S\left(\frac{\partial w}{\partial z}\cdot\frac{1}{2}\right)=\frac{c}{\mathrm{Re}}\left(\frac{1}{2}\frac{\partial}{\partial z}\right)\left((\alpha\theta+1)\left(\frac{1}{2}\frac{\partial w}{\partial z}\right)\right)-\\ \frac{Ha^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)w-Beu\right)-\frac{cP}{\mathrm{Re}}(\alpha\theta+1)w\end{array}\right\} \qquad (3.41)$$

i.e

$$\left.\begin{array}{l}\frac{\partial w}{\partial t}+\frac{S}{2}\frac{\partial w}{\partial z}=\frac{c}{4\,\mathrm{Re}}\frac{\partial}{\partial z}\left((\alpha\theta+1)\frac{\partial w}{\partial z}\right)-\\ \frac{Ha^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)w-Beu\right)-\frac{cP}{\mathrm{Re}}(\alpha\theta+1)w\end{array}\right\} \qquad (3.42)$$

Energy equation (3.28) is transformed as

$$\left.\begin{array}{l}\frac{\partial \theta}{\partial t}+S\left(\frac{\partial \theta}{\partial z}\cdot\frac{\partial z}{\partial y}\right)=\frac{c}{\mathrm{Re}\,\mathrm{Pr}}\left(\frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial y}\right)\left((\alpha\theta+1)\left(\frac{\partial \theta}{\partial z}\cdot\frac{\partial z}{\partial y}\right)\right)+\\ \frac{cEc}{\mathrm{Re}}(\alpha\theta+1)\left(\left(\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial y}\right)^2\right)+\frac{EcHa^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left(u^2+w^2\right)-Ra^2\theta\end{array}\right\} \qquad (3.43)$$

Simplifying using equation (3.36) yields

$$\left.\begin{array}{l}\frac{\partial \theta}{\partial t}+S\left(\frac{\partial \theta}{\partial z}\cdot\frac{\partial z}{\partial y}\right)=\frac{c}{\mathrm{Re}\,\mathrm{Pr}}\left(\frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial y}\right)\left((\alpha\theta+1)\left(\frac{\partial \theta}{\partial z}\cdot\frac{\partial z}{\partial y}\right)\right)+\\ \frac{cEc}{\mathrm{Re}}(\alpha\theta+1)\left(\left(\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial y}\right)^2\right)+\frac{EcHa^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left(u^2+w^2\right)-Ra^2\theta\end{array}\right\} \qquad (3.44)$$

i.e

$$\left.\begin{aligned}&\frac{\partial\theta}{\partial t}+\frac{S}{2}\frac{\partial\theta}{\partial y}=\frac{c}{4\operatorname{Re}\Pr}\frac{\partial}{\partial y}\left(\left(\alpha\theta+1\right)\frac{\partial\theta}{\partial y}\right)+\frac{cEc}{4\operatorname{Re}}\left(\alpha\theta+1\right)\left(\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right)-\\&\frac{EcHa^2}{\operatorname{Re}\left((1+BiBe)^2+Be^2\right)}\left(u^2+w^2\right)-Ra^2\theta\end{aligned}\right\} \tag{3.45}$$

Concentration equation (3.32) can be transformed as:

$$\frac{\partial\phi}{\partial t}+S\left(\frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial y}\right)=\frac{c}{Sc\operatorname{Re}}\left(\frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial y}\right)\left(\left(\alpha\theta+1\right)\left(\frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial y}\right)\right)+T_D\left(\frac{\partial^2\theta}{\partial z^2}\left(\frac{\partial z}{\partial y}\right)^2\right)-K_r\phi \tag{3.46}$$

Simplifying using equation (3.36), we get

$$\frac{\partial\phi}{\partial t}+S\left(\frac{1}{2}\frac{\partial\phi}{\partial z}\right)=\frac{c}{Sc\operatorname{Re}}\left(\frac{1}{2}\frac{\partial}{\partial z}\right)\left(\left(\alpha\theta+1\right)\left(\frac{1}{2}\frac{\partial\phi}{\partial z}\right)\right)+T_D\left(\left(\frac{1}{2}\right)^2\frac{\partial^2\theta}{\partial z^2}\right)-K_r\phi \tag{3.47}$$

This gives

$$\frac{\partial\phi}{\partial t}+\frac{S}{2}\frac{\partial\phi}{\partial z}=\frac{c}{4Sc\operatorname{Re}}\frac{\partial}{\partial z}\left(\left(\alpha\theta+1\right)\frac{\partial\phi}{\partial z}\right)+T_D\frac{\partial^2\theta}{\partial z^2}-K_r\phi \tag{3.48}$$

Next, we transform the boundary conditions using (3.36) and obtain

$$\left.\begin{array}{lll}u(y,0)=0, & u(-1,t)=0, & u(1,t)=1\\ \Rightarrow u(z,0)=0, & u(0,t)=0, & u(1,t)=1\\ w(y,0)=y(1-y), & w(-1,t)=0, & w(1,t)=0\\ \Rightarrow w(z,0)=(2z-1)(2-2z), & w(0,t)=0, & w(1,t)=0\\ \theta(y,0)=d_1, & \theta(-1,t)=0, & \theta(1,t)=1\\ \Rightarrow \theta(z,0)=d_1, & \theta(0,t)=0, & \theta(1,t)=1\\ \phi(y,0)=d_2, & \phi(-1,t)=0, & \phi(1,t)=1\\ \Rightarrow \phi(z,0)=d_2, & \phi(0,t)=0, & \phi(1,t)=1\end{array}\right\} \tag{3.49}$$

Therefore the transformed equations together with their initial and boundary conditions are given as

$$\left.\begin{array}{l}\dfrac{\partial u}{\partial t}+\dfrac{S}{2}\dfrac{\partial u}{\partial z}=-\dfrac{\partial p}{\partial x}+\dfrac{c}{4\,\mathrm{Re}}\dfrac{\partial}{\partial z}\left((\alpha\theta+1)\dfrac{\partial u}{\partial z}\right)-\\ \dfrac{Ha^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)u+Bew\right)-\dfrac{cP}{\mathrm{Re}}(\alpha\theta+1)u+Gr_\theta\theta+Gr_\phi\phi\end{array}\right\} \tag{3.50}$$

$$\left.\begin{array}{l}\dfrac{\partial w}{\partial t}+\dfrac{S}{2}\dfrac{\partial w}{\partial z}=\dfrac{c}{4\,\mathrm{Re}}\dfrac{\partial}{\partial z}\left((\alpha\theta+1)\dfrac{\partial w}{\partial z}\right)-\\ \dfrac{Ha^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)w-Beu\right)-\dfrac{cP}{\mathrm{Re}}(\alpha\theta+1)w\end{array}\right\} \tag{3.51}$$

$$\left.\begin{array}{l}\dfrac{\partial \theta}{\partial t}+\dfrac{S}{2}\dfrac{\partial \theta}{\partial z}=\dfrac{c}{4\,\mathrm{Re}\,\mathrm{Pr}}\dfrac{\partial}{\partial z}\left[(\alpha\theta+1)\dfrac{\partial \theta}{\partial z}\right]+\dfrac{cEc}{4\,\mathrm{Re}}(\alpha\theta+1)\left(\left(\dfrac{\partial u}{\partial z}\right)^2+\left(\dfrac{\partial w}{\partial z}\right)^2\right)+\\ \dfrac{EcHa^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left(u^2+w^2\right)-Ra^2\theta\end{array}\right\} \tag{3.52}$$

$$\frac{\partial \phi}{\partial t}+\frac{S}{2}\frac{\partial \phi}{\partial z}=\frac{c}{4Sc\,\mathrm{Re}}\frac{\partial}{\partial z}\left((\alpha\theta+1)\frac{\partial \phi}{\partial z}\right)+\frac{T_D}{4}\frac{\partial^2\theta}{\partial z^2}-K_r\phi \tag{3.53}$$

$$\left.\begin{array}{lll}u(z,0)=0, & u(0,t)=0, & u(1,t)=1\\ w(z,0)=(2z-1)(2-2z), & w(0,t)=0, & w(1,t)=0\\ \theta(z,0)=d_1, & \theta(0,t)=0, & \theta(1,t)=1\\ \phi(z,0)=d_2, & \phi(0,t)=0, & \phi(1,t)=1\end{array}\right\} \tag{3.54}$$

Next, we shall establish the conditions for the existence of unique solution of the transient state.

3.2.3 Existence and uniqueness of solution of transient state

Here, we consider equations (3.50) – (3.53) when μ is constant i.e when $\alpha = 0$ and $B_i = B_e = 1$. Then equations (3.50) – (3.53) reduce to

$$\frac{\partial u}{\partial t} + \frac{S}{2}\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \frac{c}{4\,\mathrm{Re}}\frac{\partial^2 u}{\partial z^2} - \frac{Ha^2}{5\,\mathrm{Re}}\left(2u + w\right) - \frac{cP}{\mathrm{Re}}u + Gr_\theta \theta + Gr_\phi \phi \tag{3.55}$$

$$\frac{\partial w}{\partial t} + \frac{S}{2}\frac{\partial w}{\partial z} = \frac{c}{4\,\mathrm{Re}}\frac{\partial^2 w}{\partial z^2} - \frac{Ha^2}{5\,\mathrm{Re}}\left(2w - u\right) - \frac{cP}{\mathrm{Re}}w \tag{3.56}$$

$$\frac{\partial \theta}{\partial t} + \frac{S}{2}\frac{\partial \theta}{\partial z} = \frac{c}{4\,\mathrm{Re}\,\mathrm{Pr}}\frac{\partial^2 \theta}{\partial z^2} + \frac{cEc}{4\,\mathrm{Re}}\left(\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right) + \frac{EcHa^2}{5\,\mathrm{Re}}\left(u^2 + w^2\right) - Ra^2\theta \tag{3.57}$$

$$\frac{\partial \phi}{\partial t} + \frac{S}{2}\frac{\partial \phi}{\partial z} = \frac{c}{4Sc\,\mathrm{Re}}\frac{\partial^2 \phi}{\partial z^2} + \frac{T_D}{4}\frac{\partial^2 \theta}{\partial z^2} - K_r \phi \tag{3.58}$$

To scientists and engineers, the question of existence and uniqueness of solution remain to be a pivot in models and designs. When a problem is formulated, we need to examine the solution(s) so as to predict the behaviour of such solution(s). We are interested in the existence and uniqueness of solution of system of equation (3.55) – (3.58) satisfying (3.54) in order to be able to predict the behavior of the solution.

This question of existence and uniqueness of solutions to these equations has been addressed by Ayeni (1978) who considered a similar set of equations and showed among other results that existence and uniqueness are somewhat well known. In his work, he studied the following system of parabolic equations

$$\left.\begin{aligned} \frac{\partial \phi}{\partial t} &= \Delta \phi + f\left(x, t, \varphi, u, v\right), \quad && x \in R^n, t > 0 \\ \frac{\partial u}{\partial t} &= \Delta u + g\left(x, t, \varphi, u, v\right), \quad && x \in R^n, t > 0 \\ \frac{\partial v}{\partial t} &= \Delta v + h\left(x, t, \varphi, u, v\right), \quad && x \in R^n, t > 0 \end{aligned}\right\} \tag{3.59}$$

$$\left.\begin{aligned} \phi(x,0) &= f_0(x) \\ u(x,0) &= g_0(x) \\ v(x,0) &= h_0(x) \\ x &= (x_1, x_2, ..., x_n) \end{aligned}\right\} \qquad (3.60)$$

(S.1) $f_0(x), g_0(x)$ and $h_0(x)$ are bounded for $x \in R^n$. Each has at most a countable number of discontinuities.

(S.2) f, g, h satisfies the uniform Lipschitz condition

$$|\varphi(x,t,\phi_1,u_1,v_1) - \varphi(x,t,\phi_2,u_2,v_2)| \le M(|\phi_1 - \phi_2| + |u_1 - u_2| + |v_1 - v_2|), \quad (x,t) \in G \quad (3.61)$$

where

$$G = \{(x,t) : x \in R^n, \ 0 < t < \tau\}.$$

Our proof of existence of unique solution of the system of parabolic equations (3.55) – (3.58) will be analogous to his proof.

Theorem 3.1: There exists a unique solution $u(z,t),\ w(z,t),\ \theta(z,t),\ \phi(z,t)$ of equation (3.55) – (3.58) which satisfies (3.54).

In the proof we shall need the following Lemma:

Lemma 3.1 (Ayeni (1978)):

Let (f_0, g_0, h_0) and (f, g, h) satisfy (S.1) and (S.2) respectively, then there exists a solution of problem (3.59).

Proof of Theorem 3.1: We rewrite the equation (3.55) – (3.58) as;

$$\frac{\partial u}{\partial t} + \frac{S}{2}\frac{\partial u}{\partial z} = \frac{c}{4\,\mathrm{Re}}\frac{\partial^2 u}{\partial z^2} + f(z,t,u,w,\theta,\phi), \quad z \in R^n, t > 0 \qquad (3.62)$$

$$\frac{\partial w}{\partial t}+\frac{S}{2}\frac{\partial w}{\partial z}=\frac{c}{4\operatorname{Re}}\frac{\partial^2 w}{\partial z^2}+g(z,t,u,w,\theta,\phi), \quad z\in R^n, t>0 \tag{3.63}$$

$$\frac{\partial \theta}{\partial t}+\frac{S}{2}\frac{\partial \theta}{\partial z}=\frac{c}{4\operatorname{Re}\Pr}\frac{\partial^2 \theta}{\partial z^2}+h(z,t,u,w,\theta,\phi), \quad z\in R^n, t>0 \tag{3.64}$$

$$\frac{\partial \phi}{\partial t}+\frac{S}{2}\frac{\partial \phi}{\partial z}=\frac{c}{4Sc\operatorname{Re}}\frac{\partial^2 \phi}{\partial z^2}+k(z,t,u,w,\theta,\phi), \quad z\in R^n, t>0\,, \tag{3.65}$$

where

$$f\left(z,t,u,w,\theta,\phi\right)=Gr_\theta\theta+Gr_\varphi\phi-\frac{\partial p}{\partial x}-\frac{Ha^2}{5\operatorname{Re}}\left(2u+w\right)-\frac{cP}{\operatorname{Re}}u \tag{3.66}$$

$$g\left(z,t,u,w,\theta,\varphi\right)=-\frac{Ha^2}{5\operatorname{Re}}\left(2w-u\right)-\frac{cP}{\operatorname{Re}}w \tag{3.67}$$

$$h(z,t,u,w,\theta,\phi)=\frac{cEc}{4\operatorname{Re}}\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right)+\frac{EcHa^2}{5\operatorname{Re}}\left(u^2+w^2\right)-Ra^2\theta \tag{3.68}$$

$$k(z,t,u,w,\theta,\phi)=\frac{T_D}{4}\frac{\partial^2\theta}{\partial z^2}-K_r\phi \tag{3.69}$$

Ignoring the second term at the right hand side, the fundamental solutions of equation (3.62) – (3.65) (Toki and Tokis (2007)) are:

$$F(z,t)=\frac{z}{2\pi^{\frac{1}{2}}\left(\frac{c}{4\operatorname{Re}}\right)^{\frac{1}{2}}t^{\frac{3}{2}}}\exp\left(\frac{\operatorname{Re}S}{c}z-\frac{\operatorname{Re}S^2t}{4c}-\frac{\operatorname{Re}z}{ct}\right) \tag{3.70}$$

$$G(z,t)=\frac{z}{2\pi^{\frac{1}{2}}\left(\frac{c}{4\operatorname{Re}}\right)^{\frac{1}{2}}t^{\frac{3}{2}}}\exp\left(\frac{\operatorname{Re}S}{c}z-\frac{\operatorname{Re}S^2t}{4c}-\frac{\operatorname{Re}z}{ct}\right) \tag{3.71}$$

$$H(z,t) = \frac{z}{2\pi^{\frac{1}{2}}\left(\frac{c}{4\operatorname{Re}\Pr}\right)^{\frac{1}{2}} t^{\frac{3}{2}}} \exp\left(\frac{\operatorname{Re}\Pr S}{c} z - \frac{\operatorname{Re}\Pr S^2 t}{4c} - \frac{\operatorname{Re}\Pr z}{ct}\right) \tag{3.72}$$

$$K(z,t) = \frac{z}{2\pi^{\frac{1}{2}}\left(\frac{c}{4Sc\operatorname{Re}}\right)^{\frac{1}{2}} t^{\frac{3}{2}}} \exp\left(\frac{Sc\operatorname{Re} S}{c} z - \frac{Sc\operatorname{Re} S^2 t}{4c} - \frac{Sc\operatorname{Re} z}{ct}\right) \tag{3.73}$$

Clearly, $f(z,t,u,w,\theta,\phi)$, $g(z,t,u,w,\theta,\phi)$, $h(z,t,u,w,\theta,\phi)$, $k(z,t,u,w,\theta,\phi)$ are Lipschitz continuous. Hence by Lemma 3.1, the result follows. This completes the proof.

Next, we shall examine the properties of solution of the transient state.

3.2.4 Properties of solution of transient state reaction

Here, we show that $u(z,t), w(z,t), \theta(z,t)\, and\, \phi(z,t)$ are bounded and increasing functions of time.

Theorem 3.2:

Let $Ec \to 0,\ Ha^2 \to 0,\ \frac{\partial p}{\partial x} \to 0,\ S > 0,\ c > 0,\ \operatorname{Re} > 0,\ P > 0,\ \Pr > 0,\ Ra^2 > 0,$

$Gr_\theta > 0,\ Gr_\phi > 0,\ Sc > 0,\ k_r > 0,\ T_D > 0$. Then, the equations (3.50) – (3.53) have a solution for all $t \geq 0$.

Proof: Equations (3.50) – (3.53) can be written respectively as:

$$Lu = f(z,t,u), \quad Lw = f(z,t,w), \quad L\theta = f(z,t,\theta), \quad L\phi = f(z,t,\phi), \tag{3.74}$$

where

$$Lu = \frac{\partial u}{\partial t} + \frac{S}{2}\frac{\partial u}{\partial z} - \frac{c}{4\mathrm{Re}}\frac{\partial^2 u}{\partial z^2} + \frac{cP}{\mathrm{Re}}u \tag{3.75}$$

$$f(z,t,u) = Gr_\theta \theta + Gr_\phi \phi$$

$$Lw = \frac{\partial w}{\partial t} + \frac{S}{2}\frac{\partial w}{\partial z} - \frac{c}{4\mathrm{Re}}\frac{\partial^2 w}{\partial z^2} + \frac{cP}{\mathrm{Re}}w \tag{3.76}$$

$$f(z,t,w) = 0$$

$$L\theta = \frac{\partial \theta}{\partial t} + \frac{S}{2}\frac{\partial \theta}{\partial z} - \frac{c}{4\mathrm{Re}\,\mathrm{Pr}}\frac{\partial^2 \theta}{\partial z^2} + Ra^2\theta \tag{3.77}$$

$$f(z,t,\theta) = 0$$

$$L\phi = \frac{\partial \phi}{\partial t} + \frac{S}{2}\frac{\partial \phi}{\partial z} = \frac{c}{4Sc\,\mathrm{Re}}\frac{\partial^2 \phi}{\partial z^2} + K_r\phi \tag{3.78}$$

$$f(z,t,\phi) = \frac{T_D}{4}\frac{\partial^2 \theta}{\partial z^2}$$

Consider

$$\underline{u}(z,t) = 0, \quad \underline{w}(z,t) = 0, \quad \underline{\theta}(z,t) = 0, \quad \underline{\phi}(z,t) = 0 \tag{3.79}$$

We shall show that (3.79) are the lower solutions to equation (3.74)

Clearly,

$$\left.\begin{array}{lll} \underline{u}(z,0) = 0, & \underline{u}(0,t) = 0, & \underline{u}(1,t) = 0 \\ \underline{w}(z,0) = 0, & \underline{w}(0,t) = 0, & \underline{w}(1,t) = 0 \\ \underline{\theta}(z,0) = 0, & \underline{\theta}(0,t) = 0, & \underline{\theta}(1,t) = 0 \\ \underline{\phi}(z,0) = 0, & \underline{\phi}(0,t) = 0, & \underline{\phi}(1,t) = 0 \end{array}\right\} \tag{3.84}$$

Now

$$\frac{\partial \underline{u}}{\partial t}=\frac{\partial \underline{u}}{\partial z}=\frac{\partial^2 \underline{u}}{\partial z^2}=\underline{u}=0 \tag{3.85}$$

$$\frac{\partial \underline{w}}{\partial t}=\frac{\partial \underline{w}}{\partial z}=\frac{\partial^2 \underline{w}}{\partial z^2}=\underline{w}=0 \tag{3.86}$$

$$\frac{\partial \underline{\theta}}{\partial t}=\frac{\partial \underline{\theta}}{\partial z}=\frac{\partial^2 \underline{\theta}}{\partial z^2}=\underline{\theta}=0 \tag{3.87}$$

$$\frac{\partial \underline{\phi}}{\partial t}=\frac{\partial \underline{\phi}}{\partial z}=\frac{\partial^2 \underline{\phi}}{\partial z^2}=\underline{\phi}=0 \tag{3.88}$$

These imply

$$L\underline{u}=0, \qquad L\underline{w}=0, \qquad L\underline{\theta}=0, \qquad L\underline{\phi}=0 \tag{3.89}$$

$$f\left(z,t,\underline{u}\right)=Gr_\theta\underline{\theta}+Gr_\varphi\underline{\varphi}, \quad f\left(z,t,\underline{w}\right)=0, \quad f\left(z,t,\underline{\theta}\right)=0, \quad f\left(z,t,\underline{\varphi}\right)=\frac{T_D}{4}\frac{\partial^2\theta}{\partial z^2} \tag{3.90}$$

Hence

$$L\underline{u}\le f\left(z,t,\underline{u}\right), \qquad L\underline{w}\le f\left(z,t,\underline{w}\right), \qquad L\underline{\theta}\le f\left(z,t,\underline{\theta}\right), \qquad L\underline{\phi}\le f\left(z,t,\underline{\phi}\right) \tag{3.91}$$

By definition 1, $\underline{u}(z,t)=0,\quad \underline{w}(z,t)=0,\quad \underline{\theta}(z,t)=0,\quad \underline{\phi}(z,t)=0$ are lower solutions of equation (3.74).

Also consider

$$\left.\begin{aligned}\bar{u}(z,t)&=1+\left(Gr_\theta+Gr_\varphi\right)t,\\ \bar{w}(z,t)&=z(1-z)+t,\\ \bar{\theta}(z,t)&=1+d_1+\frac{cP}{\mathrm{Re}}t,\\ \bar{\phi}(z,t)&=1+d_2+\frac{cP}{\mathrm{Re}}t\end{aligned}\right\} \tag{3.92}$$

We shall show that (3.92) are the upper solutions to equation (3.74). Here

$$\left.\begin{array}{lll}\bar{u}(z,0)=1, & \bar{u}(0,t)=1+\left(Gr_\theta+Gr_\varphi\right)t, & \bar{u}(1,t)=1+\left(Gr_\theta+Gr_\varphi\right)t\\ \bar{w}(z,0)=z(1-z), & \bar{w}(0,t)=t, & \bar{w}(1,t)=t\\ \bar{\theta}(z,0)=1+d_1, & \bar{\theta}(0,t)=1+d_1+\frac{cP}{\mathrm{Re}}t, & \bar{\theta}(1,t)=1+d_1+\frac{cP}{\mathrm{Re}}t\\ \bar{\phi}(z,0)=1+d_2, & \bar{\phi}(0,t)=1+d_2+\frac{cP}{\mathrm{Re}}t, & \bar{\phi}(1,t)=1+d_2+\frac{cP}{\mathrm{Re}}t\end{array}\right\} \tag{3.93}$$

Now

$$\frac{\partial\bar{u}}{\partial t}=\left(Gr_\theta+Gr_\varphi\right),\quad \frac{\partial\bar{u}}{\partial z}=0,\quad \frac{\partial^2\bar{u}}{\partial z^2}=0,\quad \bar{u}=1+\left(Gr_\theta+Gr_\varphi\right)t \tag{3.94}$$

$$\frac{\partial\bar{w}}{\partial t}=1,\quad \frac{\partial\bar{w}}{\partial z}=1-2z,\quad \frac{\partial^2\bar{w}}{\partial z^2}=-2,\quad \bar{w}=z(1-z)+t \tag{3.95}$$

$$\frac{\partial\bar{\theta}}{\partial t}=\frac{cP}{\mathrm{Re}},\quad \frac{\partial\bar{\theta}}{\partial z}=0,\quad \frac{\partial^2\bar{\theta}}{\partial z^2}=0,\quad \bar{\theta}=1+d_1+\frac{cP}{\mathrm{Re}}t \tag{3.96}$$

$$\frac{\partial\bar{\phi}}{\partial t}=\frac{cP}{\mathrm{Re}},\quad \frac{\partial\bar{\phi}}{\partial z}=0,\quad \frac{\partial^2\bar{\phi}}{\partial z^2}=0,\quad \bar{\phi}=1+d_2+\frac{cP}{\mathrm{Re}}t \tag{3.97}$$

These imply

$$\left.\begin{array}{l}L\bar{u}=\left(Gr_\theta+Gr_\varphi\right)+\frac{cP}{\mathrm{Re}}\bar{u},\\ L\bar{w}=1+\frac{S}{2}(1-2z)+\frac{c}{2\,\mathrm{Re}\,\mathrm{Pr}}+\frac{cP}{\mathrm{Re}}\bar{w},\\ L\bar{\theta}=\frac{cP}{\mathrm{Re}}+Ra^2\bar{\theta},\\ L\bar{\phi}=\frac{cP}{\mathrm{Re}}+k_r\bar{\phi}\end{array}\right\} \tag{3.98}$$

$$f\left(z,t,\bar{u}\right)=Gr_\theta\bar{\theta}+Gr_\varphi\bar{\phi},\quad f\left(z,t,\bar{w}\right)=0,\quad f\left(z,t,\bar{\theta}\right)=0,\quad f\left(z,t,\bar{\phi}\right)=\frac{T_D}{4}\frac{\partial^2\bar{\theta}}{\partial z^2} \tag{3.99}$$

Hence

$$L\underline{u} \geq f\left(z,t,\bar{u}\right), \quad L\underline{w} \geq f\left(z,t,\bar{w}\right), \quad L\underline{\theta} \geq f\left(z,t,\bar{\theta}\right), \quad L\underline{\phi} \geq f\left(z,t,\bar{\phi}\right) \quad (3.100)$$

By definition 2,

$$\bar{u}(z,t) = 1 + \left(Gr_\theta + Gr_\varphi\right)t, \quad \bar{w}(z,t) = z(1-z) + t, \quad \bar{\theta}(z,t) = 1 + d_1 + \frac{cP}{\mathrm{Re}}t, \quad \bar{\phi}(z,t) = 1 + d_2 + \frac{cP}{\mathrm{Re}}t$$

are the upper solutions of equation (3.74).

Hence, there exists a solution of problem (3.50) – (3.54). This completes the proof.

Theorem 3.3:

Let $S \to 0,\ Ec \to 0,\ Ha^2 > 0,\ \dfrac{\partial p}{\partial x} > 0,\ \ c = 4\,\mathrm{Re},\ \mathrm{Re} > 0,\ P > 0,\ \mathrm{Pr} = Sc = 1,$

$Ra^2 > 0,\ Gr_\theta > 0,\ Gr_\varphi > 0,\ Sc > 0,\ k_r > 0,\ T_D > 0$ in (3.50) – (3.54). Then $\dfrac{\partial u}{\partial t} \geq 0, \quad \dfrac{\partial w}{\partial t} \geq 0,$

$\dfrac{\partial \theta}{\partial t} \geq 0, \quad \dfrac{\partial \phi}{\partial t} \geq 0$.

In the proof, we shall make use of the following lemma of Kolodner and Pederson (1966).

Lemma (Kolodner and Pederson (1966)):

Let $v(x,t) = 0\left(e^{\alpha|x|^2}\right)$ be a solution on $R^n \times [0,t)$ of the differential inequality

$\dfrac{\partial v}{\partial t} - \Delta v + K(x,t)v \geq 0$ where K is bounded from below. If $v(x,0) \geq 0$, then $v(x,t) \geq 0$

for all $(x,t) \in R^n \times [0,t_0)$.

Proof of theorem 3.3:

Given

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial z^2}+\frac{1}{\mathrm{Re}}\left(cP+\frac{2Ha^2}{5}\right)u=Gr_\theta\theta+Gr_\phi\phi-\frac{Ha^2}{5\mathrm{Re}}w-\frac{\partial p}{\partial x} \tag{3.101}$$

$$\frac{\partial w}{\partial t}-\frac{\partial^2 w}{\partial z^2}+\frac{1}{\mathrm{Re}}\left(cP+\frac{2Ha^2}{5}\right)w=\frac{Ha^2}{5\mathrm{Re}}u \tag{3.102}$$

$$\frac{\partial \theta}{\partial t}-\frac{\partial^2 \theta}{\partial z^2}+Ra^2\theta=0 \tag{3.103}$$

$$\frac{\partial \phi}{\partial t}-\frac{\partial^2 \phi}{\partial z^2}+K_r\phi=\frac{T_D}{4}\frac{\partial^2 \theta}{\partial z^2} \tag{3.104}$$

Differentiating (3.101) – (3.104) with respect to t, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)-\frac{\partial^2}{\partial z^2}\left(\frac{\partial u}{\partial t}\right)+\frac{1}{\mathrm{Re}}\left(cP+\frac{2Ha^2}{5}\right)\left(\frac{\partial u}{\partial t}\right)=Gr_\theta\left(\frac{\partial \theta}{\partial t}\right)+Gr_\phi\left(\frac{\partial \phi}{\partial t}\right)-\frac{Ha^2}{5\mathrm{Re}}\left(\frac{\partial w}{\partial t}\right) \tag{3.105}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\right)-\frac{\partial^2}{\partial z^2}\left(\frac{\partial w}{\partial t}\right)+\frac{1}{\mathrm{Re}}\left(cP+\frac{2Ha^2}{5}\right)\left(\frac{\partial w}{\partial t}\right)=\frac{Ha^2}{5\mathrm{Re}}\left(\frac{\partial u}{\partial t}\right) \tag{3.106}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \theta}{\partial t}\right)-\frac{\partial^2}{\partial z^2}\left(\frac{\partial \theta}{\partial t}\right)+Ra^2\left(\frac{\partial \theta}{\partial t}\right)=0 \tag{3.107}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t}\right)-\frac{\partial^2}{\partial z^2}\left(\frac{\partial \phi}{\partial t}\right)+K_r\left(\frac{\partial \phi}{\partial t}\right)=\frac{T_D}{4}\frac{\partial^2}{\partial z^2}\left(\frac{\partial \theta}{\partial t}\right) \tag{3.108}$$

Let

$$h=\frac{\partial u}{\partial t},\qquad m=\frac{\partial w}{\partial t},\qquad r=\frac{\partial \theta}{\partial t},\qquad s=\frac{\partial \phi}{\partial t}$$

Then (3.105) – (3.108) become

$$\frac{\partial h}{\partial t}-\frac{\partial^2 h}{\partial z^2}+\frac{1}{\text{Re}}\left(cP+\frac{2Ha^2}{5}\right)h=Gr_\theta r+Gr_\phi s-\frac{Ha^2}{5\text{Re}}m\Rightarrow\frac{\partial h}{\partial t}-\frac{\partial^2 h}{\partial z^2}+K(z,t)h\geq 0 \tag{3.109}$$

$$\frac{\partial m}{\partial t}-\frac{\partial^2 m}{\partial z^2}+\frac{1}{\text{Re}}\left(cP+\frac{2Ha^2}{5}\right)m=\frac{Ha^2}{5\text{Re}}p\Rightarrow\frac{\partial m}{\partial t}-\frac{\partial^2 m}{\partial z^2}+K_1(z,t)m\geq 0 \tag{3.110}$$

$$\frac{\partial r}{\partial t}-\frac{\partial^2 r}{\partial z^2}+Ra^2 r=0\Rightarrow\frac{\partial r}{\partial t}-\frac{\partial^2 r}{\partial z^2}+K_2(z,t)r=0 \tag{3.111}$$

$$\frac{\partial s}{\partial t}-\frac{\partial^2 s}{\partial z^2}+K_r s=\frac{T_D}{4}\frac{\partial^2 r}{\partial z^2}\Rightarrow\frac{\partial s}{\partial t}-\frac{\partial^2 s}{\partial z^2}+K_3(z,t)s\geq 0 \tag{3.112}$$

where

$$K(z,t)=\frac{1}{\text{Re}}\left(cP+\frac{2Ha^2}{5}\right), K_1(z,t)=\frac{1}{\text{Re}}\left(cP+\frac{2Ha^2}{5}\right),\ K_2(z,t)=Ra^2,\ K_3(z,t)=K_r$$

Clearly, K, K_1, K_2, K_3 are bounded from below. Hence by Kolodner and Pederson lemma $h(z,t)\geq 0,\ m(z,t)\geq 0,\ \ r(z,t)\geq 0,\ s(z,t)\geq 0$ i.e,

$$\frac{\partial u}{\partial t}\geq 0,\ \frac{\partial w}{\partial t}\geq 0,\ \frac{\partial \theta}{\partial t}\geq 0,\ \frac{\partial \phi}{\partial t}\geq 0.$$

This completes the proof.

Equations (3.50) - (3.53) satisfying (3.54) will be considered in three forms:

Case 1: When the pressure gradient is a function of time.

Case 2: When pressure gradient is a constant.

Case 3: When the reaction is in steady state.

3.2.5 Case 1: When the pressure gradient is a function of time: $\frac{\partial p}{\partial x}=\frac{dp}{dx}=-\lambda e^{-\varepsilon t}$

In this case, equations (3.50) - (3.53) reduce to;

$$\frac{\partial u}{\partial t}+\frac{S}{2}\frac{\partial u}{\partial z}=-\lambda e^{-\varepsilon t}+\frac{c}{4\,\mathrm{Re}}\frac{\partial}{\partial z}\left((1+\alpha\theta)\frac{\partial u}{\partial z}\right)-\frac{Ha^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)^2u+Be^2w\right)-\frac{cP}{\mathrm{Re}}(\alpha\theta+1)u+Gr_\theta\theta+Gr_\phi\phi \tag{3.113}$$

$$\frac{\partial w}{\partial t}+\frac{S}{2}\frac{\partial w}{\partial z}=\frac{c}{4\,\mathrm{Re}}\frac{\partial}{\partial z}\left((1+\alpha\theta)\frac{\partial w}{\partial z}\right)-\frac{Ha^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)^2w-Be^2u\right)-\frac{cP}{\mathrm{Re}}(\alpha\theta+1)w \tag{3.114}$$

$$\frac{\partial \theta}{\partial t}+\frac{S}{2}\frac{\partial \theta}{\partial z}=\frac{c}{4\,\mathrm{Re}\,\mathrm{Pr}}\frac{\partial}{\partial z}\left((1+\alpha\theta)\frac{\partial \theta}{\partial z}\right)+\frac{cEc}{4\,\mathrm{Re}}(1+\alpha\theta)\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right)+\frac{EcHa^2}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left(u^2+w^2\right)-Ra^2\theta \tag{3.115}$$

$$\frac{\partial \phi}{\partial t}+\frac{S}{2}\frac{\partial \phi}{\partial z}=\frac{c}{4Sc\,\mathrm{Re}}\frac{\partial}{\partial z}\left((1+\alpha\theta)\frac{\partial \phi}{\partial z}\right)+\frac{T_D}{4}\frac{\partial^2\theta}{\partial z^2}-K_r\phi \tag{3.116}$$

3.2.5.1 Solution of Case 1

Here, we solve equations (3.113) – (3.116) satisfying (3.54) using perturbation method and eigenfunction expansion technique.

Let $0<\alpha<<1$ such that $Ec=b\alpha, S=e\alpha,\ Be=f\alpha, Gr_\theta=g\alpha, Gr_\phi=l\alpha$ and suppose the solutions of equations (3.113) – (3.116) satisfying (3.54) can be expressed as

$$\left.\begin{aligned} u(z,t)&=u_0(z,t)+\alpha u_1(z,t)+\ldots\\ w(z,t)&=w_0(z,t)+\alpha w_1(z,t)+\ldots\\ \theta(z,t)&=\theta_0(z,t)+\alpha\theta_1(z,t)+\ldots\\ \phi(z,t)&=\phi_0(z,t)+\alpha\phi_1(z,t)+\ldots \end{aligned}\right\} \tag{3.117}$$

Using (3.117) in (3.118) – (3.121) satisfying (3.54) and processing, we have

$$\left.\begin{array}{l}\frac{\partial\left(u_0+\alpha u_1\right)}{\partial t}+\frac{e\alpha}{2}\frac{\partial\left(u_0+\alpha u_1\right)}{\partial z}=-\lambda e^{-\varepsilon t}+\frac{c}{4\operatorname{Re}}\frac{\partial}{\partial z}\left(\left(\alpha\left(\theta_0+\alpha\theta_1\right)+1\right)\frac{\partial\left(u_0+\alpha u_1\right)}{\partial z}\right)-\\ \frac{Ha^2}{\operatorname{Re}\left(\left(1+Bif\alpha\right)^2+\left(f\alpha\right)^2\right)}\left(\left(1+Bif\alpha\right)\left(u_0+\alpha u_1\right)+f\alpha\left(w_0+\alpha w_1\right)\right)-\\ \frac{cP}{\operatorname{Re}}\left(\left(\alpha\left(\theta_0+\alpha\theta_1\right)+1\right)\left(u_0+\alpha u_1\right)\right)+g\alpha\left(\theta_0+\alpha\theta_1\right)+l\alpha\left(\phi_0+\alpha\phi_1\right)\end{array}\right\}(3.118)$$

Expanding, we have

$$\left.\begin{array}{l}\frac{\partial u_0}{\partial t}+\alpha\frac{\partial u_1}{\partial t}+\frac{e\alpha}{2}\frac{\partial u_0}{\partial z}+\frac{e\alpha^2}{2}\frac{\partial u_1}{\partial z}=-\lambda e^{-\varepsilon t}+\frac{c}{4\operatorname{Re}}\frac{\partial}{\partial z}\left(\left(\alpha\left(\theta_0+\alpha\theta_1\right)+1\right)\left(\frac{\partial u_0}{\partial z}+\alpha\frac{\partial u_1}{\partial z}\right)\right)-\\ \frac{Ha^2}{\operatorname{Re}\left(\left(1+Bif\alpha\right)^2+\left(f\alpha\right)^2\right)}\left(\left(1+Bif\alpha\right)\left(u_0+\alpha u_1\right)+f\alpha\left(w_0+\alpha w_1\right)\right)-\\ \frac{cP}{\operatorname{Re}}\left(\left(\alpha\left(\theta_0+\alpha\theta_1\right)+1\right)\left(u_0+\alpha u_1\right)\right)+g\alpha\left(\theta_0+\alpha\theta_1\right)+l\alpha\left(\phi_0+\alpha\phi_1\right)\end{array}\right\}\quad(3.119)$$

Collecting like powers of α , we obtain for

α^0 :

$$\frac{\partial u_0}{\partial t}=\frac{c}{4\operatorname{Re}}\frac{\partial}{\partial z}\left(\frac{\partial u_0}{\partial z}\right)-\frac{Ha^2}{\operatorname{Re}}u_0-\frac{cP}{\operatorname{Re}}u_0-\lambda e^{-\varepsilon t}\qquad(3.120)$$

$$\left.\begin{array}{l}\alpha^1:\\ \frac{\partial u_1}{\partial t}+\frac{e}{2}\frac{\partial u_0}{\partial z}=\frac{c}{4\operatorname{Re}}\frac{\partial}{\partial z}\left(\frac{\theta_0\partial u_0}{\partial z}+\frac{\partial u_1}{\partial z}\right)-\frac{Ha^2}{\operatorname{Re}}\left(-Bifu_0+u_1+fw_0\right)-\frac{cP}{\operatorname{Re}}\left(\theta_0u_0+u_1\right)+\\ g\theta_0+l\phi_0\end{array}\right\}\quad(3.121)$$

From equation (3.114) using equation (3.117), the second momentum equation becomes

$$\left.\begin{array}{l}\frac{\partial\left(w_0+\alpha w_1\right)}{\partial t}+\frac{e\alpha}{2}\frac{\partial\left(w_0+\alpha w_1\right)}{\partial z}=\frac{c}{4\operatorname{Re}}\frac{\partial}{\partial z}\left(\left(\alpha\left(\theta_0+\alpha\theta_1\right)+1\right)\frac{\partial\left(w_0+\alpha w_1\right)}{\partial z}\right)-\\ \frac{Ha^2}{\operatorname{Re}\left(\left(1+Bif\alpha\right)^2+\left(f\alpha\right)^2\right)}\left(\left(1+Bif\alpha\right)\left(w_0+\alpha w_1\right)-f\alpha\left(u_0+\alpha u_1\right)\right)\\ -\frac{cP}{\operatorname{Re}}\left(\left(\alpha\left(\theta_0+\alpha\theta_1\right)+1\right)\left(w_0+\alpha w_1\right)\right)\end{array}\right\}\quad(3.122)$$

Expanding, we obtain,

$$\left.\begin{aligned}&\frac{\partial w_0}{\partial t}+\alpha\frac{\partial w_1}{\partial t}+\frac{e\alpha}{2}\frac{\partial w_0}{\partial z}+\frac{e\alpha^2}{2}\frac{\partial w_1}{\partial z}=\frac{c}{4\operatorname{Re}}\frac{\partial}{\partial z}\left(\left(\alpha\left(\theta_0+\alpha\theta_1\right)+1\right)\frac{\partial\left(w_0+\alpha w_1\right)}{\partial z}\right)-\\&\frac{Ha^2}{\operatorname{Re}\left(\left(1+Bif\alpha\right)^2+f\alpha^2\right)}\left(\left(1+Bif\alpha\right)\left(w_0+\alpha w_1\right)-f\alpha\left(u_0+\alpha u_1\right)\right)\\&-\frac{cP}{\operatorname{Re}}\left(\left(\alpha\left(\theta_0+\alpha\theta_1\right)+1\right)\left(w_0+\alpha w_1\right)\right)\end{aligned}\right\} \quad (3.123)$$

Collecting like powers of α we obtain for:

α^0 :

$$\frac{\partial w_0}{\partial t}=\frac{c}{4\operatorname{Re}}\frac{\partial}{\partial z}\left(\frac{\partial w_0}{\partial z}\right)-\frac{Ha^2}{\operatorname{Re}}\left[w_0\right]-\frac{cP}{\operatorname{Re}}w_0 \quad (3.124)$$

α^1 :

$$\frac{\partial w_1}{\partial t}+\frac{e}{2}\frac{\partial w_0}{\partial z}=\frac{c}{4\operatorname{Re}}\frac{\partial}{\partial z}\left(\theta_0\frac{\partial w_0}{\partial z}+\frac{\partial w_1}{\partial z}\right)-\frac{Ha^2}{\operatorname{Re}}\left(-Bifw_0+w_1-fu_0\right)-\frac{cP}{\operatorname{Re}}\left(\theta_0 w_0+w_1\right) \quad (3.125)$$

From equation (3.115) using equation (3.117), the energy equation becomes

$$\left.\begin{aligned}&\frac{\partial}{\partial t}\left(\theta_0+\alpha\theta_1\right)+\frac{e\alpha}{2}\frac{\partial}{\partial z}\left(\theta_0+\alpha\theta_1\right)=\frac{c}{4\operatorname{Re}\Pr}\frac{\partial}{\partial z}\left(\left(\alpha\left(\theta_0+\alpha\theta_1\right)+1\right)\frac{\partial}{\partial z}\left(\theta_0+\alpha\theta_1\right)\right)+\\&\frac{cb\alpha}{4\operatorname{Re}}\left(\left(\theta_0+\alpha\theta_1\right)+1\right)\left(\left(\frac{\partial\left(u_0+\alpha u_1\right)}{\partial z}\right)^2+\left(\frac{\partial\left(w_0+\alpha w_1\right)}{\partial z}\right)^2\right)-\\&\frac{b\alpha Ha^2}{\operatorname{Re}\left(\left(1+Bif\alpha\right)^2+\left(f\alpha\right)^2\right)}\left(\left(u_0+\alpha u_1\right)^2+\left(w_0+\alpha w_1\right)^2\right)-Ra^2\left(\theta_0+\alpha\theta_1\right)\end{aligned}\right\} \quad (3.126)$$

Expanding we obtain,

$$\left.\begin{array}{l}\frac{\partial\theta_0}{\partial t}+\alpha\frac{\partial\theta_1}{\partial t}+\frac{e\alpha}{2}\frac{\partial\theta_0}{\partial z}+\frac{e\alpha^2}{2}\frac{\partial\theta_1}{\partial z}=\frac{c}{4\operatorname{Re}\Pr}\frac{\partial}{\partial z}\left(\left(\alpha\left(\theta_0+\alpha\theta_1\right)+1\right)\frac{\partial}{\partial z}\left(\theta_0+\alpha\theta_1\right)\right)+\\ \frac{cb\alpha}{4\operatorname{Re}}\left(\alpha\left(\theta_0+\alpha\theta_1\right)+1\right)\left(\left(\frac{\partial\left(u_0+\alpha u_1\right)}{\partial z}\right)^2+\left(\frac{\partial\left(w_0+\alpha w_1\right)}{\partial z}\right)^2\right)-\\ \frac{b\alpha Ha^2}{\operatorname{Re}\left((1+Bifa)^2+\left(fa\right)^2\right)}\left(\left(u_0+\alpha u_1\right)^2+\left(w_0+\alpha w_1\right)^2\right)-Ra^2\left(\theta_0+\alpha\theta_1\right)\end{array}\right\} \quad (3.127)$$

Collecting like powers of α we obtain for:

α^0 :

$$\frac{\partial\theta_0}{\partial t}=\frac{c}{4\operatorname{Re}\Pr}\frac{\partial}{\partial z}\left(\frac{\partial\theta_0}{\partial z}\right)-Ra^2\theta_0 \quad (3.128)$$

$$\left.\begin{array}{l}\alpha^1 :\\ \frac{\partial\theta_1}{\partial t}+\frac{e}{2}\frac{\partial\theta_0}{\partial z}=\frac{c}{4\operatorname{Re}\Pr}\frac{\partial}{\partial y}\left(e\theta_0\frac{\partial\theta_0}{\partial z}+\frac{\partial\theta_1}{\partial z}\right)+\frac{bc}{4\operatorname{Re}}\left(\left(\frac{\partial u_0}{\partial z}\right)^2+\left(\frac{\partial w_0}{\partial z}\right)^2\right)-\\ \frac{bHa^2}{\operatorname{Re}}\left(\left(u_0\right)^2+\left(w_0\right)^2\right)-Ra^2\theta_1\end{array}\right\} \quad (3.129)$$

From equation (3.116) using equation (3.117), the concentration equation becomes

$$\left.\begin{array}{l}\frac{\partial}{\partial t}\left(\phi_0+\alpha\phi_1\right)+\frac{e}{2}\frac{\partial\left(\phi_0+\alpha\phi_1\right)}{\partial z}=\frac{c}{4Sc\operatorname{Re}}\frac{\partial}{\partial z}\left(\left(\alpha\left(\theta_0+\alpha\theta_1\right)+1\right)\frac{\partial}{\partial z}\left(\phi_0+\alpha\phi_1\right)\right)+\\ \frac{T_D}{4}\frac{\partial^2}{\partial z^2}\left(\theta_0+\alpha\theta_1\right)-K_r\left(\phi_0+\alpha\phi_1\right)\end{array}\right\} \quad (3.130)$$

Expanding, we obtain

$$\left.\begin{array}{l}\frac{\partial\phi_0}{\partial t}+\alpha\frac{\partial\phi_1}{\partial t}+\frac{e\alpha}{2}\frac{\partial\phi_0}{\partial z}+\frac{e\alpha^2}{2}\frac{\partial\phi_1}{\partial z}=\frac{c}{4Sc\operatorname{Re}}\frac{\partial}{\partial z}\left(\left(\alpha\left(\theta_0+\alpha\theta_1\right)+1\right)\frac{\partial}{\partial z}\left(\phi_0+\alpha\phi_1\right)\right)+\\ \frac{T_D}{4}\frac{\partial^2}{\partial z^2}\left(\theta_0+\alpha\theta_1\right)-K_r\left(\phi_0+\alpha\phi_1\right)\end{array}\right\} \quad (3.131)$$

Collecting like powers of α we obtain

α^0 :

$$\frac{\partial \phi_0}{\partial t} = \frac{c}{4Sc\,\mathrm{Re}} \frac{\partial}{\partial z}\left(\frac{\partial \phi_0}{\partial z}\right) + \frac{T_D}{4}\frac{\partial^2 \theta_0}{\partial z^2} - K_r \phi_0 \tag{3.132}$$

α^1 :

$$\frac{\partial \phi_1}{\partial t} + \frac{e}{2}\frac{\partial \phi_0}{\partial z} = \frac{c}{4Sc\,\mathrm{Re}} \frac{\partial}{\partial z}\left(\theta_0 \frac{\partial \phi_0}{\partial z} + \frac{\partial \phi_1}{\partial z}\right) + \frac{T_D}{4}\frac{\partial^2 \theta_1}{\partial z^2} - K_r \phi_1 \tag{3.133}$$

Therefore, the equations for $u_0, w_0, \theta_0, \phi_0, u_1, w_1, \theta_1$ *and* ϕ_1 together with their initial and boundary conditions are given by

$$\left.\begin{array}{l} \dfrac{\partial \theta_0}{\partial t} = \dfrac{c}{4\,\mathrm{Re}\,\mathrm{Pr}} \dfrac{\partial}{\partial z}\left(\dfrac{\partial \theta_0}{\partial z}\right) - Ra^2 \theta_0 \\ \theta_0(z,0) = d_1, \quad \theta_0(0,t) = 0, \quad \theta_0(1,t) = 1 \end{array}\right\} \tag{3.134}$$

$$\left.\begin{array}{l} \dfrac{\partial \phi_0}{\partial t} = \dfrac{c}{4Sc\,\mathrm{Re}} \dfrac{\partial}{\partial z}\left(\dfrac{\partial \phi_0}{\partial z}\right) + T_D \dfrac{\partial^2 \theta_0}{\partial z^2} - K_r \phi_0 \\ \phi_0(z,0) = d_2, \quad \phi_0(0,t) = 0, \quad \phi_0(1,t) = 1 \end{array}\right\} \tag{3.135}$$

$$\left.\begin{array}{l} \dfrac{\partial w_0}{\partial t} = \dfrac{c}{4\,\mathrm{Re}} \dfrac{\partial}{\partial z}\left(\dfrac{\partial w_0}{\partial z}\right) - \dfrac{Ha^2}{\mathrm{Re}} w_0 - \dfrac{cP}{\mathrm{Re}} w_0 \\ w_0(z,0) = (2z-1)(2-2z), \quad w_0(0,t) = 0, \quad w_0(1,t) = 0 \end{array}\right\} \tag{3.136}$$

$$\left.\begin{array}{l} \dfrac{\partial u_0}{\partial t} = \dfrac{c}{4\,\mathrm{Re}} \dfrac{\partial}{\partial z}\left(\dfrac{\partial u_0}{\partial z}\right) - \dfrac{Ha^2}{\mathrm{Re}} u_0 - \dfrac{cP}{\mathrm{Re}}(u_0) - \lambda e^{-\varepsilon t} \\ u_0(z,0) = 0, \quad u_0(0,t) = 0, \quad u_0(1,t) = 1 \end{array}\right\} \tag{3.137}$$

$$
\left.\begin{aligned}
&\frac{\partial u_1}{\partial t}+\frac{e}{2}\frac{\partial u_0}{\partial z}=\frac{c}{4\operatorname{Re}}\frac{\partial}{\partial z}\left(\frac{\theta_0\partial u_0}{\partial z}+\frac{\partial u_1}{\partial z}\right)-\frac{Ha^2}{\operatorname{Re}}\left(-Bifu_0+u_1+fw_0\right)-\frac{cP}{\operatorname{Re}}\left(\theta_0 u_0+u_1\right)+\\
&g\theta_0+l\phi_0\\
&u_1(z,0)=0,\qquad u_1(0,t)=0,\qquad u_1(1,t)=0
\end{aligned}\right\}\quad(3.138)
$$

$$
\left.\begin{aligned}
&\frac{\partial \theta_1}{\partial t}+\frac{e}{2}\frac{\partial \theta_0}{\partial z}=\frac{c}{4\operatorname{Re}\Pr}\frac{\partial}{\partial y}\left(\theta_0\frac{\partial\theta_0}{\partial z}+\frac{\partial \theta_1}{\partial z}\right)+\frac{bc}{4\operatorname{Re}}\left(\left(\frac{\partial u_0}{\partial z}\right)^2+\left(\frac{\partial w_0}{\partial z}\right)^2\right)-\\
&\frac{bHa^2}{\operatorname{Re}}\left(\left(u_0\right)^2+\left(w_0\right)^2\right)-Ra^2\theta_1\\
&\theta_1(z,0)=0,\qquad \theta_1(0,t)=0,\qquad \theta_1(1,t)=0
\end{aligned}\right\}\quad(3.139)
$$

$$
\left.\begin{aligned}
&\frac{\partial w_1}{\partial t}+\frac{e}{2}\frac{\partial w_0}{\partial z}=\frac{c}{4\operatorname{Re}}\frac{\partial}{\partial z}\left(\theta_0\frac{\partial w_0}{\partial z}+\frac{\partial w_1}{\partial z}\right)-\frac{Ha^2}{\operatorname{Re}}\left(-Bifw_0+w_1-fu_0\right)-\frac{cP}{\operatorname{Re}}\left(\theta_0 w_0+w_1\right)\\
&w_1(z,0)=0,\qquad w_1(0,t)=0,\qquad w_1(1,t)=0
\end{aligned}\right\}\quad(3.140)
$$

$$
\left.\begin{aligned}
&\frac{\partial \phi_1}{\partial t}+\frac{e}{2}\frac{\partial \phi_0}{\partial z}=\frac{c}{4Sc\operatorname{Re}}\frac{\partial}{\partial z}\left(\theta_0\frac{\partial\phi_0}{\partial z}+\frac{\partial \phi_1}{\partial z}\right)+T_D\frac{\partial^2\theta_1}{\partial z^2}-K_r\phi_1\\
&\phi_1(z,0)=0,\qquad \phi_1(0,t)=0,\qquad \phi_1(1,t)=0
\end{aligned}\right\}\quad(3.141)
$$

Consider equation (3.134) given by

$$
\left.\begin{aligned}
&\frac{\partial \theta_0}{\partial t}=\frac{c}{4\operatorname{Re}\Pr}\frac{\partial}{\partial z}\left(\frac{\partial\theta_0}{\partial z}\right)-Ra^2\theta_0\\
&\theta_0(z,0)=d_1,\quad \theta_0(0,t)=0,\quad \theta_0(1,t)=1
\end{aligned}\right\}
$$

This is a non homogenous boundary condition problem. We first find a function, $\mu_1(z,t)$ which satisfies the boundary conditions. We let

$$
\mu_1(z,t)=\alpha(t)+\frac{z}{L}\left(\beta(t)-\alpha(t)\right)=0+z\left(1-0\right)=zt^0 \quad (3.142)
$$

We make the change of variables

$$\theta_0(z,t) = v_1(z,t) + \mu_1(z,t) \tag{3.143}$$

Then

$$\frac{\partial \theta_0}{\partial t} = \frac{\partial v_1}{\partial t} + \frac{\partial \mu_1}{\partial t} = \frac{\partial v_1}{\partial t} + 0 = \frac{\partial v_1}{\partial t} \tag{3.144}$$

$$\frac{\partial \theta_0}{\partial z} = \frac{\partial v_1}{\partial z} + \frac{\partial \mu_1}{\partial z} = \frac{\partial v_1}{\partial z} + 1 \tag{3.145}$$

$$\frac{\partial^2 \theta_0}{\partial z^2} = \frac{\partial^2 v_1}{\partial z^2} + \frac{\partial^2 \mu_1}{\partial z^2} = \frac{\partial^2 v_1}{\partial z^2} \tag{3.146}$$

$$\theta_0(z,0) = v_1(z,0) + \mu_1(z,0) = v_1(z,0) + z = d_1 \Rightarrow v_1(z,0) = d_1 - z \tag{3.147}$$

$$\theta_0(0,t) = v_1(0,t) + \mu_1(0,t) = v_1(0,t) + 0 = 0 \Rightarrow v_1(0,t) = 0 \tag{3.148}$$

$$\theta_0(1,t) = v_1(1,t) + \mu_1(1,t) = v_1(1,t) + 1 = 1 \Rightarrow v_1(1,t) = 0 \tag{3.149}$$

So, equation (3.134) reduce to

$$\left.\begin{array}{l} \dfrac{\partial v_1}{\partial t} = \dfrac{c}{4\,\mathrm{Re}\,\mathrm{Pr}} \dfrac{\partial^2 v_1}{\partial z^2} - Ra^2 v_1 - Ra^2 z \\ v_1(z,0) = d_1 - z, \qquad v_1(0,t) = 0 \qquad v_1(1,t) = 0 \end{array}\right\} \tag{3.150}$$

Now, consider the problem (Myint-U and Debnath, (1987))

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2} + \alpha u + F(x,t) \\ u(x,0) = f(x), \qquad u(0,t) = 0, \qquad u(L,t) = 0 \end{array}\right\} \tag{3.151}$$

For the solution of problem (3.151), we assume a solution of the form

$$u(x,t)=\sum_{n=1}^{\infty}u_n(t)\sin\frac{n\pi}{L}x \tag{3.152}$$

where

$$u_n(t)=\int_0^t e^{\left(\alpha-k\left(\frac{n\pi}{L}\right)^2\right)(t-\tau)}F_n(\tau)d\tau+b_n e^{\left(\alpha-k\left(\frac{n\pi}{L}\right)^2\right)t} \tag{3.153}$$

$$F_n(t)=\frac{2}{L}\int_0^L F(x,t)\sin\frac{n\pi}{L}xdx \tag{3.154}$$

$$b_n=\frac{2}{L}\int_0^L F(x)\sin\frac{n\pi}{L}xdx \tag{3.155}$$

Comparing equation (3.150) with equation (3.151) gives

$$u=v_1,\ \ k=\frac{c}{4\operatorname{Re}\Pr},\quad \alpha=-Ra^2,\quad F(z,t)=-Ra^2z,\quad x=z,\quad f(z)=d_1-z,\ \ L=1$$

.

Then,

$$b_n=2\int_0^1(d_1-z)\sin n\pi zdz \tag{3.156}$$

$$=2d_1\int_0^1\sin n\pi zdz-2\int_0^1 z\sin n\pi zdz$$

$$=\left(\frac{2(d_1+1)(-1)^n}{n\pi}-2d_1\right)$$

$$F_n(t)=-2Ra^2\int_0^L z.\sin n\pi zdz$$

$$=-2Ra^2\left(\left(\frac{-z\cos n\pi z}{n\pi}\right)_0^1+\int_0^1\frac{\cos n\pi z}{n\pi}dz\right)=-2Ra^2\left(\frac{-\cos n\pi}{n\pi}+\frac{\sin n\pi}{n^2\pi^2}\Bigg|_0^1\right) \tag{3.157}$$

$$F_n(\mathrm{t}) = -2Ra^2\left(\frac{-\cos n\pi}{n\pi}\right) = \frac{2Ra^2(-1)^n}{n\pi} \tag{3.158}$$

Then

$$v_{1n}(t) = \int_0^t e^{-\left(Ra^2+\frac{c}{4\mathrm{Re}\,\mathrm{Pr}}(n\pi)^2\right)(t-\tau)} \cdot \frac{2Ra^2(-1)^n}{n\pi}\,d\tau + b_n e^{-\left(Ra^2+\frac{c}{4\mathrm{Re}\,\mathrm{Pr}}(n\pi)^2\right)} \tag{3.159}$$

Let $q_0 = \left(Ra^2+\frac{c}{4\mathrm{Re}\,\mathrm{Pr}}(n\pi)^2\right)$

Then

$$v_{1n}(t) = \frac{2Ra^2(-1)^n}{n\pi} e^{-q_1 t}\int_0^t e^{q_0\tau}\,d\tau + b_n e^{-q_0 t} \tag{3.160}$$

i.e

$$v_{1n}(t) = \frac{2Ra^2(-1)^n}{n\pi q_0}\left(1-e^{-q_0 t}\right) + b_n e^{-q_0 t} \tag{3.161}$$

Therefore,

$$v_{1n}(t) = \left(b_n - q_1\right)e^{-q_0 t} + q_1 \tag{3.162}$$

where

$$q_1 = \frac{2Ra^2(-1)^n}{q_0 n\pi} \tag{3.163}$$

Therefore, from equation (3.152)

$$v_1(\mathrm{z},\mathrm{t}) = \sum_{n=1}^{\infty}\left(q_1 + \left(b_n - q_1\right)e^{-q_0 t}\right)\sin n\pi z \tag{3.164}$$

Thus,

$$\theta_0(\mathrm{z,t}) = \mathrm{z} + \sum_{n=1}^{\infty}\left(q_1 + \left(b_n - q_1\right)e^{-q_0 t}\right)\sin n\pi z \tag{3.165}$$

Consider equation (3.135) given by

$$\left.\begin{aligned}&\frac{\partial\phi_0}{\partial t} = \frac{c}{4Sc\,\mathrm{Re}}\frac{\partial}{\partial z}\left(\frac{\partial\phi_0}{\partial z}\right) + T_D\frac{\partial^2\theta_0}{\partial z^2} - K_r\phi_0\\ &\phi_0\left(z,0\right) = d_2,\quad \phi_0\left(0,t\right) = 0,\quad \phi_0\left(1,t\right) = 1\end{aligned}\right\}$$

Then

$$\frac{\partial\theta_0}{\partial z} = 1 + \sum_{n=1}^{\infty}\left(q_1 + \left(b_n - q_1\right)e^{-q_0 t}\right)n\pi\cos n\pi z \tag{3.166}$$

$$\frac{\partial^2\theta_0}{\partial z^2} = -\sum_{n=1}^{\infty}\left(q_1 + \left(b_n - q_1\right)e^{-q_0 t}\right)\left(n\pi\right)^2\sin n\pi z \tag{3.167}$$

Substituting equations (3.166) and (3.167) into equation (3.135) gives

$$\left.\begin{aligned}&\frac{\partial\phi_0}{\partial t} = \frac{c}{4\,\mathrm{Re}\,Sc}\frac{\partial^2\phi_0}{\partial z^2} - K_r\phi_0 - \frac{T_D}{4}\sum_{n=1}^{\infty}\left(q_1 + \left(b_n - q_1\right)e^{-q_0 t}\right)\left(n\pi\right)^2\sin n\pi z\\ &\phi_0\left(z,0\right) = d_2,\quad \phi_0\left(0,t\right) = 0,\quad \phi_0\left(1,t\right) = 1\end{aligned}\right\} \tag{3.168}$$

This is a non homogenous boundary condition problem. We first find a function, $\mu_2(z,t)$ which satisfies the boundary conditions. We let

$$\mu_2(z,t) = \alpha(t) + \frac{z}{L}\left(\beta(t) - \alpha(t)\right) = 0 + z\left(1 - 0\right) = zt^0 \tag{3.169}$$

We make the change of variables

$$\phi_0(z,t) = v_2(z,t) + \mu_2(z,t)$$

Then

$$\frac{\partial \phi_0}{\partial t} = \frac{\partial v_2}{\partial t} + \frac{\partial \mu_2}{\partial t} = \frac{\partial v_2}{\partial t} + 0 = \frac{\partial v_2}{\partial t} \tag{3.170}$$

$$\frac{\partial \phi_0}{\partial z} = \frac{\partial v_2}{\partial z} + \frac{\partial \mu_2}{\partial z} = \frac{\partial v_2}{\partial z} + 1 \tag{3.171}$$

$$\frac{\partial^2 \phi_0}{\partial z^2} = \frac{\partial^2 v_2}{\partial z^2} + \frac{\partial^2 \mu_2}{\partial z^2} = \frac{\partial^2 v_2}{\partial z^2} \tag{3.172}$$

$$\phi_0(z,0) = v_2(z,0) + \mu_2(z,0) = v_2(z,0) + z = d_2 \Rightarrow v_2(z,0) = d_2 - z \tag{3.173}$$

$$\phi_0(0,t) = v_2(0,t) + \mu_2(0,t) = v_2(0,t) + 0 = 0 \Rightarrow v_2(0,t) = 0 \tag{3.174}$$

$$\phi_0(1,t) = v_2(1,t) + \mu_2(1,t) = v_2(1,t) + 1 = 1 \Rightarrow v_2(1,t) = 0 \tag{3.175}$$

Substituting equations (3.173) – (3.175), equation (3.135) reduces to

$$\left.\begin{array}{l} \dfrac{\partial v_2}{\partial t} = \dfrac{c}{4\operatorname{Re} Sc}\dfrac{\partial^2 v_2}{\partial z^2} - K_r v_2 - K_r z - \dfrac{T_D}{4}\displaystyle\sum_{n=1}^{\infty}\left(q_1 + (b_n - q_1)e^{-q_0 t}\right)(n\pi)^2 \sin n\pi z \\ v_2(z,0) = d_2 - z, \qquad v_2(0,t) = 0, \qquad v_2(1,t) = 0 \end{array}\right\} \tag{3.176}$$

Comparing equations (3.176) with (3.151), we have

$$k = \frac{c}{4\operatorname{Re} sc}, \quad \alpha = -K_r, \quad f(\mathrm{x}) = d_2 - z, \quad \mathrm{L} = 1$$

$$F(z,t) = -\left(K_r z + \frac{T_D}{4}\sum_{n=1}^{\infty}\left(q_1 + (b_n - q_1)e^{-q_0 t}\right)(n\pi)^2 \sin n\pi z\right) \tag{3.177}$$

$$\mathrm{b}_{2n} = 2\int_0^1 (d_2 - z)\sin n\pi z\,dz = \left(\frac{2(d_2 + 1)(-1)^n}{n\pi} - 2d_2\right)$$

Then

$$F_{2n}(t)=-2\left(K_r\int_0^1 z\sin n\pi z dz+\frac{T_D}{4}\sum_{n=1}^{\infty}\left(q_1+\left(b_n-q_1\right)e^{-q_0 t}\right)\left(n\pi\right)^2\int_0^1 \sin n\pi z dz\right)$$

$$F_{2n}(t)=-2\left(-\frac{K_r\left(-1\right)^n}{n\pi}+\frac{T_D}{8}\sum_{n=1}^{\infty}\left(q_1+\left(b_n-q_1\right)e^{-q_0 t}\right)\left(n\pi\right)^2\right)$$

$$F_{2n}(t)=\left(\frac{2K_r\left(-1\right)^n}{n\pi}-\frac{T_D}{4}\sum_{n=1}^{\infty}\left(q_1+\left(b_n-q_1\right)e^{-q_0 t}\right)\left(n\pi\right)^2\right) \tag{3.178}$$

Let $q_2=\left(K_r+\frac{c}{4\operatorname{Re}Sc}\left(n\pi\right)^2\right)$

Then

$$v_{2n}(t)=e^{-q_2 t}\left(\frac{2K_r\left(-1\right)^n}{n\pi}\int_0^t e^{q_2\tau}d\tau-\frac{T_D}{4}\sum_{n=1}^{\infty}\left(n\pi\right)^2\left(q_1\int_0^t e^{q_2\tau}+\left(b_n-q_1\right)\int_0^t e^{-\left(q_2-q_0\right)\tau}d\tau\right)\right)+b_{2n}e^{-q_2 t} \tag{3.179}$$

$$v_{2n}(t)=\frac{2K_r\left(-1\right)^n}{n\pi q_2}\left(1-e^{-q_2 t}\right)-\frac{T_D}{4}\sum_{n=1}^{\infty}\left(n\pi\right)^2\left(\frac{q_1}{q_2}\left(1-e^{-q_2 t}\right)+\frac{\left(b_n-q_1\right)}{\left(q_2-q_1\right)}\left(e^{-q_0 t}-e^{-q_2 t}\right)\right)+b_{2n}e^{-q_2 t} \tag{3.180}$$

That is

$$v_{2n}(t)=q_3\left(1-e^{-q_2 t}\right)-\sum_{n=1}^{\infty}\left(q_4-q_5e^{-q_2 t}+q_6e^{-q_0 t}\right)+b_{2n}e^{-q_2 t}$$

where

$$q_3=\frac{2K_r\left(-1\right)^n}{n\pi q_2},\quad q_4=\frac{T_D\left(n\pi\right)^2 q_1}{4q_2},\quad q_5=\frac{T_D\left(n\pi\right)^2}{4}\left(\frac{q_1}{q_2}+\frac{\left(b_n-q_1\right)}{\left(q_2-q_0\right)}\right),$$

$$q_6=\frac{T_D\left(n\pi\right)^2}{4}\left(\frac{\left(b_n-q_1\right)}{\left(q_2-q_0\right)}\right)$$

Therefore, from equation (3.152)

$$v_2(z,t)=\sum_{n=1}^{\infty}\left(q_3\left(1-e^{-q_2t}\right)-\sum_{n=1}^{\infty}\left(q_4-q_5e^{-q_2t}+q_6e^{-q_0t}\right)+b_{2n}e^{-q_2t}\right)\sin n\pi z \qquad (3.181)$$

Thus

$$\phi_0(z,t)=z+\sum_{n=1}^{\infty}\left(q_3\left(1-e^{-q_2t}\right)-\sum_{n=1}^{\infty}\left(q_4-q_5e^{-q_2t}+q_6e^{-q_0t}\right)+b_{2n}e^{-q_2t}\right)\sin n\pi z \quad (3.182)$$

Consider equation (3.136) given as

$$\left.\begin{aligned}&\frac{\partial w_0}{\partial t}=\frac{c}{4\operatorname{Re}}\frac{\partial}{\partial z}\left(\frac{\partial w_0}{\partial z}\right)-\left(\frac{Ha^2+cP}{\operatorname{Re}}\right)w_0\\&w_0\left(z,0\right)=\left(2z-1\right)\left(2-2z\right),\quad w_0\left(0,t\right)=0,\quad w_0\left(1,t\right)=0\end{aligned}\right\}$$

This is a homogenous boundary condition problem. Compare (3.136) with (3.151), gives

$$u=w_0,\ k=\frac{c}{4\operatorname{Re}},\alpha=-\left(\frac{Ha^2+cP}{\operatorname{Re}}\right)=-q_7,\mathrm{L}=1,F(\mathrm{z,t})=0,\ \ \mathrm{x}=\mathrm{z},\ \ \mathrm{f(z)}=-(4\mathrm{z}^2-6\mathrm{z}+2)$$

Then

$$b_{3n}=2\int_0^1-(4\mathrm{z}^2-6\mathrm{z}+2)\,\mathrm{Sinn}\,\pi\,\mathrm{z}\,\mathrm{dz} \qquad (3.183)$$

$$=\left(-8\int_0^1 z^2 \mathrm{Sinn}\,\pi\,z\,dz+12\int_0^1 z\mathrm{Sinn}\,\pi\,z\,dz-4\int_0^1 \mathrm{Sinn}\,\pi\,z\,dz\right)$$

$$=\frac{-8\left(\left(2-(n\pi)^2\right)\cos n\pi-2\right)}{(n\pi)^3}-\frac{12\cos n\pi}{n\pi}-\frac{4(1-\cos n\pi)}{n\pi} \tag{3.184}$$

$$b_{3n}=\frac{\left(\left(8(n\pi)^2-16\right)(-1)^n+16\right)}{(n\pi)^3}-\frac{8(-1)^n}{n\pi}-\frac{4}{n\pi} \tag{3.185}$$

Similarly,

$$F_{3n}(t)=2\int_0^1 0\cdot\sin n\pi z dz=0 \tag{3.186}$$

Let

$$q_8=\left(q_7+\frac{c}{4\,\mathrm{Re}}(n\pi)^2\right)$$

So,

$$u_{3n}(t)=0+b_{3n}e^{-q_8 t}$$

From equation (3.152), we obtain

$$u_{3n}(t)=b_{3n}e^{-q_8 t}$$

Thus,

$$w_0(\mathrm{z},t)=\sum_{n=1}^{\infty} b_{3n}e^{-q_8 t}Sinn\pi z \tag{3.187}$$

Consider equation (3.137) given as

$$\left.\begin{aligned}&\frac{\partial u_0}{\partial t}=\frac{c}{4\,\mathrm{Re}}\frac{\partial}{\partial z}\left(\frac{\partial u_0}{\partial z}\right)-q_7u_0-\lambda e^{-\varepsilon t}\\&u_0\left(z,0\right)=0,\quad u_0\left(0,t\right)=0,\quad u_0\left(1,t\right)=1\end{aligned}\right\}$$

This is a non-homogenous boundary condition problem. We first find a function, $\mu_3(z,t)$ which satisfies the boundary conditions. We let

$$\mu_3(z,t)=\alpha(t)+\frac{z}{L}\left(\beta(t)-\alpha(t)\right)=0+z\left(1-0\right)=zt^0 \tag{3.188}$$

We make the change of variables

$$u_0(z,t)=v_4(z,t)+\mu_3(z,t) \tag{3.189}$$

Then

$$\frac{\partial u_0}{\partial t}=\frac{\partial v_4}{\partial t}+\frac{\partial \mu_3}{\partial t}=\frac{\partial v_4}{\partial t}+0=\frac{\partial v_4}{\partial t} \tag{3.190}$$

$$\frac{\partial u_0}{\partial z}=\frac{\partial v_4}{\partial z}+\frac{\partial \mu_3}{\partial z}=\frac{\partial v_4}{\partial z}+1 \tag{3.191}$$

$$\frac{\partial^2 u_0}{\partial z^2}=\frac{\partial^2 v_4}{\partial z^2}+\frac{\partial^2 \mu_3}{\partial z^2}=\frac{\partial^2 v_4}{\partial z^2} \tag{3.192}$$

$$u_0\left(z,0\right)=v_4\left(z,0\right)+\mu_3\left(z,0\right)=\quad v_4\left(z,0\right)+z=0\Rightarrow v_4\left(z,0\right)=-z$$

$$u_0\left(0,t\right)=\quad v_4\left(0,t\right)+\mu_3\left(0,t\right)=\quad v_4\left(0,t\right)+0=0\Rightarrow\quad v_4\left(0,t\right)=0 \tag{3.193}$$

$$u_0\left(1,t\right)=\quad v_4\left(1,t\right)+\mu_3\left(1,t\right)=\quad v_4\left(1,t\right)+1=1\Rightarrow\quad v_4\left(1,t\right)=0 \tag{3.194}$$

Then equation (3.137) reduces to

$$\left.\begin{aligned}&\frac{\partial v_4}{\partial t}=\frac{c}{4\,\mathrm{Re}}\frac{\partial^2 v_4}{\partial z^2}-q_7 v_4-q_7 z-\lambda e^{-\varepsilon t}\\&v_4(z,0)=-z,\quad v_4(0,t)=0,\qquad v_4(1,t)=0\end{aligned}\right\}\tag{3.195}$$

Comparing equations (3.195) and (3.151), we have

$$k=\frac{c}{4\,\mathrm{Re}},\qquad \alpha=-q_7,\qquad f(x)=-z,\qquad \mathrm{L}=1,\quad x=z,\quad F_{4n}(t)=-\left(q_7 z+\sigma e^{-\varepsilon t}\right)$$

Then,

$$\mathrm{b}_{4n}=-2\int_0^1 z\sin n\pi z dz=\frac{2\cos n\pi}{n\pi}=\frac{2(-1)^n}{n\pi}$$

$$F_{4n}(t)=-2\left(q_7\int_0^1 z\sin n\pi z dz+\sigma e^{-\varepsilon t}\int_0^1 \sin n\pi z dz\right)$$

$$=-2\left(-\frac{q_7(-1)^n}{n\pi}+\sigma e^{-\varepsilon t}\left(\frac{1-\cos n\pi}{n\pi}\right)\right)$$

$$F_{4n}(t)=\left(\frac{2q_7(-1)^n}{n\pi}+\frac{\sigma e^{-\varepsilon t}\left((-1)^n-1\right)}{n\pi}\right)\tag{3.196}$$

Then

$$v_{4n}(t)=e^{-q_8 t}\left(\frac{2q_7(-1)^n}{n\pi}\int_0^t e^{q_8\tau}d\tau+\frac{\sigma\left((-1)^n-1\right)}{n\pi}\int_0^t e^{(q_8-\varepsilon)\tau}d\tau\right)+b_{4n}e^{-q_8 t}\tag{3.197}$$

$$v_{4n}(t)=\frac{2q_7(-1)^n}{n\pi q_8}\left(1-e^{-q_8 t}\right)+\frac{\sigma\left((-1)^n-1\right)}{n\pi(q_8-\varepsilon)}\left(e^{-\varepsilon t}-e^{-q_8 t}\right)+b_{4n}e^{-q_8 t}$$

Therefore from equation (3.152), we obtain

$$v_4(z,t)=\sum_{n=1}^{\infty}\left(q_9+q_{10}e^{-q_8t}+q_{11}e^{-\varepsilon t}\right)\sin n\pi z \tag{3.198}$$

Thus

$$u_0\left(z,t\right)=z+\sum_{n=1}^{\infty}\left(q_9+q_{10}e^{-q_8t}+q_{11}e^{-\varepsilon t}\right)\sin n\pi z \tag{3.199}$$

where

$$q_9=\frac{2q_7\left(-1\right)^n}{n\pi q_8},\quad q_{10}=\left(b_n-\frac{2q_7\left(-1\right)^n}{n\pi q_8}-\frac{\sigma\left(\left(-1\right)^n-1\right)}{n\pi\left(q_8-\varepsilon\right)}\right),\ q_{11}=\frac{\sigma\left(\left(-1\right)^n-1\right)}{n\pi\left(q_8-\varepsilon\right)}$$

Consider the equation (3.138) given below as

$$\left.\begin{array}{l}\dfrac{\partial u_1}{\partial t}+\dfrac{e}{2}\dfrac{\partial u_0}{\partial z}=\dfrac{c}{4\operatorname{Re}}\dfrac{\partial}{\partial z}\left(\dfrac{\theta_0\partial u_0}{\partial z}+\dfrac{\partial u_1}{\partial z}\right)-\dfrac{Ha^2}{\operatorname{Re}}\left(-Bifu_0+u_1+fw_0\right)-\dfrac{cP}{\operatorname{Re}}\left(\theta_0u_0+u_1\right)+\\ g\theta_0+l\phi_0\\ u_1\left(z,0\right)=0,\qquad u_1\left(0,t\right)=0,\qquad u_1\left(1,t\right)=0\end{array}\right\}$$

This is a homogenous boundary condition problem. Rearranging the above equation, we

have

$$\left.\begin{array}{l}\frac{\partial u_1}{\partial t}=\frac{c}{4\operatorname{Re}}\frac{\partial^2 u_1}{\partial z^2}-q_7u_1-\frac{e}{2}\frac{\partial u_0}{\partial z}+\frac{c}{4\operatorname{Re}}\frac{\theta_0\partial^2 u_0}{\partial z^2}+\frac{c}{4\operatorname{Re}}\left(\frac{\partial u_0}{\partial z}\right)\left(\frac{\partial\theta_0}{\partial z}\right)-\\ \frac{Ha^2}{\operatorname{Re}}\left(Bifu_0+fw_0\right)-\frac{cP}{\operatorname{Re}}\theta_0u_0+g\theta_0+l\phi_0\\ u_1\left(z,0\right)=0,\qquad u_1\left(0,t\right)=0,\qquad u_1\left(1,t\right)=0\end{array}\right\}\qquad(3.200)$$

But,

$$\frac{\partial\theta_0}{\partial z}=1+\sum_{n=1}^{\infty}\left(q_1+\left(b_n-q_1\right)e^{-q_0t}\right)n\pi\cos n\pi z\qquad(3.201)$$

$$\frac{\partial u_0}{\partial z}=1+\sum_{n=1}^{\infty}\left(q_9+q_{10}e^{-q_8t}+q_{11}e^{-\varepsilon t}\right)n\pi C\operatorname{os}n\pi z\qquad(3.202)$$

$$\theta_0\frac{\partial^2 u_0}{\partial z^2}=-\left(z+\sum_{n=1}^{\infty}\left(q_1+\left(b_n-q_1\right)e^{-q_0t}\right)\sin n\pi z\right)\sum_{n=1}^{\infty}\left(q_9+q_{10}e^{-q_8t}+q_{11}e^{-\varepsilon t}\right)\left(n\pi\right)^2\sin n\pi z$$

$$\begin{aligned}&=-\sum_{n=1}^{\infty}\left(q_9+q_{10}e^{-q_8t}+q_{11}e^{-\varepsilon t}\right)\left(n\pi\right)^2 z\sin n\pi z-\\&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\begin{pmatrix}q_1q_9+q_1q_{10}e^{-q_8t}+q_1q_{11}e^{-\varepsilon t}+q_9\left(b_n-q_1\right)e^{-q_0t}+\\q_{10}\left(b_n-q_1\right)e^{-\left(q_0+q_8\right)t}+q_{11}\left(b_n-q_1\right)e^{-\left(q_0+\varepsilon\right)t}\end{pmatrix}\left(n\pi\right)^2\sin^2 n\pi z\end{aligned}\qquad(3.203)$$

$$\left(\frac{\partial u_0}{\partial z}\right)\left(\frac{\partial\theta_0}{\partial z}\right)=\left(1+\sum_{n=1}^{\infty}\left(q_1+\left(b_n-q_1\right)e^{-q_0t}\right)n\pi\cos n\pi z\right)\left(1+\sum_{n=1}^{\infty}\left(q_9+q_{10}e^{-q_8t}+q_{11}e^{-\varepsilon t}\right)n\pi\cos n\pi z\right)$$

$$\begin{aligned}&=1+\sum_{n=1}^{\infty}\left(q_9+q_{10}e^{-q_8t}+q_{11}e^{-\varepsilon t}\right)n\pi\cos n\pi z+\sum_{n=1}^{\infty}\left(q_1+\left(b_n-q_1\right)e^{-q_0t}\right)n\pi\cos n\pi z+\\&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\begin{pmatrix}q_1q_9+q_1q_{10}e^{-q_8t}+q_1q_{11}e^{-\varepsilon t}+q_9\left(b_n-q_1\right)e^{-q_0t}+\\q_{10}\left(b_n-q_1\right)e^{-\left(q_0+q_8\right)t}+q_{11}\left(b_n-q_1\right)e^{-\left(q_0+\varepsilon\right)t}\end{pmatrix}\left(n\pi\right)^2\cos^2 n\pi z\end{aligned}\qquad(3.204)$$

$$u_0\theta_0=\left(z+\sum_{n=1}^{\infty}\left(q_9+q_{10}e^{-q_8 t}+q_{11}e^{-\varepsilon t}\right)\sin n\pi z\right)\left(z+\sum_{n=1}^{\infty}\left(q_1+\left(b_n-q_1\right)e^{-q_0 t}\right)\sin n\pi z\right)$$

$$=z^2+\sum_{n=1}^{\infty}\left(q_1+\left(b_n-q_1\right)e^{-q_0 t}\right)z\sin n\pi z+\sum_{n=1}^{\infty}\left(q_9+q_{10}e^{-q_8 t}+q_{11}e^{-\varepsilon t}\right)z\sin n\pi z+$$
$$\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\begin{pmatrix}q_1q_9+q_1q_{10}e^{-q_8 t}+q_1q_{11}e^{-\varepsilon t}+q_9\left(b_n-q_1\right)e^{-q_0 t}+\\ q_{10}\left(b_n-q_1\right)e^{-(q_0+q_8)t}+q_{11}\left(b_n-q_1\right)e^{-(q_0+\varepsilon)t}\end{pmatrix}\sin^2 n\pi z \tag{3.205}$$

$$f\left(z\right)=0\Rightarrow b_{5n}=2\int_0^1 0\cdot\sin n\pi z dz=0 \tag{3.206}$$

Then,

$$F_{5n}\left(t\right)=2\left(\begin{array}{l}q_{12}\int_0^1\sin n\pi z dz+\sum_{n=1}^{\infty}\left(q_{12}q_9+q_{12}q_{10}e^{-q_8 t}+q_{12}q_{11}e^{-\varepsilon t}+\frac{cq_1}{4\operatorname{Re}}+\frac{c\left(b_n-q_1\right)}{4\operatorname{Re}}e^{-q_0 t}\right)n\pi\int_0^1\cos n\pi z\sin n\pi z dz-\\ \sum_{n=1}^{\infty}\left(q_{13}q_9+q_{13}q_{10}e^{-q_8 t}+q_{13}q_{11}e^{-\varepsilon t}+\frac{Pq_1}{\operatorname{Re}}+\frac{Pc\left(b_n-q_1\right)}{\operatorname{Re}}e^{-q_0 t}\right)\int_0^1 z\sin^2 n\pi z dz+\\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\begin{pmatrix}\frac{cq_1q_9\left(n\pi\right)^2}{4\operatorname{Re}}+\frac{cq_1q_{10}\left(n\pi\right)^2}{4\operatorname{Re}}e^{-q_8 t}+\frac{cq_1q_{10}\left(n\pi\right)^2}{4\operatorname{Re}}e^{-\varepsilon t}+\frac{cq_9\left(b_n-q_1\right)\left(n\pi\right)^2}{4\operatorname{Re}}e^{-q_0 t}+\\ \frac{cq_{10}\left(b_n-q_1\right)\left(n\pi\right)^2}{4\operatorname{Re}}e^{-(q_0+q_8)t}+\frac{cq_{11}\left(b_n-q_1\right)\left(n\pi\right)^2}{4\operatorname{Re}}e^{-(q_0+\varepsilon)t}\end{pmatrix}.\\ \quad\int_0^1\left(\sin^2 n\pi z+\cos^2 n\pi z\sin n\pi\right)dz\\ +q_{14}\int_0^1 z\sin n\pi z dz+\\ \sum_{n=1}^{\infty}\begin{pmatrix}\left(-\frac{Ha^2Bifq_9}{\operatorname{Re}}+gq_1+lq_3\right)+\frac{Ha^2 f}{\operatorname{Re}}\left(-Biq_{10}+b_n\right)e^{-q_8 t}-\frac{Ha^2Bifq_{11}}{\operatorname{Re}}e^{-\varepsilon t}+\\ g\left(b_n-q_1\right)e^{-q_0 t}+l\left(b_n-q_3\right)e^{-q_2 t}\end{pmatrix}\int_0^1\sin^2 n\pi z dz-\\ \frac{Pc}{\operatorname{Re}}\int_0^1 z^2\sin n\pi z dz+\\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\begin{pmatrix}q_1q_9+q_1q_{10}e^{-q_8 t}+q_1q_{11}e^{-\varepsilon t}+q_9\left(b_n-q_1\right)e^{-q_0 t}+\\ q_{10}\left(b_n-q_3\right)e^{-(q_0+q_8)t}+q_{11}\left(b_n-q_3\right)e^{-(q_0+\varepsilon)t}\end{pmatrix}\int_0^1\sin^3 n\pi z dz-\\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(lq_4-lq_5e^{-q_2 t}+lq_6e^{-q_0 t}\right)\int_0^1\sin^2 n\pi z dz\end{array}\right) \tag{3.207}$$

i.e

$$F_{5n}(t)=2\begin{pmatrix} q_{15}+\sum_{n=1}^{\infty}\left(q_{16}+q_{17}e^{-q_8 t}+q_{18}e^{-\varepsilon t}+q_{19}e^{-q_0 t}+q_{20}e^{-q_2 t}\right)+ \\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(q_{21}+q_{22}e^{-q_8 t}+q_{23}e^{-\varepsilon t}+q_{24}e^{-q_0 t}+q_{25}e^{-(q_0-q_8)t}+q_{26}e^{-(q_0-\varepsilon)t}+q_{27}e^{-q_2 t}\right) \end{pmatrix}$$

(3.208)

Then,

$$v_{5n}(t)=2e^{-q_8 t}\begin{pmatrix} q_{15}\int_0^t e^{q_8\tau}d\tau+\sum_{n=1}^{\infty}\left(q_{16}\int_0^t e^{q_8\tau}d\tau+q_{17}\int_0^t d\tau+q_{18}\int_0^t e^{(q_8-\varepsilon)\tau}d\tau+q_{19}\int_0^t e^{(q_8-q_0)\tau}d\tau+q_{20}\int_0^t e^{(q_8-q_2)\tau}d\tau\right)+ \\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\begin{pmatrix} q_{21}\int_0^t e^{q_8\tau}d\tau+q_{22}\int_0^t d\tau+q_{23}\int_0^t e^{(q_8-\varepsilon)\tau}d\tau+q_{24}\int_0^t e^{(q_8-q_0)\tau}d\tau+ \\ q_{25}\int_0^t e^{-q_0\tau}d\tau+q_{26}\int_0^t e^{(q_8-q_0-\varepsilon)\tau}d\tau+q_{27}\int_0^t e^{(q_8-q_2)\tau}d\tau \end{pmatrix} \end{pmatrix}$$

(3.209)

i.e

$$v_{5n}(t)=2\begin{pmatrix} \frac{q_{15}}{q_8}\left(1-e^{q_8 t}\right)+\sum_{n=1}^{\infty}\begin{pmatrix} \frac{q_{16}}{q_8}\left(1-e^{-q_8 t}\right)+q_{17}te^{-q_8 t}+\frac{q_{18}}{(q_8-\varepsilon)}\left(e^{-\varepsilon t}-e^{-q_8 t}\right)+ \\ \frac{q_{19}}{(q_8-q_0)}\left(e^{-q_0 t}-e^{-q_8 t}\right)+\frac{q_{20}}{(q_8-q_2)}\left(e^{-q_2 t}-e^{-q_8 t}\right) \end{pmatrix}+ \\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\begin{pmatrix} \frac{q_{21}}{q_8}\left(1-e^{-q_8 t}\right)+q_{22}te^{-q_8 t}+\frac{q_{23}}{(q_8-\varepsilon)}\left(e^{-\varepsilon t}-e^{-q_8 t}\right)+\frac{q_{24}}{(q_8-q_0)}\left(e^{-q_0 t}-e^{-q_8 t}\right)- \\ \frac{q_{25}}{q_0}\left(e^{-(q_8+q_0)t}-e^{-q_8 t}\right)+\frac{q_{26}}{(q_8-q_0-\varepsilon)}\left(e^{-(q_0+\varepsilon)t}-e^{-q_8 t}\right)+\frac{q_{27}}{(q_8-q_2)}\left(e^{-q_2 t}-e^{-q_8 t}\right) \end{pmatrix} \end{pmatrix}$$

Therefore,

$$u_1(z,t)=\sum_{n=1}^{\infty}v_{5n}(t)\sin n\pi z \tag{3.210}$$

Consider equation (3.139), given below

$$\left.\begin{aligned}
&\frac{\partial\theta_1}{\partial t}+\frac{e}{2}\frac{\partial\theta_0}{\partial z}=\frac{c}{4\operatorname{Re}\Pr}\frac{\partial}{\partial y}\left(\theta_0\frac{\partial\theta_0}{\partial z}+\frac{\partial\theta_1}{\partial z}\right)+\frac{bc}{4\operatorname{Re}}\left(\frac{\partial u_0}{\partial z}\right)^2+\left(\frac{\partial w_0}{\partial z}\right)^2-\\
&\frac{bHa^2}{\operatorname{Re}}\left(\left(u_0\right)^2+\left(w_0\right)^2\right)-Ra^2\theta_1\\
&\theta_1(z,0)=0,\qquad \theta_1(0,t)=0,\qquad \theta_1(1,t)=0
\end{aligned}\right\}$$

Rearranging the above equation gives

$$\left.\begin{aligned}
&\frac{\partial\theta_1}{\partial t}=\frac{c}{4\operatorname{Re}\Pr}\frac{\partial^2\theta_1}{\partial z^2}-Ra^2\theta+\frac{c}{4\operatorname{Re}\Pr}\theta_0\frac{\partial^2\theta_0}{\partial z^2}+\frac{c}{4\operatorname{Re}\Pr}\left(\frac{\partial\theta_0}{\partial y}\right)^2+\\
&\frac{bc}{4\operatorname{Re}}\left(\left(\frac{\partial u_0}{\partial z}\right)^2+\left(\frac{\partial w_0}{\partial z}\right)^2\right)-\frac{bHa^2}{\operatorname{Re}}\left(\left(u_0\right)^2+\left(w_0\right)^2\right)-\frac{e}{2}\frac{\partial\theta_0}{\partial z}\\
&\theta_1(z,0)=0,\qquad \theta_1(0,t)=0,\qquad \theta_1(1,t)=0
\end{aligned}\right\} \tag{3.211}$$

i.e

$$\left.\begin{aligned}
&\frac{\partial\theta_1}{\partial t}=\frac{c}{4\operatorname{Re}\Pr}\frac{\partial^2\theta_1}{\partial z^2}-Ra^2\theta+q_{28}+q_{29}z-q_{30}z^2+\sum_{n=1}^{\infty}\begin{pmatrix}\left(\frac{c}{4\operatorname{Re}\Pr}-\frac{e}{2}\right)\left(q_1+\left(b_n-q_1\right)e^{-q_0t}\right)+\\ \frac{2bc}{4\operatorname{Re}}\left(q_9+q_{10}e^{-q_8t}+q_{11}e^{-\varepsilon t}\right)\end{pmatrix}n\pi\cos n\pi z+\\
&\sum_{n=1}^{\infty}\frac{c}{4\operatorname{Re}\Pr}\left(q_1+\left(b_n-q_1\right)e^{-q_0t}\right)\sin n\pi z+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{c}{4\operatorname{Re}\Pr}\begin{pmatrix}q_1^2+2q_1\left(b_n-q_1\right)e^{-q_0t}+\\ \left(b_n-q_1\right)^2e^{-2q_0t}\end{pmatrix}n\pi\sin n\pi z\cos n\pi z+\\
&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{bc}{4\operatorname{Re}}\left(\left(q_9+q_{10}e^{-q_8t}+q_{11}e^{-\varepsilon t}\right)^2+b_{3n}^2e^{-2q_8t}\right)\left(n\pi\right)^2\cos^2 n\pi z-\sum_{n=1}^{\infty}2\left(q_9+q_{10}e^{-q_8t}+q_{11}e^{-\varepsilon t}\right)q_{30}z\sin n\pi z-\\
&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}q_{30}\left(\left(q_9+q_{10}e^{-q_8t}+q_{11}e^{-\varepsilon t}\right)^2+b_{3n}^2e^{-2q_8t}\right)\sin^2 n\pi z\\
&\theta_1(z,0)=0,\qquad \theta_1(0,t)=0,\qquad \theta_1(1,t)=0
\end{aligned}\right\} \tag{3.212}$$

$$f(z)=0\Rightarrow b_n=0 \tag{3.213}$$

Then

$$F_{6n}(t)=2\left(\begin{array}{l} q_{28}\int_0^1 \sin n\pi z dz + q_{29}\int_0^1 z\sin n\pi z dz - q_{30}\int_0^1 z^2 \sin n\pi z dz + \\ \sum_{n=1}^{\infty}\left(\begin{array}{l}\left(\frac{c}{4\operatorname{Re}\Pr}-\frac{e}{2}\right)\left(q_1+\left(b_n-q_1\right)e^{-q_0 t}\right)+ \\ \frac{2b_1 c}{4\operatorname{Re}}\left(q_9+q_{10}e^{-q_8 t}+q_{11}e^{-\varepsilon t}\right)\end{array}\right) n\pi\int_0^1 \cos n\pi z \sin n\pi z dz + \\ \sum_{n=1}^{\infty}\frac{c}{4\operatorname{Re}\Pr}\left(q_1+\left(b_n-q_1\right)e^{-q_0 t}\right)\int_0^1 \sin^2 n\pi z dz + \\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{c}{4\operatorname{Re}\Pr}\left(\begin{array}{l} q_1^2+2q_1\left(b_n-q_1\right)e^{-q_0 t}+ \\ \left(b_n-q_1\right)^2 e^{-2q_0 t}\end{array}\right) n\pi\int_0^1 \sin^2 n\pi z \cos n\pi z dz + \\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{b_1 c}{4\operatorname{Re}}\left(\left(q_9+q_{10}e^{-q_8 t}+q_{11}e^{-\varepsilon t}\right)^2+b_{3n}^2 e^{-2q_8 t}\right)\left(n\pi\right)^2\int_0^1 \cos^2 n\pi z \sin n\pi z - \\ \sum_{n=1}^{\infty} 2\left(q_9+q_{10}e^{-q_8 t}+q_{11}e^{-\varepsilon t}\right) q_{30}\int_0^1 z\sin^2 n\pi z dz - \\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} q_{30}\left(\left(q_9+q_{10}e^{-q_8 t}+q_{11}e^{-\varepsilon t}\right)^2+b_{3n}^2 e^{-2q_8 t}\right)\int_0^1 \sin^3 n\pi z dz \end{array}\right) \tag{3.214}$$

$$F_{6n}(t)=2\left(\begin{array}{l} \frac{q_{28}\left(1-(-1)^n\right)}{n\pi}+\frac{q_{29}(-1)^n}{n\pi}-\frac{q_{30}\left(\left(2-(n\pi)^2\right)(-1)^2-2\right)}{(n\pi)^3}+ \\ \sum_{n=1}^{\infty}\left(\frac{1-(-1)^{2n}}{2}\right)\left(\begin{array}{l}\left(\frac{c}{4\operatorname{Re}\Pr}-\frac{e}{2}\right)\left(q_1+\left(b_n-q_1\right)e^{-q_0 t}\right)+ \\ \frac{2b_1 c}{4\operatorname{Re}}\left(q_9+q_{10}e^{-q_8 t}+q_{11}e^{-\varepsilon t}\right)\end{array}\right)+ \\ \sum_{n=1}^{\infty}\frac{c}{8\operatorname{Re}\Pr}\left(q_1+\left(b_n-q_1\right)e^{-q_0 t}\right)\int_0^1 \sin^2 n\pi z dz + \\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{b_1 c}{4\operatorname{Re}}\left(\frac{1-(-1)^{3n}}{3n\pi}\right)\left(\left(q_9+q_{10}e^{-q_8 t}+q_{11}e^{-\varepsilon t}\right)^2+b_{3n}^2 e^{-2q_8 t}\right)\left(n\pi\right)^2 - \\ \sum_{n=1}^{\infty}\frac{2q_{30}\left(1+(n\pi)^2-(-1)^{2n}\right)}{4(n\pi)^2}\left(q_9+q_{10}e^{-q_8 t}+q_{11}e^{-\varepsilon t}\right)- \\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{q_{30}\left((-1)^{3n}-3(-1)^n+2\right)}{3n\pi}\left(\left(q_9+q_{10}e^{-q_8 t}+q_{11}e^{-\varepsilon t}\right)^2+b_{3n}^2 e^{-2q_8 t}\right)\int_0^1 \sin^3 n\pi z dz \end{array}\right) \tag{3.215}$$

$$v_{6n}(t)=2\left(\begin{array}{l}\frac{1}{q_0}\left(\frac{q_{28}\left(1-(-1)^n\right)}{n\pi}+\frac{q_{29}(-1)^n}{n\pi}-\frac{q_{30}\left(\left(2-(n\pi)^2\right)(-1)^n-2\right)}{(n\pi)^3}\right)\left(1-e^{-q_0t}\right)+\\ \sum_{n=1}^{\infty}\left(\frac{\left(1-(-1)^{2n}\right)}{2}\right)\left(\begin{array}{l}\left(\frac{c}{4\operatorname{Re}\Pr}-\frac{e}{2}\right)\left(\frac{q_1}{q_0}\left(1-e^{-q_0t}\right)+\left(b_n-q_1\right)te^{-q_0t}\right)+\\ \frac{2b_1c}{4\operatorname{Re}}\left(\frac{q_9}{q_0}\left(1-e^{-q_0t}\right)+\frac{q_{10}}{(q_0-q_8)}\left(e^{-q_8t}-e^{-q_0t}\right)+\frac{q_{11}}{(q_0-\varepsilon)}\left(e^{-\varepsilon t}-e^{-q_0t}\right)\right)\end{array}\right)+\\ \sum_{n=1}^{\infty}\frac{c}{8\operatorname{Re}\Pr}\left(\frac{q_1}{q_0}\left(1-e^{-q_0t}\right)+\left(b_n-q_1\right)te^{-q_0t}\right)+\\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{b_1c(n\pi)^2}{4\operatorname{Re}}\left(\left(\frac{\left(1-(-1)^{3n}\right)}{3n\pi}\right)+\left(\frac{q_{30}\left((-1)^{3n}-3(-1)^n+2\right)}{3n\pi}\right)\right)\left(\begin{array}{l}\frac{q_9}{q_0}\left(1-e^{-q_0t}\right)+\frac{2q_9q_{10}}{(q_0-q_8)}\left(e^{-q_8t}-e^{-q_0t}\right)+\\ \frac{2q_9q_{11}}{(q_0-\varepsilon)}\left(e^{-\varepsilon t}-e^{-q_0t}\right)+\\ \frac{\left(b_{3n}^2+q_{10}^2\right)}{(q_0-2q_8)}\left(e^{-2q_8t}-e^{-q_0t}\right)+\\ \frac{2q_{10}q_{11}}{\left(q_0-(q_8+\varepsilon)\right)}\left(e^{-2\varepsilon t}-e^{-q_0t}\right)+\\ \frac{q_{11}^2}{(q_0-2\varepsilon)}\left(e^{-2\varepsilon t}-e^{-q_0t}\right)\end{array}\right)-\\ \sum_{n=1}^{\infty}\left(\frac{2q_{30}\left(1-(-1)^2+(-1)^{2n}\right)}{4(n\pi)^2}\right)\left(\frac{q_9}{q_0}\left(1-e^{-q_0t}\right)+\frac{q_{10}}{(q_0-q_8)}\left(e^{-q_8t}-e^{-q_0t}\right)+\frac{q_{11}}{(q_0-\varepsilon)}\left(e^{-\varepsilon t}-e^{-q_0t}\right)\right)\end{array}\right) \tag{3.216}$$

i.e

$$v_{6n}(t)=2\left(\begin{array}{l}q_{31}\left(1-e^{-q_0t}\right)+\\ \sum_{n=1}^{\infty}\left(q_{48}\left(1-e^{-q_0t}\right)+q_{49}te^{-q_0t}+q_{50}\left(e^{-q_8t}-e^{-q_0t}\right)-q_{51}\left(e^{-\varepsilon t}-e^{-q_0t}\right)\right)+\\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(\begin{array}{l}q_{39}\left(1-e^{-q_0t}\right)+q_{40}\left(e^{-q_8t}-e^{-q_0t}\right)+q_{41}\left(e^{-\varepsilon t}-e^{-q_0t}\right)+q_{42}\left(e^{-2q_8t}-e^{-q_0t}\right)+\\ q_{43}\left(e^{-(q_8+\varepsilon)t}-e^{-q_0t}\right)+q_{44}\left(e^{-2\varepsilon t}-e^{-q_0t}\right)\end{array}\right)\end{array}\right) \tag{3.217}$$

where

$$q_{12}=\left(\frac{c}{4\,\mathrm{Re}}-\frac{e}{2}\right),q_{13}=\left(\frac{cP}{\mathrm{Re}}+\frac{c\left(n\pi\right)^2}{4\,\mathrm{Re}}\right),q_{14}=\left(g+l-\frac{Ha^2Bif}{\mathrm{Re}}\right),$$

$$q_{15}=\left(\frac{q_{12}\left(1-\left(-1\right)^n\right)}{n\pi}-\frac{q_{14}\left(-1\right)^n}{n\pi}-\frac{Pc}{\mathrm{Re}}\left(\frac{2\left(-1\right)^n-\left(-1\right)^n\left(n\pi\right)-2}{\left(n\pi\right)^3}\right)\right),$$

$$q_{16}=\left(\frac{\left(q_9q_{12}+\frac{cq_1}{4\,\mathrm{Re}}\right)\left(1-\left(-1\right)^{2n}\right)}{2}-\frac{\left(q_9q_{13}+\frac{cPq_1}{\mathrm{Re}}\right)\left(\left(n\pi\right)^2+1-\left(-1\right)^{2n}\right)}{4\left(n\pi\right)^2}+\frac{1}{2}\left(-\frac{Ha^2Bifq_9}{\mathrm{Re}}+gq_1+lq_3\right)\right),$$

$$q_{17}=\left(\frac{\left(q_{10}q_{12}\right)\left(1-\left(-1\right)^{2n}\right)}{2}-\frac{\left(q_{10}q_{13}\right)\left(\left(n\pi\right)^2+1-\left(-1\right)^{2n}\right)}{4\left(n\pi\right)^2}+\frac{Ha^2f}{2\,\mathrm{Re}}\left(-Biq_{10}+b_n\right)\right),$$

$$q_{18}=\left(\frac{\left(q_{11}q_{12}\right)\left(1-\left(-1\right)^{2n}\right)}{2}-\frac{\left(q_{11}q_{13}\right)\left(\left(n\pi\right)^2+1-\left(-1\right)^{2n}\right)}{4\left(n\pi\right)^2}-\frac{1}{2}\frac{Ha^2Bifq_{11}}{\mathrm{Re}}\right),$$

$$q_{19}=\left(\frac{c\left(b_n-q_1\right)\left(1-\left(-1\right)^{2n}\right)}{8\,\mathrm{Re}}-\frac{Pc\left(b_n-q_1\right)\left(\left(n\pi\right)^2+1-\left(-1\right)^{2n}\right)}{4\,\mathrm{Re}\left(n\pi\right)^2}+\frac{g\left(b_n-q_1\right)}{2}\right),$$

$$q_{20}=\frac{1}{2}l\left(b_{2n}-q_3\right),$$

$$q_{21}=\left(\frac{cq_1q_9\left(n\pi\right)^2}{4\,\mathrm{Re}}\left(\frac{\left(-1\right)^{3n}-3\left(-1\right)^n+2+1-\left(-1\right)^{3n}}{3\left(n\pi\right)}\right)+\frac{q_1q_9\left(\left(-1\right)^{3n}-3\left(-1\right)^n+2\right)}{3\left(n\pi\right)}-\frac{lq_4}{2}\right),$$

$$q_{22}=\left(\frac{cq_1q_{10}\left(n\pi\right)^2}{4\,\mathrm{Re}}\left(\frac{3-3\left(-1\right)^n}{3\left(n\pi\right)}\right)+\frac{q_1q_{10}\left(\left(-1\right)^{3n}-3\left(-1\right)^n+2\right)}{3\left(n\pi\right)}\right),$$

$$q_{23}=\left(\frac{cq_1q_{11}\left(n\pi\right)^2}{4\,\mathrm{Re}}\left(\frac{1-\left(-1\right)^n}{\left(n\pi\right)}\right)+\frac{q_1q_{11}\left(\left(-1\right)^{3n}-3\left(-1\right)^n+2\right)}{3\left(n\pi\right)}\right),$$

$$q_{24}=\left(\frac{cq_9\left(b_n-q_1\right)\left(n\pi\right)^2}{4\,\mathrm{Re}}\left(\frac{1-\left(-1\right)^n}{\left(n\pi\right)}\right)+\frac{q_9\left(b_n-q_1\right)\left(\left(-1\right)^{3n}-3\left(-1\right)^n+2\right)}{3\left(n\pi\right)}-\frac{lq_6}{2}\right),$$

$$q_{25}=\left(\frac{cq_{10}\left(b_n-q_1\right)\left(n\pi\right)^2}{4\,\mathrm{Re}}\left(\frac{1-\left(-1\right)^n}{\left(n\pi\right)}\right)+\frac{q_{10}\left(b_n-q_1\right)\left(\left(-1\right)^{3n}-3\left(-1\right)^n+2\right)}{3\left(n\pi\right)}\right),$$

$$q_{26}=\left(\frac{cq_{11}\left(b_n-q_1\right)\left(n\pi\right)^2}{4\,\mathrm{Re}}\left(\frac{1-\left(-1\right)^n}{\left(n\pi\right)}\right)+\frac{q_{11}\left(b_n-q_1\right)\left(\left(-1\right)^{3n}-3\left(-1\right)^n+2\right)}{3\left(n\pi\right)}\right),q_{27}=\frac{lq_5}{2},q_{28}=\left(\frac{bc}{4\,\mathrm{Re}}-\frac{e}{2}\right),$$

$$q_{29}=\frac{c}{4\,\mathrm{Re}\,\mathrm{Pr}},q_{30}=\frac{bHa^2}{\mathrm{Re}},q_{31}=\frac{1}{q_0}\left(\frac{q_{28}\left(1-\left(-1\right)^n\right)}{n\pi}-\frac{q_{29}\left(-1\right)^n}{n\pi}-\frac{q_{30}\left(\left(2-\left(n\pi\right)^2\right)\left(-1\right)^n-2\right)}{\left(n\pi\right)^3}\right),$$

$$q_{32}=\frac{q_1}{q_0}\left(\frac{\left(1-(-1)^{2n}\right)}{2}\right)\left(\frac{c}{4\operatorname{Re}\Pr}-\frac{e}{2}\right), q_{33}=\left(b_n-q_1\right)\left(\frac{\left(1-(-1)^{2n}\right)}{2}\right)\left(\frac{c}{4\operatorname{Re}\Pr}-\frac{e}{2}\right),$$

$$q_{34}=\frac{q_9}{q_0}\left(\frac{\left(1-(-1)^{2n}\right)}{2}\right)\left(\frac{2b_1c}{4\operatorname{Re}}\right), q_{35}=\frac{q_{10}}{\left(q_0-q_8\right)}\left(\frac{\left(1-(-1)^{2n}\right)}{2}\right)\left(\frac{2b_1c}{4\operatorname{Re}}\right), q_{36}=\frac{q_{11}}{\left(q_0-\varepsilon\right)}\left(\frac{\left(1-(-1)^{2n}\right)}{2}\right)\left(\frac{2b_1c}{4\operatorname{Re}}\right),$$

$$q_{37}=\frac{q_1c}{8\operatorname{Re}\Pr q_0}, q_{38}=\frac{c\left(b_n-q_1\right)}{8\operatorname{Re}\Pr}, q_{39}=\frac{q_9}{q_{10}}\left(\frac{b_1c(n\pi)^2}{4\operatorname{Re}}\left(\frac{1-(-1)^{3n}}{3(n\pi)}\right)+\frac{q_{30}\left((-1)^{3n}-3(-1)^n+2\right)}{3(n\pi)}\right),$$

$$q_{40}=\frac{2q_9q_{10}}{\left(q_0-q_8\right)}\left(\frac{b_1c(n\pi)^2}{4\operatorname{Re}}\left(\frac{1-(-1)^{3n}}{3(n\pi)}\right)+\frac{q_{30}\left((-1)^{3n}-3(-1)^n+2\right)}{3(n\pi)}\right),$$

$$q_{41}=\frac{2q_9q_{11}}{\left(q_0-\varepsilon\right)}\left(\frac{b_1c(n\pi)^2}{4\operatorname{Re}}\left(\frac{1-(-1)^{3n}}{3(n\pi)}\right)+\frac{q_{30}\left((-1)^{3n}-3(-1)^n+2\right)}{3(n\pi)}\right),$$

$$q_{42}=\frac{\left(b_{3n}^2+q_{10}^2\right)}{\left(q_0-2q_8\right)}\left(\frac{b_1c(n\pi)^2}{4\operatorname{Re}}\left(\frac{1-(-1)^{3n}}{3(n\pi)}\right)+\frac{q_{30}\left((-1)^{3n}-3(-1)^n+2\right)}{3(n\pi)}\right),$$

$$q_{43}=\frac{2q_{10}q_{11}}{\left(q_0-\left(q_8+\varepsilon\right)\right)}\left(\frac{b_1c(n\pi)^2}{4\operatorname{Re}}\left(\frac{1-(-1)^{3n}}{3(n\pi)}\right)+\frac{q_{30}\left((-1)^{3n}-3(-1)^n+2\right)}{3(n\pi)}\right),$$

$$q_{44}=\frac{q_{11}^2}{\left(q_0-2\varepsilon\right)}\left(\frac{b_1c(n\pi)^2}{4\operatorname{Re}}\left(\frac{1-(-1)^{3n}}{3(n\pi)}\right)+\frac{q_{30}\left((-1)^{3n}-3(-1)^n+2\right)}{3(n\pi)}\right),$$

Thus from equation (3.152)

$$\theta_1(z,t)=\sum_{n=1}^{\infty}v_{6n}(t)\,Sinn\pi z \tag{3.218}$$

Consider equation (3.140) given by

$$\left.\begin{aligned}&\frac{\partial w_1}{\partial t}+\frac{e}{2}\frac{\partial w_0}{\partial z}=\frac{c}{4\operatorname{Re}}\frac{\partial}{\partial z}\left(\theta_0\frac{\partial w_0}{\partial z}+\frac{\partial w_1}{\partial z}\right)-\frac{Ha^2}{\operatorname{Re}}\left(-Bifw_0+w_1-fu_0\right)-\frac{cP}{\operatorname{Re}}\left(\theta_0w_0+w_1\right)\\&w_1(z,0)=0,\qquad w_1(0,t)=0,\qquad w_1(1,t)=0\end{aligned}\right\}$$

Rearranging the above equation gives,

$$\left.\begin{aligned}
&\frac{\partial w_1}{\partial t}=\frac{c}{4\operatorname{Re}}\frac{\partial^2 w_1}{\partial z^2}-q_7 w_1\frac{c\theta_0}{4\operatorname{Re}}\frac{\partial^2 w_0}{\partial z^2}+\frac{c}{4\operatorname{Re}}\left(\frac{\partial w_0}{\partial z}\right)\left(\frac{\partial \theta_0}{\partial z}\right)-\frac{Ha^2}{\operatorname{Re}}\left(Bifw_0-fu_0\right)-\\
&\frac{cP}{\operatorname{Re}}\left(\theta_0 w_0\right)-\frac{e}{2}\frac{\partial w_0}{\partial z}\\
&w_1(z,0)=0,\qquad w_1(0,t)=0,\qquad w_1(1,t)=0
\end{aligned}\right\}\tag{3.219}$$

Then

$$\left.\begin{aligned}
&\frac{\partial w_1}{\partial t}=\frac{c}{4\operatorname{Re}}\frac{\partial^2 w_1}{\partial z^2}-q_7 w_1-\sum_{n=1}^{\infty}b_{3n}\left((n\pi)^2+1+\left(\frac{c(n\pi)^2}{4\operatorname{Re}}+\frac{cP}{\operatorname{Re}}\right)\right)e^{-q_8 t}z\sin n\pi z+\\
&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}b_{3n}\left((n\pi)^2+1+\left(\frac{c(n\pi)^2}{4\operatorname{Re}}+\frac{cP}{\operatorname{Re}}\right)\right)e^{-q_8 t}\left(q_1+\left(b_n-q_1\right)e^{-q_0 t}\right)\sin^2 n\pi z+\\
&\sum_{n=1}^{\infty}q_{12}b_{3n}e^{-q_8 t}(n\pi)\cos n\pi z+\\
&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}b_{3n}e^{-q_8 t}\left(q_1+\left(b_n-q_1\right)e^{-q_0 t}\right)\left(\frac{c(n\pi)^2\cos^2 n\pi z}{4\operatorname{Re}}-\frac{cP}{\operatorname{Re}}\sin^2 n\pi z\right)+\\
&\sum_{n=1}^{\infty}\frac{Ha^2 f}{\operatorname{Re}}\left(q_9+\left(q_{10}+Bib_{3n}\right)e^{-q_8 t}+q_{11}e^{-\varepsilon t}\right)\sin n\pi z+\frac{Ha^2 f}{\operatorname{Re}}z\\
&w_1(z,0)=0,\qquad w_1(0,t)=0,\qquad w_1(1,t)=0
\end{aligned}\right\}\tag{3.220}$$

$$F_{7n}(t)=2\left(\begin{aligned}
&\frac{Ha^2 f}{\operatorname{Re}}\int_0^1 z\sin n\pi z dz-\sum_{n=1}^{\infty}b_{3n}\left(\frac{c(n\pi)^2}{4\operatorname{Re}}+\frac{cP}{\operatorname{Re}}\right)e^{-q_8 t}\int_0^1 z\sin^2 n\pi z dz+\\
&\sum_{n=1}^{\infty}q_{12}b_{3n}e^{-q_8 t}(n\pi)\int_0^1\cos n\pi z\sin n\pi z dz\\
&\sum_{n=1}^{\infty}\frac{Ha^2 f}{\operatorname{Re}}\left(q_9+\left(q_{10}+Bib_{3n}\right)e^{-q_8 t}+q_{11}e^{-\varepsilon t}\right)\int_0^1\sin^2 n\pi z dz+\\
&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}b_{3n}\left((n\pi)^2+1+\left(\frac{c(n\pi)^2}{4\operatorname{Re}}+\frac{cP}{\operatorname{Re}}\right)\right)e^{-q_8 t}\left(q_1+\left(b_n-q_1\right)e^{-q_0 t}\right)\int_0^1\sin^3 n\pi z dz+\\
&\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}b_{3n}e^{-q_8 t}\left(q_1+\left(b_n-q_1\right)e^{-q_0 t}\right)\left(\frac{c(n\pi)^2}{4\operatorname{Re}}\int_0^1\cos^2 n\pi z\sin n\pi z dz-\frac{cP}{\operatorname{Re}}\int_0^1\sin^3 n\pi z\right)
\end{aligned}\right)\tag{3.221}$$

i.e

$$F_{7n}(t)=2\begin{pmatrix} -\frac{Ha^2 f(-1)^n}{n\pi \operatorname{Re}}-\sum_{n=1}^{\infty}b_{3n}\frac{\left(1+n^2\pi^2-(-1)^{2n}\right)}{4(n\pi)^2}\left(\frac{c(n\pi)^2}{4\operatorname{Re}}+\frac{cP}{\operatorname{Re}}\right)e^{-q_8t}+\sum_{n=1}^{\infty}\frac{q_{12}b_{3n}\left(1-(-1)^{2n}\right)}{2}e^{-q_8t} \\ \sum_{n=1}^{\infty}\frac{Ha^2 f}{2\operatorname{Re}}\left(q_9+\left(q_{10}+Bib_{3n}\right)e^{-q_8t}+q_{11}e^{-\varepsilon t}\right)+ \\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{b_{3n}\left((-1)^{3n}-3(-1)^n+2\right)}{3n\pi}\left(\frac{c(n\pi)^2}{4\operatorname{Re}}+\frac{cP}{\operatorname{Re}}\right)e^{-q_8t}\left(q_1+\left(b_n-q_1\right)e^{-q_0t}\right)+ \\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}b_{3n}e^{-q_8t}\left(q_1+\left(b_n-q_1\right)e^{-q_0t}\right)\left(\frac{c(n\pi)^2}{4\operatorname{Re}}\left(\frac{1-(-1)^{3n}}{3n\pi}\right)-\frac{Pc}{\operatorname{Re}}\left(\frac{\left((-1)^{3n}-3(-1)^n+2\right)}{3n\pi}\right)\right) \end{pmatrix} \tag{3.222}$$

i.e

$$F_{7n}(t)=2\begin{pmatrix} -\frac{Ha^2 f(-1)^n}{n\pi \operatorname{Re} q_8}\left(1-e^{-q_8t}\right)+\sum_{n=1}^{\infty}\begin{pmatrix} \frac{Ha^2 fq_9}{2\operatorname{Re} q_8}\left(1-e^{-q_8t}\right)+\begin{pmatrix} \frac{Ha^2 f}{2\operatorname{Re}}\left(q_{10}+Bib_{3n}\right)+\frac{q_{12}b_{3n}\left(1-(-1)^{2n}\right)}{2}- \\ b_{3n}\frac{\left(1+n^2\pi^2-(-1)^{2n}\right)}{4(n\pi)^2}\left(\frac{c(n\pi)^2}{4\operatorname{Re}}+\frac{cP}{\operatorname{Re}}\right) \end{pmatrix}te^{-q_8t}+ \\ \frac{Ha^2 fq_9}{2\operatorname{Re}\left(q_8-\varepsilon\right)}\left(e^{-\varepsilon t}-e^{-q_8t}\right) \end{pmatrix}+ \\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\begin{pmatrix} \frac{b_{3n}\left((-1)^{3n}-3(-1)^n+2\right)}{3n\pi}\left(\frac{c(n\pi)^2}{4\operatorname{Re}}+\frac{cP}{\operatorname{Re}}\right)+ \\ b_{3n}\left(\frac{c(n\pi)^2}{4\operatorname{Re}}\left(\frac{1-(-1)^{3n}}{3n\pi}\right)-\frac{cP\left((-1)^{3n}-3(-1)^n+2\right)}{3n\pi \operatorname{Re}}\right) \end{pmatrix}\left(q_1te^{-q_8t}+\frac{\left(b_n-q_1\right)}{q_0}\left(e^{-(q_0+q_8)t}-e^{-q_8t}\right)\right) \end{pmatrix} \tag{3.223}$$

Therefore, from equation (3.152)

$$w_1(z,t)=\sum_{n=1}^{\infty}v_{7n}(t)\sin n\pi z \tag{3.224}$$

Consider equation (3.141) given as

$$\left.\begin{array}{l} \frac{\partial \phi_1}{\partial t}+\frac{e}{2}\frac{\partial \phi_0}{\partial z}=\frac{c}{4Sc\operatorname{Re}}\frac{\partial}{\partial z}\left(\theta_0\frac{\partial \phi_0}{\partial z}+\frac{\partial \phi_1}{\partial z}\right)+T_D\frac{\partial^2\theta_1}{\partial z^2}-Kr\phi_1 \\ \phi_1(z,0)=0, \qquad \phi_1(0,t)=0, \qquad \phi_1(1,t)=0 \end{array}\right\}$$

Rearranging the above equation gives

$$\left.\begin{array}{l}\dfrac{\partial \phi_1}{\partial t}=\dfrac{c}{4Sc\,\mathrm{Re}}\dfrac{\partial^2\phi_1}{\partial z^2}-Kr\phi_1+T_D\dfrac{\partial^2\theta_1}{\partial z^2}-\dfrac{e}{2}\dfrac{\partial \phi_0}{\partial z}+\dfrac{ce\theta_0}{4Sc\,\mathrm{Re}}\dfrac{\partial^2\phi_0}{\partial z^2}+\dfrac{c}{4Sc\,\mathrm{Re}}\left(\dfrac{\partial \phi_0}{\partial z}\right)\left(\dfrac{\partial \theta_0}{\partial z}\right)\\ \phi_1(z,0)=0,\qquad \phi_1(0,t)=0,\qquad \phi_1(1,t)=0\end{array}\right\} \tag{3.225}$$

$$\left.\begin{array}{l}\dfrac{\partial \phi_1}{\partial t}=\dfrac{c}{4Sc\,\mathrm{Re}}\dfrac{\partial^2\phi_1}{\partial z^2}-Kr\phi_1+T_D\sum_{n=1}^{\infty}v_{6n}(t)(n\pi)^2\sin n\pi z-\\ \dfrac{e}{2}\left(1+\sum_{n=1}^{\infty}\left(q_3+(b_{2n}-q_3)e^{-q_2t}-\sum_{n=1}^{\infty}\left(q_4-q_5e^{-q_2t}+q_6e^{-q_0t}\right)\right)\right)n\pi\cos n\pi z+\\ \dfrac{c}{4\,\mathrm{Re}\,Sc}\left(z+\sum_{n=1}^{\infty}\left(q_3+(b_{2n}-q_3)e^{-q_2t}-\sum_{n=1}^{\infty}\left(q_4-q_5e^{-q_2t}+q_6e^{-q_0t}\right)\right)\right)n\pi z\cos n\pi z+\\ \sum_{n=1}^{\infty}\left(q_1+(b_n-q_1)e^{-q_0t}\right)\sin n\pi z+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(q_1+(b_n-q_1)e^{-q_0t}\right)\left(q_3+(b_{2n}-q_3)e^{-q_2t}\right)-\\ \sum_{n=1}^{\infty}\left(q_4-q_5e^{-q_2t}+q_6e^{-q_0t}\right)n\pi\cos n\pi z\sin n\pi z+\\ \dfrac{c}{4\,\mathrm{Re}\,Sc}\left(1+\sum_{n=1}^{\infty}\left(q_1+(b_n-q_1)e^{-q_0t}\right)n\pi\cos n\pi z+\sum_{n=1}^{\infty}\left(q_3+(b_{2n}-q_3)e^{-q_2t}\right)-\sum_{n=1}^{\infty}\left(q_4-q_5e^{-q_2t}+q_6e^{-q_0t}\right)\right)n\pi\cos n\pi z+\\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(q_1+(b_n-q_1)e^{-q_0t}\right)\left(q_3+(b_{2n}-q_3)e^{-q_2t}-\sum_{n=1}^{\infty}\left(q_4-q_5e^{-q_2t}+q_6e^{-q_0t}\right)\right)(n\pi)^2\cos^2 n\pi z\\ \phi_1(z,0)=0,\qquad \phi_1(0,t)=0,\qquad \phi_1(1,t)=0\end{array}\right\} \tag{3.226}$$

$$f(z)=0\Rightarrow b_{8n}=0 \tag{3.227}$$

Then

$$F_{8n}(t)=2\left(\begin{array}{l}q_{52}\int_0^1\sin n\pi z dz+\sum_{n=1}^{\infty}\left(q_1+(b_n-q_1)e^{-q_0t}-T_Dv_{6n}(t)(n\pi)^2\right)\int_0^1\sin^2 n\pi z dz+\\ \sum_{n=1}^{\infty}\left(\begin{array}{l}\left(\dfrac{c}{4\,\mathrm{Re}\,Sc}-\dfrac{e}{2}\right)\left(q_3+(b_{2n}-q_3)e^{-q_2t}-\sum\left(q_4-q_5e^{-q_2t}+q_6e^{-q_0t}\right)\right)+\\ \left(q_1+(b_n-q_1)e^{-q_0t}\right)\dfrac{c}{4\,\mathrm{Re}\,Sc}\end{array}\right)n\pi\int_0^1\cos n\pi z\sin n\pi z dz+\\ +\dfrac{c}{4\,\mathrm{Re}\,Sc}\int_0^1 z\sin n\pi z dz+\\ \sum_{n=1}^{\infty}\dfrac{c}{4\,\mathrm{Re}\,Sc}\left(q_3+(b_{2n}-q_3)e^{-q_2t}-\sum_{n=1}^{\infty}\left(q_4-q_5e^{-q_2t}+q_6e^{-q_0t}\right)\right)n\pi\int_0^1 z\cos n\pi z\sin n\pi z dz+\\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}n\pi\left(q_1+(b_n-q_1)e^{-q_0t}\right)\left(q_3+(b_{2n}-q_3)e^{-q_2t}-\sum_{n=1}^{\infty}\left(q_4-q_5e^{-q_2t}+q_6e^{-q_0t}\right)\right)\cdot\\ \left(\int_0^1\cos n\pi z\sin^2 n\pi z dz+n\pi\int_0^1\cos^2 n\pi z\sin n\pi z dz\right)\end{array}\right) \tag{3.228}$$

i.e

$$F_{8n}\left(t\right)=2\left(\begin{array}{l}\frac{q_{52}\left(1-\left(-1\right)^{n}\right)}{n\pi}+\sum_{n=1}^{\infty}\frac{1}{2}\left(q_{1}+\left(b_{n}-q_{1}\right)e^{-q_{0}t}-T_{D}v_{6n}\left(t\right)\left(n\pi\right)^{2}\right)+\\ \sum_{n=1}^{\infty}\left(\begin{array}{l}\left(\frac{c}{4\operatorname{Re}Sc}-\frac{e}{2}\right)\left(q_{3}+\left(b_{2n}-q_{3}\right)e^{-q_{2}t}-\sum_{n=1}^{\infty}\left(q_{4}-q_{5}e^{-q_{2}t}+q_{6}e^{-q_{0}t}\right)\right)+\\ \left(q_{1}+\left(b_{n}-q_{1}\right)e^{-q_{0}t}\right)\frac{c}{4\operatorname{Re}Sc}\end{array}\right)\left(\frac{1-\left(-1\right)^{2n}}{2}\right)+\\ -\frac{c\left(-1\right)^{n}}{4\operatorname{Re}Scn\pi}-\\ \sum_{n=1}^{\infty}\frac{c\left(1-2\left(-1\right)^{2n}\right)}{16\operatorname{Re}Sc}\left(q_{3}+\left(b_{2n}-q_{3}\right)e^{-q_{2}t}-\sum_{n=1}^{\infty}\left(q_{4}-q_{5}e^{-q_{2}t}+q_{6}e^{-q_{0}t}\right)\right)+\\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{n\pi\left(1-\left(-1\right)^{3n}\right)}{3}\left(q_{1}+\left(b_{n}-q_{1}\right)e^{-q_{0}t}\right)\left(q_{3}+\left(b_{2n}-q_{3}\right)e^{-q_{2}t}-\sum_{n=1}^{\infty}\left(q_{4}-q_{5}e^{-q_{2}t}+q_{6}e^{-q_{0}t}\right)\right)\end{array}\right) \qquad (3.229)$$

i.e

$$F_{8n}\left(t\right)=2\left(\begin{array}{l}q_{53}+\sum_{n=1}^{\infty}\left(q_{54}+q_{55}e^{-q_{0}t}-q_{56}v_{6n}\left(t\right)\right)+\\ \sum_{n=1}^{\infty}\left(\left(q_{57}+q_{58}e^{-q_{2}t}-\sum_{n=1}^{\infty}\left(q_{59}-q_{60}e^{-q_{2}t}+q_{61}e^{-q_{0}t}\right)\right)+\left(q_{62}+q_{63}e^{-q_{0}t}\right)\right)-\\ \sum_{n=1}^{\infty}\left(q_{64}+q_{65}e^{-q_{2}t}-\sum_{n=1}^{\infty}\left(q_{66}-q_{67}e^{-q_{2}t}+q_{68}e^{-q_{0}t}\right)\right)+\\ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(\begin{array}{l}q_{69}+q_{70}e^{-q_{2}t}-\sum_{n=1}^{\infty}\left(q_{71}-q_{72}e^{-q_{2}t}+q_{73}e^{-q_{0}t}\right)+q_{74}e^{-q_{2}t}+q_{75}e^{-\left(q_{0}+q_{2}\right)t}-\\ \sum_{n=1}^{\infty}\left(q_{76}e^{-q_{0}t}-q_{77}e^{-\left(q_{0}+q_{2}\right)t}+q_{78}e^{-2q_{0}t}\right)\end{array}\right)\end{array}\right) \qquad (3.230)$$

Then

$$
v_{8n}\left(t\right)=2\left(\begin{array}{l}
\dfrac{q_{53}}{q_2}\left(1-e^{-q_2t}\right)+\sum\limits_{n=1}^{\infty}\left(\begin{array}{l}
\dfrac{q_{54}}{q_2}\left(1-e^{-q_2t}\right)+\dfrac{q_{53}}{\left(q_2-q_0\right)}\left(e^{-q_0t}-e^{-q_2t}\right)-\\
2q_{56}\left(\begin{array}{l}
\dfrac{q_{31}}{q_2}\left(1-e^{-q_2t}\right)-\dfrac{q_{31}}{\left(q_2-q_0\right)}\left(e^{-q_0t}-e^{-q_2t}\right)+\\
\sum\limits_{n=1}^{\infty}\dfrac{q_{48}}{q_0}\left(1-e^{-q_2t}\right)-\dfrac{q_{48}}{\left(q_2-q_0\right)}\left(e^{-q_ot}-e^{-q_2t}\right)+q_{49}\left(\dfrac{te^{-q_0t}}{\left(q_2-q_0\right)}-\dfrac{\left(e^{-q_0t}-e^{-q_2t}\right)}{\left(q_2-q_0\right)^2}\right)+\\
\dfrac{q_{50}}{\left(q_2-q_8\right)}\left(e^{-q_8t}-e^{-q_2t}\right)-\dfrac{q_{50}}{\left(q_2-q_0\right)}\left(e^{-q_0t}-e^{-q_2t}\right)+\dfrac{q_{51}}{\left(q_2-\varepsilon\right)}\left(e^{-\varepsilon t}-e^{-q_2t}\right)-\\
\dfrac{q_{51}}{\left(q_2-q_0\right)}\left(e^{-q_ot}-e^{-q_2t}\right)+
\end{array}\right)
\end{array}\right)\\
\qquad\sum\limits_{n=1}^{\infty}\sum\limits_{n=1}^{\infty}\left(\dfrac{q_{39}}{q_2}\left(1-e^{-q_2t}\right)-\dfrac{q_{39}}{\left(q_2-q_0\right)}\left(e^{-q_0t}-e^{-q_2t}\right)+\dfrac{q_{40}}{\left(q_2-q_8\right)}\left(e^{-q_8t}-e^{-q_2t}\right)\right)-\dfrac{q_{40}}{\left(q_2-q_0\right)}\left(e^{-q_0t}-e^{-q_2t}\right)-\\
\qquad\dfrac{q_{41}}{\left(q_2-\varepsilon\right)}\left(e^{-\varepsilon t}-e^{-q_2t}\right)-\dfrac{q_{41}}{\left(q_2-q_0\right)}\left(e^{-q_0t}-e^{-q_2t}\right)+\dfrac{q_{42}}{\left(q_2-2q_8\right)}\left(e^{-2q_8t}-e^{-q_2t}\right)-\dfrac{q_{42}}{\left(q_2-q_0\right)}\left(e^{-q_0t}-e^{-q_2t}\right)+\\
\qquad\dfrac{q_{43}}{\left(q_2-\left(q_8+\varepsilon\right)\right)}\left(e^{-\left(q_8+\varepsilon\right)t}-e^{-q_2t}\right)-\dfrac{q_{43}}{\left(q_2-q_0\right)}\left(e^{-q_0t}-e^{-q_2t}\right)+\dfrac{q_{44}}{\left(q_2-2\varepsilon\right)}\left(e^{-2\varepsilon t}-e^{-q_2t}\right)-\dfrac{q_{44}}{\left(q_2-q_0\right)}\left(e^{-q_0t}-e^{-q_2t}\right)\\
\sum\limits_{n=1}^{\infty}\left(\left(\dfrac{q_{57}}{q_2}\left(1-e^{-q_2t}\right)+q_{58}te^{-q_2t}-\sum\limits_{n=1}^{\infty}\dfrac{q_{59}}{q_2}\left(1-e^{-q_2t}\right)-q_{60}te^{-q_2t}+\dfrac{q_{61}}{\left(q_2-q_0\right)}\left(e^{-q_0t}-e^{-q_2t}\right)\right)\right)+\left(\dfrac{q_{62}}{q_2}\left(1-e^{-q_2t}\right)+\dfrac{q_{63}}{\left(q_2-q_0\right)}\left(e^{-q_0t}-e^{-q_2t}\right)\right)-\\
\sum\limits_{n=1}^{\infty}\left(\dfrac{q_{64}}{q_2}\left(1-e^{-q_2t}\right)+q_{65}te^{-q_2t}-\sum\limits_{n=1}^{\infty}\left(\dfrac{q_{66}}{q_2}\left(1-e^{-q_2t}\right)-q_{67}te^{-q_2t}+\dfrac{q_{68}}{\left(q_2-q_0\right)}\left(e^{-q_0t}-e^{-q_2t}\right)\right)\right)+\\
\sum\limits_{n=1}^{\infty}\sum\limits_{n=1}^{\infty}\left(\begin{array}{l}
\dfrac{q_{69}}{q_2}\left(1-e^{-q_2t}\right)+q_{70}te^{-q_2t}-\sum\limits_{n=1}^{\infty}\left(\dfrac{q_{71}}{q_2}\left(1-e^{-q_2t}\right)-q_{72}te^{-q_2t}+\dfrac{q_{73}}{\left(q_2-q_0\right)}\left(e^{-q_0t}-e^{-q_2t}\right)\right)\dfrac{q_{74}}{\left(q_2-q_0\right)}\left(e^{-q_0t}-e^{-q_2t}\right)-\\
\dfrac{q_{75}}{q_0}\left(e^{-\left(q_2+q_0\right)t}-e^{-q_2t}\right)-\sum\limits_{n=1}^{\infty}\left(\dfrac{q_{76}}{\left(q_2-q_0\right)}\left(e^{-q_0t}-e^{-q_2t}\right)+\dfrac{q_{77}}{q_0}\left(e^{-\left(q_2+q_0\right)t}-e^{-q_2t}\right)+\dfrac{q_{78}}{\left(q_2-2q_0\right)}\left(e^{-2q_0t}-e^{-q_2t}\right)\right)
\end{array}\right)
\end{array}\right)
\tag{3.231}
$$

where

$$
q_{45}=\frac{2q_9q_{30}\left(1+\left(n\pi\right)^2-\left(-1\right)^{2n}\right)}{4\left(n\pi\right)^2q_0},q_{46}=\frac{2q_{10}q_{30}\left(1+\left(n\pi\right)^2-\left(-1\right)^{2n}\right)}{4\left(n\pi\right)^2\left(q_0-q_8\right)},q_{47}=\frac{2q_{11}q_{30}\left(1+\left(n\pi\right)^2-\left(-1\right)^{2n}\right)}{4\left(n\pi\right)^2\left(q_0-\varepsilon\right)},
$$

$$
q_{48}=\left(q_{31}+q_{37}-q_{45}+q_{34}\right),q_{49}=\left(q_{33}+q_{38}\right),q_{50}=\left(q_{35}-q_{46}\right),q_{51}=\left(q_{36}-q_{47}\right),q_{52}=\left(\frac{c}{4\operatorname{Re}Sc}-\frac{e}{2}\right),
$$

$$
q_{53}=\left(\frac{q_{52}\left(1-\left(-1\right)^n\right)}{n\pi}-\frac{c\left(-1\right)^n}{4\operatorname{Re}Scn\pi}\right),q_{54}=\frac{q_1}{2},q_{55}=\frac{b_n-q_1}{2},q_{56}=\frac{T_D\left(n\pi\right)^2}{2},
$$

$$
q_{57}=\left(\frac{1-\left(-1\right)^{2n}}{2}\right)q_3\left(\frac{c}{4\operatorname{Re}Sc}-\frac{e}{2}\right),q_{58}=\left(\frac{1-\left(-1\right)^{2n}}{2}\right)\left(b_{2n}-q_3\right)\left(\frac{c}{4\operatorname{Re}Sc}-\frac{e}{2}\right),
$$

$$
q_{59}=\left(\frac{1-\left(-1\right)^{2n}}{2}\right)q_4\left(\frac{c}{4\operatorname{Re}Sc}-\frac{e}{2}\right),q_{60}=\left(\frac{1-\left(-1\right)^{2n}}{2}\right)q_5\left(\frac{c}{4\operatorname{Re}Sc}-\frac{e}{2}\right),q_{61}=\left(\frac{1-\left(-1\right)^{2n}}{2}\right)q_6\left(\frac{c}{4\operatorname{Re}Sc}-\frac{e}{2}\right),
$$

$$
q_{62}=\left(\frac{1-\left(-1\right)^{2n}}{2}\right)q_1\left(\frac{c}{4\operatorname{Re}Sc}\right),q_{63}=\left(\frac{1-\left(-1\right)^{2n}}{2}\right)\left(b_n-q_1\right)\left(\frac{c}{4\operatorname{Re}Sc}\right),q_{64}=\frac{cq_3\left(1-2\left(-1\right)^{2n}\right)}{16\operatorname{Re}Sc},
$$

$$
q_{65}=\frac{c\left(b_{2n}-q_3\right)\left(1-2\left(-1\right)^{2n}\right)}{16\operatorname{Re}Sc},q_{66}=\frac{cq_4\left(1-2\left(-1\right)^{2n}\right)}{16\operatorname{Re}Sc},q_{67}=\frac{cq_5\left(1-2\left(-1\right)^{2n}\right)}{16\operatorname{Re}Sc},q_{68}=\frac{cq_6\left(1-2\left(-1\right)^{2n}\right)}{16\operatorname{Re}Sc},
$$

$$q_{69}=\frac{q_1q_3n\pi\left(1-(-1)^{3n}\right)}{3},q_{70}=\frac{q_1\left(b_{2n}-q_3\right)n\pi\left(1-(-1)^{3n}\right)}{3},q_{71}=\frac{q_1q_4n\pi\left(1-(-1)^{3n}\right)}{3},$$
$$q_{72}=\frac{q_1q_5n\pi\left(1-(-1)^{3n}\right)}{3},q_{73}=\frac{q_1q_6n\pi\left(1-(-1)^{3n}\right)}{3},q_{74}=\frac{q_3\left(b_n-q_3\right)n\pi\left(1-(-1)^{3n}\right)}{3},$$
$$q_{75}=\frac{\left(b_n-q_1\right)\left(b_{2n}-q_3\right)n\pi\left(1-(-1)^{3n}\right)}{3},q_{76}=\frac{q_4\left(b_n-q_1\right)n\pi\left(1-(-1)^{3n}\right)}{3},$$
$$q_{77}=\frac{q_5\left(b_n-q_1\right)n\pi\left(1-(-1)^{3n}\right)}{3},q_{78}=\frac{q_6\left(b_n-q_1\right)n\pi\left(1-(-1)^{3n}\right)}{3}.$$

Hence from equation (3.152) we obtain

$$\phi_1\left(z,t\right)=\sum_{n=1}^{\infty}v_{8n}\left(t\right)\sin n\pi z \tag{3.232}$$

Therefore the solutions to the governing equations for case 1 are given as:

$$\theta(\mathrm{z,t})=\mathrm{z}+\sum_{n=1}^{\infty}\left(q_1+\left(b_n-q_1\right)e^{-q_0t}\right)\sin n\pi z+a\sum_{n=1}^{\infty}v_{6n}\left(t\right)Sinn\pi z \tag{3.233}$$

$$\phi(\mathrm{z,t})=z+\sum_{n=1}^{\infty}\left(q_3\left(1-e^{-q_2t}\right)-\sum_{n=1}^{\infty}\left(q_4-q_5e^{-q_2t}-q_6e^{-q_0t}\right)+b_{2n}e^{-q_2t}\right)s\text{in}\, n\pi z+ \\ a\sum_{n=1}^{\infty}v_{8n}\left(t\right)sinn\pi z \tag{3.234}$$

$$w(\mathrm{z},t)=\sum_{n=1}^{\infty}b_{3n}e^{-q_8t}Sinn\pi z+a\sum_{n=1}^{\infty}v_{7n}\left(t\right)Sinn\pi z \tag{3.235}$$

$$u\left(z,t\right)=z+\sum_{n=1}^{\infty}\left(q_9+q_{10}e^{-q_8t}-q_{11}e^{-\varepsilon t}\right)Sinn\pi z+a\sum_{n=1}^{\infty}v_{5n}\left(t\right)Sinn\pi z \tag{3.236}$$

3.2.6Case 2: When the pressure gradient is a constant: $\frac{\partial p}{\partial x}=\frac{dp}{dx}=\lambda$

Since we are taking pressure gradient to be a constant, then solutions to the governing equations of case 2 are obtained by setting $\varepsilon=0$ in case 1.

In this case, equations (3.50) - (3.53) reduce to

$$\frac{\partial u}{\partial t}+\frac{S}{2}\frac{\partial u}{\partial z}=-\lambda+\frac{c}{4\operatorname{Re}}\frac{\partial}{\partial z}\left((1+\alpha\theta)\frac{\partial u}{\partial z}\right)-\frac{Ha^2}{\operatorname{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)^2+Be^2\right)- \\ \frac{cP}{\operatorname{Re}}(1+\alpha\theta)u+Gr_\theta\theta+Gr_\phi\phi \tag{3.237}$$

$$\frac{\partial w}{\partial t}+\frac{S}{2}\frac{\partial w}{\partial z}=\frac{c}{4\operatorname{Re}}\frac{\partial}{\partial z}\left((1+\alpha\theta)\frac{\partial w}{\partial z}\right)-\frac{Ha^2}{\operatorname{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)w-Beu\right)- \\ \frac{cP}{\operatorname{Re}}\left((1+\alpha\theta)w\right) \tag{3.238}$$

$$\frac{\partial \theta}{\partial t}+\frac{S}{2}\frac{\partial \theta}{\partial z}=\frac{c}{4\operatorname{Re}\Pr}\frac{\partial}{\partial z}\left((1+\alpha\theta)\frac{\partial \theta}{\partial z}\right)+\frac{cEc}{4\operatorname{Re}}(1+\alpha\theta)\left(\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right)+ \\ \frac{EcHa^2}{\operatorname{Re}\left((1+BiBe)^2+Be^2\right)}\left(u^2+w^2\right)-Ra^2\theta \tag{3.239}$$

$$\frac{\partial \phi}{\partial t}+\frac{S}{2}\frac{\partial \phi}{\partial z}=\frac{c}{4Sc\operatorname{Re}}\frac{\partial}{\partial z}\left((1+\alpha\theta)\frac{\partial \phi}{\partial z}\right)+\frac{T_D}{4}\frac{\partial^2\theta}{\partial z^2}-K_r\phi \tag{3.240}$$

3.2.7 Case 3: Steady State Reaction: $\frac{\partial^*}{\partial t}=0,\ *=\{u,w,\theta,\phi\}$

In this case, equations (3.50) - (3.53) reduce to;

$$\frac{c}{4\operatorname{Re}}\frac{d}{dz}\left((1+\alpha\theta)\frac{du}{dz}\right)-\frac{S}{2}\frac{du}{dz}-\lambda-\frac{Ha^2}{\operatorname{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)u+Bew\right)- \\ \frac{cP}{\operatorname{Re}}(1+\alpha\theta)u+Gr_\theta\theta+Gr_\phi\phi=0 \tag{3.241}$$

$$\frac{c}{4\operatorname{Re}}\frac{d}{dz}\left((1+\alpha\theta)\frac{dw}{dz}\right)-\frac{S}{2}\frac{dw}{dz}-\frac{Ha^2}{\operatorname{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)w-Beu\right)- \\ \frac{cP}{\operatorname{Re}}(1+\alpha\theta)w=0 \tag{3.242}$$

$$\frac{c}{4\operatorname{Re}\Pr}\frac{d}{dz}\left((1+\alpha\theta)\frac{d\theta}{dz}\right)-\frac{S}{2}\frac{d\theta}{dz}+\frac{cEc}{4\operatorname{Re}}(1+\alpha\theta)\left(\left(\frac{du}{dz}\right)^2+\left(\frac{dw}{dz}\right)^2\right)+ \frac{EcHa^2}{\operatorname{Re}\left((1+BiBe)^2+Be^2\right)}\left(u^2+w^2\right)-Ra^2\theta=0 \tag{3.243}$$

$$\frac{c}{4Sc\operatorname{Re}}\frac{d}{dz}\left((1+\alpha\theta)\frac{d\phi}{dz}\right)-\frac{S}{2}\frac{d\phi}{dz}+\frac{T_D}{4}\frac{d^2\theta}{dz^2}-Kr\phi=0 \tag{3.244}$$

Next, we shall examine the properties of solution of the steady state.

3.2.7.1 Properties of solution of steady state reaction

Here, we consider equations (3.241) – (3.244) when μ is constant i.e when $\alpha=0$ and $B_i=B_e=1$. Then equations (3.241) – (3.244) reduce to

$$\frac{c}{4\operatorname{Re}}\frac{d^2u}{dz^2}-\frac{S}{2}\frac{du}{dz}-\lambda-\frac{Ha^2}{5\operatorname{Re}}(2u+w)-\frac{cP}{\operatorname{Re}}u+Gr_\theta\theta+Gr_\phi\phi=0 \tag{3.245}$$

$$\frac{c}{4\operatorname{Re}}\frac{d^2w}{dz^2}-\frac{S}{2}\frac{dw}{dz}-\frac{Ha^2}{5R\mathrm{e}}(2w-u)-\frac{cP}{\operatorname{Re}}w=0 \tag{3.246}$$

$$\frac{c}{4\operatorname{Re}\Pr}\frac{d^2\theta}{dz^2}-\frac{S}{2}\frac{d\theta}{dz}+\frac{cEc}{4\operatorname{Re}}\left(\left(\frac{du}{dz}\right)^2+\left(\frac{dw}{dz}\right)^2\right)+\frac{EcHa^2}{5\operatorname{Re}}\left(u^2+w^2\right)-Ra^2\theta=0 \tag{3.247}$$

$$\frac{c}{4Sc\operatorname{Re}}\frac{d^2\phi}{dz^2}-\frac{S}{2}\frac{d\phi}{dz}+\frac{T_D}{4}\frac{d^2\theta}{dz^2}-K_r\phi=0 \tag{3.248}$$

Theorem 3.4: Let $Ec\to 0,\ Ha^2\to 0,\ \lambda\to 0,\ S>0,\ c>0,\ \operatorname{Re}>0$ $P>0,\ \Pr>0,\ Ra^2>0\ Gr_\theta>0,\ Gr_\phi>0,\ Sc>0,\ k_r>0,\ T_D>0$. Then the equations (3.245) – (3.248) have a solution.

Proof of Theorem 3.4:

Equations (3.245) – (3.248) can be written respectively as:

$$Lu = f(z,u), \qquad Lw = f(z,w), \qquad L\theta = f(z,\theta), \qquad L\phi = f(z,\phi)$$

where

$$Lu = \frac{c}{4\,\mathrm{Re}}\frac{d^2u}{dz^2} - \frac{S}{2}\frac{du}{dz} - \frac{cP}{\mathrm{Re}}u \tag{3.249}$$

$$f(z,u) = -Gr_\theta \theta - Gr_\phi \phi$$

$$Lw = \frac{c}{4\,\mathrm{Re}}\frac{d^2w}{dz^2} - \frac{S}{2}\frac{dw}{dz} - \frac{cP}{\mathrm{Re}}w \tag{3.250}$$

$$f(z,w) = 0$$

$$L\theta = \frac{c}{4\,\mathrm{Re}\,\mathrm{Pr}}\frac{d^2\theta}{dz^2} - \frac{S}{2}\frac{d\theta}{dz} - Ra^2\theta \tag{3.251}$$

$$f(z,\theta) = 0$$

$$L\phi = \frac{c}{4Sc\,\mathrm{Re}}\frac{d^2\phi}{dz^2} - \frac{S}{2}\frac{d\phi}{dz} - K_r\phi \tag{3.252}$$

$$f(z,\phi) = -\frac{T_D}{4}\frac{\partial^2\theta}{\partial z^2}$$

Let

$$\underline{u}(z) = 0, \qquad \underline{w}(z) = 0, \qquad \underline{\theta}(z) = 0, \qquad \underline{\phi}(z) = 0 \tag{3.253}$$

We shall show that (3.253) are the lower solutions of equations (3.249 – 2.252) respectively.

Clearly,

$$\left.\begin{array}{ll} \underline{u}(0)=0, & \underline{u}(1)=0 \\ \underline{w}(0)=0, & \underline{w}(1)=0 \\ \underline{\theta}(0)=0, & \underline{\theta}(1)=0 \\ \underline{\phi}(0)=0, & \underline{\phi}(1)=0 \end{array}\right\} \tag{3.254}$$

Now

$$\frac{d\underline{u}}{dz}=\frac{d^2\underline{u}}{dz^2}=\underline{u}=0 \tag{3.255}$$

$$\frac{d\underline{w}}{dz}=\frac{d^2\underline{w}}{dz^2}=\underline{w}=0 \tag{3.256}$$

$$\frac{d\underline{\theta}}{dz}=\frac{d^2\underline{\theta}}{dz^2}=\underline{\theta}=0 \tag{3.257}$$

$$\frac{d\underline{\phi}}{dz}=\frac{d^2\underline{\phi}}{dz^2}=\underline{\phi}=0 \tag{3.258}$$

This imply

$$L\underline{u}=0, \qquad L\underline{w}=0, \qquad L\underline{\theta}=0, \qquad L\underline{\phi}=0 \tag{3.259}$$

$$f\left(z,\underline{u}\right)=-Gr_{\theta}\underline{\theta}-Gr_{\varphi}\underline{\phi}, \quad f\left(z,\underline{w}\right)=0,\ f\left(z,\underline{\theta}\right)=0,\ f\left(z,\underline{\phi}\right)=-\frac{T_D}{4}\frac{\partial^2\underline{\theta}}{\partial z^2} \tag{3.260}$$

Hence

$$L\underline{u}\geq f\left(z,\underline{u}\right), \qquad L\underline{w}\geq f\left(z,\underline{w}\right), \qquad L\underline{\theta}\geq f\left(z,\underline{\theta}\right), \qquad L\underline{\phi}\geq f\left(z,\underline{\phi}\right) \tag{3.261}$$

By definition 3, $\underline{u}(z)=0,\ \underline{w}(z)=0, \underline{\theta}(z)=0, \phi(z)=0$ are the lower solutions of equations (3.249 – 3.252) respectively.

Also consider

$$\bar{u}(z)=1+\left(Gr_{\theta}+Gr_{\varphi}\right)z, \quad \bar{w}(z)=1+z, \quad \bar{\theta}(z)=1+\frac{cP}{\text{Re}}z, \quad \bar{\varphi}(z)=1+\frac{cP}{\text{Re}}z \quad (3.262)$$

We shall show that (3.262) are the upper solutions of equations (3.249 – 3.252) respectively.

Clearly,

$$\left.\begin{array}{ll} \bar{u}(0)=1, & \bar{u}(1)=1+\left(Gr_{\theta}+Gr_{\varphi}\right) \\ \bar{w}(0)=1, & \bar{w}(1)=2 \\ \bar{\theta}(0)=1, & \bar{\theta}(1)=1+\dfrac{cP}{\text{Re}} \\ \bar{\varphi}(0)=1, & \bar{\varphi}(1)=1+\dfrac{cP}{\text{Re}} \end{array}\right\} \quad (3.263)$$

Now

$$\frac{d\bar{u}}{dz}=\left(Gr_{\theta}+Gr_{\phi}\right), \quad \frac{d^{2}\bar{u}}{dz^{2}}=0, \ \bar{u}=1+\left(Gr_{\theta}+Gr_{\phi}\right)z \quad (3.264)$$

$$\frac{d\bar{w}}{dz}=1, \frac{d^{2}\bar{w}}{dz^{2}}=0, \bar{w}=1+z \quad (3.265)$$

$$\frac{d\bar{\theta}}{dz}=\frac{Pc}{\text{Re}}, \ \frac{d^{2}\bar{\theta}}{dz^{2}}=0, \bar{\theta}=1+\frac{cP}{\text{Re}}z \quad (3.266)$$

$$\frac{d\bar{\phi}}{dz}=\frac{Pc}{\text{Re}}, \ \frac{d^{2}\bar{\phi}}{dz^{2}}=0, \ \bar{\phi}=1+\frac{cP}{\text{Re}}z \quad (3.267)$$

These imply

$$\begin{array}{ll} L\bar{u}=-\dfrac{S}{2}\left(Gr_{\theta}+Gr_{\phi}\right)-\dfrac{cP}{\text{Re}}\bar{u}, & L\bar{w}=-\dfrac{S}{2}-\dfrac{cP}{\text{Re}}\bar{w}, \\ L\bar{\theta}=-\dfrac{S}{2}\dfrac{cP}{\text{Re}}-Ra^{2}\bar{\theta}, & L\bar{\varphi}=-\dfrac{S}{2}\dfrac{cP}{\text{Re}}-K_{r}\bar{\phi} \end{array} \quad (3.268)$$

$$f\left(z,\bar{u}\right)=-Gr_\theta\bar{\theta}-Gr_\phi\bar{\phi},\ f\left(z,\bar{w}\right)=0,\ f\left(z,\bar{\theta}\right)=0,\ f\left(z,\bar{\phi}\right)=-\frac{T_D}{4}\frac{\partial^2\bar{\theta}}{\partial z^2} \tag{3.269}$$

Hence

$$L\underline{u}\le f\left(z,\bar{u}\right),L\underline{w}\le f\left(z,\bar{w}\right),L\underline{\theta}\le f\left(z,\bar{\theta}\right),L\underline{\varphi}\le f\left(z,\bar{\phi}\right) \tag{3.270}$$

By definition 4,

$\bar{u}(z)=1+\left(Gr_\theta+Gr_\phi\right)z,\bar{w}(z)=1+z,\bar{\theta}(z)=1+\frac{cP}{\text{Re}}z,\bar{\phi}(z)=1+\frac{cP}{\text{Re}}z$ are the upper solutions to equations (3.249 – 3.252) respectively.

Hence, there exists a solution of problem (3.241) – (3.244). This completes the proof.

3.2.7.2 Solution of Case 3

Here, we solve equations (3.241) – (3.244) satisfies (3.54) using perturbation method.

Let $0<\alpha<<1$ such that $K_r=g_1\alpha, Ra^2=b_1\alpha, Ha^2=d\alpha, P=e_1\alpha, Gr_\theta=g\alpha, Gr_\phi=l\alpha$ and suppose the solution of equations (3.241) – (3.244) satisfies (3.54) can be expressed as

$$\left.\begin{aligned}u&=u_0+\alpha u_1+...\\w&=w_0+\alpha w_1+...\\\theta&=\theta_0+\alpha\theta_1+...\\\phi&=\phi_0+\alpha\phi_1+...\end{aligned}\right\} \tag{3.271}$$

Processing and collecting like powers of α we have for:

α^0 :

$$\left.\begin{aligned}&\frac{c}{4\text{Re}}\frac{d^2u_0}{dz^2}-\frac{S}{2}\frac{du_0}{dz}-\lambda=0\\&u_0(0)=0,u_0(1)=1\end{aligned}\right\} \tag{3.272}$$

$$\left.\begin{aligned}&\frac{c}{4\,\mathrm{Re}}\frac{d^2w_0}{dz^2}-\frac{S}{2}\frac{dw_0}{dz}=0\\&w_0(0)=0,w_0(1)=0\end{aligned}\right\} \tag{3.273}$$

$$\left.\begin{aligned}&\frac{c}{4\,\mathrm{Re}\,\mathrm{Pr}}\frac{d^2\theta_0}{dz^2}-\frac{S}{2}\frac{d\theta_0}{dz}+\frac{cEc}{4\,\mathrm{Re}}\left(\left(\frac{du_0}{dz}\right)^2+\left(\frac{dw_0}{dz}\right)^2\right)=0\\&\theta_0(0)=0,\theta_0(1)=1\end{aligned}\right\} \tag{3.274}$$

$$\left.\begin{aligned}&\frac{c}{4\,\mathrm{Re}\,Sc}\frac{d^2\phi_0}{dz^2}-\frac{S}{2}\frac{d\phi_0}{dz}+\frac{T_D}{4}\frac{d^2\theta_0}{dz^2}=0\\&\phi_0(0)=0,\phi_0(1)=1\end{aligned}\right\} \tag{3.275}$$

α^1:

$$\left.\begin{aligned}&\frac{c}{4\,\mathrm{Re}}\left(\frac{d^2u_1}{dz^2}+\theta_0\frac{d^2u_0}{dz^2}+\frac{d\theta_0}{dz}\frac{du_0}{dz}\right)-\frac{S}{2}\frac{du_1}{dz}-\\&\frac{d}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)u_0+Bew_0\right)-\frac{e_1c}{\mathrm{Re}}u_0+g\theta_0+l\phi_0=0\\&u_0(0)=0,u_1(1)=1\end{aligned}\right\} \tag{3.276}$$

$$\left.\begin{aligned}&\frac{c}{4\,\mathrm{Re}}\left(\frac{d^2w_1}{dz^2}+\theta_0\frac{d^2w_0}{dz^2}+\frac{d\theta_0}{dz}\frac{dw_0}{dz}\right)-\frac{S}{2}\frac{dw_1}{dz}-\\&\frac{d}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left((1+BiBe)w_0-Beu_0\right)-\frac{e_1c}{\mathrm{Re}}w_0=0\\&w_0(0)=0,w_1(1)=0\end{aligned}\right\} \tag{3.277}$$

$$\left.\begin{aligned}&\frac{c}{4\,\mathrm{Re}\,\mathrm{Pr}}\left(\frac{d^2\theta_1}{dz^2}+\theta_0\frac{d^2\theta_0}{dz^2}+\left(\frac{d\theta_0}{dz}\right)^2\right)-\frac{S}{2}\frac{d\theta_1}{dz}-+\frac{2cEc}{4\,\mathrm{Re}}\left(\frac{du_0}{dz}\frac{du_1}{dz}+\frac{dw_0}{dz}\frac{dw_1}{dz}\right)+\\&\frac{cEc}{4\,\mathrm{Re}}\theta_0\left(\left(\frac{du_0}{dz}\right)^2+\left(\frac{dw_0}{dz}\right)^2\right)+\frac{dEc}{\mathrm{Re}\left((1+BiBe)^2+Be^2\right)}\left(u_0^2+w_0^2\right)-b_1\theta_0=0\\&\theta_0(0)=0,\theta_1(1)=0\end{aligned}\right\} \tag{3.278}$$

$$\left.\begin{aligned}&\frac{c}{4\,\mathrm{Re}}\left(\frac{d^2\phi_1}{dz^2}+\theta_0\frac{d^2\phi_0}{dz^2}+\frac{d\theta_0}{dz}\frac{d\phi_0}{dz}\right)-\frac{S}{2}\frac{d\phi_1}{dz}+\frac{T_D}{4}\frac{d^2\theta_1}{dz^2}-g_1\phi_0=0\\&\phi_0(0)=0,\phi_1(1)=0\end{aligned}\right\}\tag{3.279}$$

Consider equation (3.272),

$$\left.\begin{aligned}&\frac{d^2u_0}{dz^2}-\beta\frac{du_0}{dz}=\gamma,\\&u_0(0)=0,u_0(1)=1\end{aligned}\right\}$$

where

$$\beta=\frac{2\,\mathrm{Re}\,S}{c},\gamma=\frac{4\,\mathrm{Re}\,\sigma}{c}$$

Let

$$q=\frac{du_0}{dz}\tag{3.280}$$

then

$$\frac{dq}{dz}-\beta q=\gamma\tag{3.281}$$

So

$$q(z)=e^{\beta z}\int_0^z\gamma e^{-\beta x}dx+c_1e^{\beta z}=e^{\beta z}-\frac{\gamma}{\beta}e^{-\beta x}\bigg|_0^z+c_1e^{\beta z}=\frac{\gamma}{\beta}\left(e^{\beta z}-1\right)+c_1e^{\beta z}$$

$$q(z)=\left(\frac{\gamma}{\beta}+c_1\right)e^{\beta z}-\frac{\gamma}{\beta}\tag{3.282}$$

i.e

$$\frac{du_0}{dz} = \left(\frac{\gamma}{\beta} + c_1\right)e^{\beta z} - \frac{\gamma}{\beta}$$

$$u_0 = \frac{1}{\beta}\left(\frac{\gamma}{\beta} + c_1\right)e^{\beta z} - \frac{\gamma}{\beta}z + c_2 \tag{3.283}$$

$$u_0(0) = 0 \Rightarrow c_2 = -\frac{1}{\beta}\left(\frac{\gamma}{\beta} + c_1\right)$$

$$u_1(1) = 1 \Rightarrow \frac{1}{\beta}\left(\frac{\gamma}{\beta} + c_1\right)\left(e^{\beta} - 1\right) - \frac{\gamma}{\beta} = 1$$

i.e

$$\left(\frac{\gamma}{\beta} + c_1\right) = \frac{\beta\left(\frac{\gamma}{\beta} + 1\right)}{\left(e^{\beta} - 1\right)}$$

$$c_1 = \frac{\beta\left(\frac{\gamma}{\beta} + 1\right)}{\left(e^{\beta} - 1\right)} - \frac{\gamma}{\beta} \tag{3.284}$$

$$c_2 = -\frac{\left(\frac{\gamma}{\beta} + 1\right)}{\left(e^{\beta} - 1\right)} \tag{3.285}$$

Then

$$u_0(z) = \frac{\left(\frac{\gamma}{\beta} + 1\right)}{\left(e^{\beta} - 1\right)}e^{\beta z} - \frac{\gamma}{\beta}z - \frac{\left(\frac{\gamma}{\beta} + 1\right)}{\left(e^{\beta} - 1\right)}$$

i.e

$$u_0(z) = A\left(e^{\beta z} - 1\right) - Bz \tag{3.286}$$

where

$$A = \frac{\left(\frac{\gamma}{\beta} + 1\right)}{\left(e^{\beta} - 1\right)}, B = \frac{\gamma}{\beta}$$

Also consider equation (3.273)

$$\left.\begin{aligned} &\frac{d^2 w_0}{dz^2} - \beta \frac{dw_0}{dz} = 0 \\ &w_0(0) = 0, w_0(1) = 0 \end{aligned}\right\}$$

Let

$$q_1 = \frac{dw_0}{dz} \tag{3.287}$$

Then,

$$\frac{dq_1}{dz} - \beta q_1 = 0 \tag{3.288}$$

So

$$q_1(z) = e^{\beta z} \int_0^z 0 \cdot e^{-\beta x} dx + c_3 e^{\beta z}$$

$$= e^{\beta z} \cdot 0 + c_3 e^{\beta z} = c_3 e^{\beta z} \tag{3.289}$$

That is

$$\frac{dw_0}{dz} = c_3 e^{\beta z}$$

$$w_0(z) = \frac{c_3}{\beta} e^{\beta z} + c_4 \tag{3.290}$$

$$w_0(0) = \frac{c_3}{\beta} + c_4 = 0 \Rightarrow c_4 = -\frac{c_3}{\beta}$$

$$w_0(1) = \frac{c_3}{\beta} e^{\beta} + c_4 = 0 \Rightarrow \frac{c_3}{\beta}\left(e^{\beta} - 1\right) = 0$$

$$\Rightarrow c_3 = 0 \, and \, c_4 = 0$$

Therefore,

$$w_0(z) = 0 \tag{3.291}$$

Also consider equation (3.274) given as

$$\left.\begin{array}{l} \dfrac{d^2\theta_0}{dz^2} - \beta_1 \dfrac{d\theta_0}{dz} = \gamma_1 e^{2\beta z} + \gamma_2 e^{\beta z} + \gamma_3 \\ \theta_0(0) = 0, \theta_0(1) = 1 \end{array}\right\} \tag{3.292}$$

where

$$\beta_1 = \frac{2\,\mathrm{Re}\,\mathrm{Pr}\,S}{c}, \gamma_1 = -Ec\,\mathrm{Pr}\,A^2B^2, \gamma_2 = 2Ec\,\mathrm{Pr}\,AB\beta, \gamma_3 = -Ec\,\mathrm{Pr}\,B^2$$

Let

$$q_2 = \frac{d\theta_0}{dz} \tag{3.293}$$

Then

$$\frac{dq_2}{dz} - \beta_1 q_2 = \gamma_1 e^{2\beta z} + \gamma_2 e^{\beta z} + \gamma_3 \tag{3.294}$$

So

$$q_2(z) = e^{\beta_1 z}\int_0^z e^{-\beta_1 x}\left(\gamma_1 e^{2\beta x} + \gamma_2 e^{\beta x} + \gamma_3\right)dx + c_5 e^{\beta_1 z}$$

$$= e^{\beta_1 z}\cdot\left(\frac{\gamma_1}{(2\beta-\beta_1)}e^{(2\beta-\beta_1)x}\Big|_0^z + \frac{\gamma_2}{(\beta-\beta_1)}e^{(\beta-\beta_1)x}\Big|_0^z - \frac{\gamma_3}{\beta_1}e^{(-\beta_1)x}\Big|_0^z\right) + c_5 e^{\beta_1 z}$$

$$= \frac{\gamma_1}{(2\beta-\beta_1)}\left(e^{2\beta z} - e^{\beta_1 z}\right) + \frac{\gamma_2}{(\beta-\beta_1)}\left(e^{\beta z} - e^{\beta_1 z}\right) + \frac{\gamma_3}{\beta_1}\left(e^{\beta_1 z} - 1\right) + c_5 e^{\beta_1 z}$$

$$q_2(z) = \frac{\gamma_1}{(2\beta-\beta_1)}e^{2\beta z} + \frac{\gamma_2}{(\beta-\beta_1)}e^{\beta z} + \left(\frac{\gamma_3}{\beta_1} - \frac{\gamma_1}{(2\beta-\beta_1)} - \frac{\gamma_2}{(\beta-\beta_1)} + c_5\right)e^{\beta_1 z} - \frac{\gamma_3}{\beta_1} \quad (3.295)$$

i.e

$$\frac{d\theta_0}{dz} = \frac{\gamma_1}{(2\beta-\beta_1)}e^{2\beta z} + \frac{\gamma_2}{(\beta-\beta_1)}e^{\beta z} + \left(\frac{\gamma_3}{\beta_1} - \frac{\gamma_1}{(2\beta-\beta_1)} - \frac{\gamma_2}{(\beta-\beta_1)} + c_5\right)e^{\beta_1 z} - \frac{\gamma_3}{\beta_1}$$

$$\left.\begin{aligned}&\theta_0(z) = \frac{\gamma_1}{2\beta(2\beta-\beta_1)}e^{2\beta z} + \frac{\gamma_2}{\beta(\beta-\beta_1)}e^{\beta z} + \\ &\left(\frac{\gamma_3}{\beta_1} - \frac{\gamma_1}{(2\beta-\beta_1)} - \frac{\gamma_2}{(\beta-\beta_1)} + c_5\right)\frac{1}{\beta_1}e^{\beta_1 z} - \frac{\gamma_3}{\beta_1}z + c_6\end{aligned}\right\} \quad (3.296)$$

$$\theta_0(0) = 0 \Rightarrow c_6 = \left(\frac{\gamma_1}{\beta_1(2\beta-\beta_1)} + \frac{\gamma_2}{\beta_1(\beta-\beta_1)} - \frac{\gamma_3}{\beta_1} - \frac{\gamma_1}{2\beta(2\beta-\beta_1)} - \frac{\gamma_2}{\beta(\beta-\beta_1)}\right) - \frac{c_5}{\beta_1}$$

$$\theta_0(1) = 1 \Rightarrow \frac{\gamma_1}{2\beta(2\beta-\beta_1)}e^{2\beta} + \frac{\gamma_2}{\beta(\beta-\beta_1)}e^{\beta} + \left(\frac{\gamma_3}{\beta_1} - \frac{\gamma_1}{(2\beta-\beta_1)} - \frac{\gamma_2}{(\beta-\beta_1)} + c_5\right)\frac{1}{\beta_1}e^{\beta_1} - \frac{\gamma_3}{\beta_1} +$$
$$\left(\frac{\gamma_1}{\beta_1(2\beta-\beta_1)} + \frac{\gamma_2}{\beta_1(\beta-\beta_1)} - \frac{\gamma_3}{\beta_1} - \frac{\gamma_1}{2\beta(2\beta-\beta_1)} - \frac{\gamma_2}{\beta(\beta-\beta_1)}\right) - \frac{c_5}{\beta_1} = 1$$

That is

$$c_5\left(\frac{e^{\beta_1}-1}{\beta_1}\right)=1+\frac{\gamma_3}{\beta_1}-\frac{\gamma_1\left(e^{2\beta}-1\right)}{2\beta\left(2\beta-\beta_1\right)}-\frac{\gamma_2\left(e^{\beta}-1\right)}{\beta\left(\beta-\beta_1\right)}+\frac{\gamma_1\left(e^{\beta}-1\right)}{\beta_1\left(2\beta-\beta_1\right)}+\frac{\gamma_2\left(e^{\beta_1}-1\right)}{\beta\left(\beta-\beta_1\right)}$$

$$\Rightarrow c_5=\frac{\beta_1\left(1+\frac{\gamma_3}{\beta_1}-\frac{\gamma_1\left(e^{2\beta}-1\right)}{2\beta\left(2\beta-\beta_1\right)}-\frac{\gamma_2\left(e^{\beta}-1\right)}{\beta\left(\beta-\beta_1\right)}+\frac{\gamma_1\left(e^{\beta}-1\right)}{\beta_1\left(2\beta-\beta_1\right)}+\frac{\gamma_2\left(e^{\beta_1}-1\right)}{\beta_1\left(\beta-\beta_1\right)}\right)}{\left(e^{\beta_1}-1\right)} \tag{3.297}$$

And

$$c_6=\left(\frac{\gamma_1}{\beta_1\left(2\beta-\beta_1\right)}+\frac{\gamma_2}{\beta_1\left(\beta-\beta_1\right)}-\frac{\gamma_3}{\beta_1^2}-\frac{\gamma_1}{2\beta\left(2\beta-\beta_1\right)}-\frac{\gamma_2}{\beta\left(\beta-\beta_1\right)}\right)-\frac{c_5}{\beta_1} \tag{3.298}$$

Therefore,

$$\theta_0(z)=A_1e^{2\beta z}+A_2e^{\beta z}+A_3e^{\beta_1 z}-B_1z+c_6 \tag{3.299}$$

where

$$A_1=\frac{\gamma_1}{2\beta\left(2\beta-\beta_1\right)},A_2=\frac{\gamma_2}{\beta\left(\beta-\beta_1\right)},A_3=\frac{1}{\beta_1}\left(\frac{\gamma_3}{\beta_1}-\frac{\gamma_1}{\left(2\beta-\beta_1\right)}-\frac{\gamma_2}{\left(\beta-\beta_1\right)}+c_5\right),B_1=\frac{\gamma_3}{\beta_1}$$

Also consider (3.275) given as

$$\left.\begin{array}{l}\frac{d^2\phi_0}{dz^2}-\beta_2\frac{d\phi_0}{dz}=\gamma_4e^{2\beta z}+\gamma_5e^{\beta z}+\gamma_6e^{\beta_1 z}\\ \phi_0\left(0\right)=0,\phi_0\left(1\right)=1\end{array}\right\}$$

where

$$\beta_2=\frac{2\operatorname{Re}ScS}{c},\gamma_4=-\frac{4\operatorname{Re}ScT_D\beta^3A_1}{c},\gamma_5=-\frac{\operatorname{Re}ScT_D\beta^2A_2}{c},\gamma_6=-\frac{\operatorname{Re}ScT_D\beta_1^2A_3}{c}$$

Let

$$q_3 = \frac{d\phi_0}{dz} \tag{3.300}$$

Then

$$\frac{dq_3}{dz} - \beta_2 q_3 = \gamma_4 e^{2\beta z} + \gamma_5 e^{\beta z} + \gamma_6 e^{\beta_1 z} \tag{3.301}$$

So

$$q_3(z) = e^{\beta_2 z}\int_0^z \left(\gamma_4 e^{x(2\beta-\beta_2)} + \gamma_5 e^{x(\beta-\beta_2)} + \gamma_6 e^{x(\beta_1-\beta_2)}\right)dx + c_7 e^{\beta_2 z}$$

$$= e^{\beta_2 z}\left(\frac{\gamma_4}{(2\beta-\beta_2)} e^{(2\beta-\beta_2)x}\Big|_0^z + \frac{\gamma_5}{(\beta-\beta_2)} e^{(\beta-\beta_2)x}\Big|_0^z + \frac{\gamma_6}{(\beta_1-\beta_2)} e^{(\beta_1-\beta_2)x}\Big|_0^z\right) + c_7 e^{\beta_2 z}$$

$$= \frac{\gamma_4}{(2\beta-\beta_2)}\left(e^{2\beta z} - e^{\beta_2 z}\right) + \frac{\gamma_5}{(\beta-\beta_2)}\left(e^{\beta z} - e^{\beta_2 z}\right) + \frac{\gamma_6}{(\beta_1-\beta_2)}\left(e^{\beta_1 z} - e^{\beta_2 z}\right) + c_7 e^{\beta_2 z}$$

$$q_3(z) = \frac{\gamma_4}{(2\beta-\beta_2)} e^{2\beta z} + \frac{\gamma_5}{(\beta-\beta_2)} e^{\beta z} + \frac{\gamma_6}{(\beta_1-\beta_2)} e^{\beta_1 z} + \left(c_7 - \frac{\gamma_4}{(2\beta-\beta_2)} - \frac{\gamma_5}{(\beta-\beta_2)} - \frac{\gamma_6}{(\beta_1-\beta_2)}\right) e^{\beta_2 z}$$

That is

$$\frac{d\phi_0}{dz} = \frac{\gamma_4}{(2\beta-\beta_2)} e^{2\beta z} + \frac{\gamma_5}{(\beta-\beta_2)} e^{\beta z} + \frac{\gamma_6}{(\beta_1-\beta_2)} e^{\beta_1 z} + \left(c_7 - \frac{\gamma_4}{(2\beta-\beta_2)} - \frac{\gamma_5}{(\beta-\beta_2)} - \frac{\gamma_6}{(\beta_1-\beta_2)}\right) e^{\beta_2 z}$$

$$\begin{aligned}\phi_0(z) = {} & \frac{\gamma_4}{(2\beta-\beta_2)} e^{2\beta z} + \frac{\gamma_5}{(\beta-\beta_2)} e^{\beta z} + \frac{\gamma_6}{(\beta_1-\beta_2)} e^{\beta_1 z} + \\ & \frac{1}{\beta_2}\left(c_7 - \frac{\gamma_4}{(2\beta-\beta_2)} - \frac{\gamma_5}{(\beta-\beta_2)} - \frac{\gamma_6}{(\beta_1-\beta_2)}\right) e^{\beta_2 z} + c_8\end{aligned} \tag{3.302}$$

$$\phi_0(0)=0 \Rightarrow c_8=\frac{1}{\beta_2}\left(\frac{\gamma_4}{\left(2\beta-\beta_2\right)}+\frac{\gamma_5}{\left(\beta-\beta_2\right)}+\frac{\gamma_6}{\left(\beta_1-\beta_2\right)}-c_7\right)-$$

$$\frac{\gamma_4}{2\beta\left(2\beta-\beta_2\right)}-\frac{\gamma_5}{\beta\left(\beta-\beta_2\right)}-\frac{\gamma_6}{\beta_1\left(\beta_1-\beta_2\right)}$$

$$\phi_0(1)=1 \Rightarrow \frac{\gamma_4}{2\beta\left(2\beta-\beta_2\right)}e^{2\beta}+\frac{\gamma_5}{\beta\left(\beta-\beta_2\right)}e^{\beta}+\frac{\gamma_6}{\beta_1\left(\beta_1-\beta_2\right)}e^{\beta_1}+\frac{c_7}{\beta_2}e^{\beta_2}-$$

$$\left(\frac{\gamma_4}{2\beta\left(2\beta-\beta_2\right)}+\frac{\gamma_5}{\beta\left(\beta-\beta_2\right)}+\frac{\gamma_6}{\beta_1\left(\beta_1-\beta_2\right)}\right)e^{\beta_2}+\left(\begin{array}{l}\frac{\gamma_4}{2\beta\left(2\beta-\beta_2\right)}+\\ \frac{\gamma_5}{\beta\left(\beta-\beta_2\right)}+\frac{\gamma_6}{\beta_1\left(\beta_1-\beta_2\right)}\\ -\frac{c_7}{\beta_2}\end{array}\right)-$$

$$\frac{\gamma_4}{2\beta\left(2\beta-\beta_2\right)}-\frac{\gamma_5}{\beta\left(\beta-\beta_2\right)}-\frac{\gamma_6}{\beta_1\left(\beta_1-\beta_2\right)}=1$$

$$c_7\left(\frac{e^{\beta_2}-1}{\beta_2}\right)=1+\left(\frac{\gamma_4}{\beta_2\left(2\beta-\beta_2\right)}+\frac{\gamma_5}{\beta_2\left(\beta-\beta_2\right)}+\frac{\gamma_6}{\beta_2\left(\beta_1-\beta_2\right)}\right)\left(e^{\beta_2}-1\right)-$$

$$\frac{\gamma_4\left(e^{2\beta}-1\right)}{2\beta\left(2\beta-\beta_2\right)}-\frac{\gamma_5\left(e^{\beta}-1\right)}{\beta\left(\beta-\beta_2\right)}-\frac{\gamma_6\left(e^{\beta_1}-1\right)}{\beta_1\left(\beta_1-\beta_2\right)}$$

$\Rightarrow$

$$c_7=\frac{\beta_2\left(\begin{array}{l}1+\left(\frac{\gamma_4}{\beta_2\left(2\beta-\beta_2\right)}+\frac{\gamma_5}{\beta_2\left(\beta-\beta_2\right)}+\frac{\gamma_6}{\beta_2\left(\beta_1-\beta_2\right)}\right)\left(e^{\beta_2}-1\right)-\\ \frac{\gamma_4\left(e^{2\beta}-1\right)}{2\beta\left(2\beta-\beta_2\right)}-\frac{\gamma_5\left(e^{\beta}-1\right)}{\beta\left(\beta-\beta_2\right)}-\frac{\gamma_6\left(e^{\beta_1}-1\right)}{\beta_1\left(\beta_1-\beta_2\right)}\end{array}\right)}{\left(e^{\beta_2}-1\right)} \tag{3.303}$$

and

$$c_8=\frac{1}{\beta_2}\left(\frac{\gamma_4}{\left(2\beta-\beta_2\right)}+\frac{\gamma_5}{\left(\beta-\beta_2\right)}+\frac{\gamma_6}{\left(\beta_1-\beta_2\right)}-c_7\right)-\frac{\gamma_4}{2\beta\left(2\beta-\beta_2\right)}-\frac{\gamma_5}{\beta\left(\beta-\beta_2\right)}-\frac{\gamma_6}{\beta_1\left(\beta_1-\beta_2\right)} \tag{3.304}$$

Therefore,

$$\phi_0(z) = A_4 e^{2\beta z} + A_5 e^{\beta z} + A_6 e^{\beta_1 z} + A_7 e^{\beta_2 z} + c_8 \tag{3.305}$$

where

$$A_4 = \frac{\gamma_4}{2\beta(2\beta-\beta_2)}, A_5 = \frac{\gamma_5}{\beta(\beta-\beta_2)}, A_6 = \frac{\gamma_6}{\beta_1(\beta_1-\beta_2)},$$
$$A_7 = \frac{1}{\beta_2}\left(c_7 - \frac{\gamma_4}{(2\beta-\beta_2)} - \frac{\gamma_5}{(\beta-\beta_2)} - \frac{\gamma_6}{(\beta_1-\beta_2)}\right)$$

–

Also consider equation (3.276),

$$\left.\begin{array}{l} \frac{d^2u_1}{dz^2} - \beta\frac{du_1}{dz} = \gamma_7 e^{\beta z} + \gamma_8 z - \gamma_9 - \gamma_{10}e^{3\beta z} - \gamma_{11}e^{2\beta z} - \gamma_{12}e^{(\beta+\beta_1)z} + \gamma_{13}ze^{\beta z} + \gamma_{14}e^{\beta_1 z} - \gamma_{15}e^{\beta_2 z} \\ u_1(0) = 0, u_1(1) = 0 \end{array}\right\}$$

where

$$\gamma_4 = \left(\begin{array}{l} \frac{4\operatorname{Re}A}{c}\left(\frac{e_1 c}{\operatorname{Re}} + \frac{d(1+BiBe)}{\operatorname{Re}\left[(1+BiBe)^2 + Be^2\right]}\right) - A\beta^2 c_6 + A\beta B_1 + \beta A_2 B - \\ \frac{4\operatorname{Re}gA_2}{c} - \frac{4\operatorname{Re}lA_5}{c} \end{array}\right),$$
$$\gamma_8 = \left(\frac{4\operatorname{Re}gB_1}{c} - \frac{4\operatorname{Re}B}{c}\left(\frac{e_1 c}{\operatorname{Re}} + \frac{d(1+BiBe)}{\operatorname{Re}\left((1+BiBe)^2 + Be^2\right)}\right)\right),$$
$$\gamma_9 = \left(\frac{4\operatorname{Re}A}{c}\left(\frac{e_1 c}{\operatorname{Re}} + \frac{d(1+BiBe)}{\operatorname{Re}\left[(1+BiBe)^2 + Be^2\right]}\right) + BB_1 + \frac{4\operatorname{Re}gc_6}{c} + \frac{4\operatorname{Re}lc_8}{c}\right),$$
$$\gamma_{10} = A\beta^2 A_1, \gamma_{11} = \left(A\beta^2 A_2 + \frac{4\operatorname{Re}gA_1}{c} + \frac{4\operatorname{Re}lA_4}{c} - 2\beta A_1 B\right),$$
$$\gamma_{12} = A\beta A_3(\beta+\beta_1), \gamma_{13} = A\beta^2 B_1,$$
$$\gamma_{14} = \left(\beta_1 A_3 B - \frac{4\operatorname{Re}gA_3}{c} - \frac{4\operatorname{Re}lA_6}{c}\right), \gamma_{15} = \frac{4\operatorname{Re}lA_7}{c}.$$

Let

$$q_4 = \frac{du_1}{dz} \tag{3.307}$$

Then

$$\frac{dq_4}{dz} - \beta q_4 = \gamma_7 e^{\beta z} + \gamma_8 z - \gamma_9 - \gamma_{10} e^{3\beta z} - \gamma_{11} e^{2\beta z} - \gamma_{12} e^{(\beta+\beta_1)z} + \gamma_{13} z e^{\beta z} + \gamma_{14} e^{\beta_1 z} - \gamma_{15} e^{\beta_2 z} \tag{3.308}$$

So

$$q_4(z) = e^{\beta z} \int_1^z \begin{pmatrix} \gamma_7 + \gamma_8 x e^{-\beta x} - \gamma_9 e^{-\beta x} - \gamma_{10} e^{2\beta x} - \gamma_{11} e^{\beta x} - \gamma_{12} e^{\beta_1 x} + \\ \gamma_{13} x + \gamma_{14} e^{(\beta_1 - \beta)x} - \gamma_{15} e^{(\beta_2 - \beta)x} \end{pmatrix} dx + c_9 e^{-\beta z}$$

$$= e^{\beta z} \begin{pmatrix} \gamma_7 x\Big|_0^z + \gamma_8 \left(\dfrac{x e^{-\beta x}}{\beta}\Bigg|_0^z + \dfrac{1}{\beta^2} e^{-\beta x}\Bigg|_0^z \right) + \dfrac{\gamma_9}{\beta} e^{-\beta x}\Bigg|_0^z - \dfrac{\gamma_{10}}{2\beta} e^{2\beta x}\Bigg|_0^z - \dfrac{\gamma_{11}}{\beta} e^{\beta x}\Bigg|_0^z - \\ \dfrac{\gamma_{12}}{\beta_1} e^{\beta_1 x}\Bigg|_0^z + \dfrac{\gamma_{13}}{\beta} x^2\Bigg|_0^z + \dfrac{\gamma_{14}}{(\beta_1 - \beta)} e^{(\beta_1 - \beta)x} - \dfrac{\gamma_{15}}{(\beta_2 - \beta)} e^{(\beta_2 - \beta)x} \end{pmatrix} + c_9 e^{\beta z}$$

$$q_4(z) = \gamma_7 z e^{\beta z} - \frac{\gamma_8}{\beta} z - \frac{\gamma_8}{\beta^2}\left(1 - e^{\beta z}\right) + \frac{\gamma_9}{\beta}\left(1 - e^{\beta z}\right) - \frac{\gamma_{10}}{2\beta}\left(e^{3\beta z} - e^{\beta z}\right) - \frac{\gamma_{11}}{\beta}\left(e^{2\beta z} - e^{\beta z}\right) - \frac{\gamma_{12}}{\beta_1}\left(e^{(\beta+\beta_1)z} - e^{-\beta z}\right) + \frac{\gamma_{13}}{2} z^2 e^{\beta z} + \frac{\gamma_{14}}{(\beta_1 - \beta)}\left(e^{\beta_1 z} - e^{\beta z}\right) - \frac{\gamma_{15}}{(\beta_2 - \beta)}\left(e^{\beta_2 z} - e^{\beta z}\right) + c_9 e^{\beta z}$$

That is

$$q_4(z) = \frac{\gamma_{13}}{2} z^2 e^{\beta z} + \gamma_7 z e^{\beta z} - \frac{\gamma_{10}}{2\beta} e^{3\beta z} - \frac{\gamma_{11}}{\beta} e^{2\beta z} - \frac{\gamma_{12}}{\beta_1} e^{(\beta+\beta_1)z} + \frac{\gamma_{14}}{(\beta_1 - \beta)} e^{\beta_1 z} - \frac{\gamma_{15}}{(\beta_2 - \beta)} e^{\beta_2 z} + \frac{\gamma_{12}}{\beta_1} e^{-\beta z} - \frac{\gamma_8}{\beta} z + \left(c_9 + \frac{\gamma_8}{\beta^2} - \frac{\gamma_9}{\beta} + \frac{\gamma_{10}}{2\beta} + \frac{\gamma_{11}}{\beta} - \frac{\gamma_{14}}{(\beta_1 - \beta)} + \frac{\gamma_{15}}{(\beta_2 - \beta)} \right) e^{\beta z} \tag{3.309}$$

$$\frac{du_1}{dz} = q_4(z)$$

That is

$$
\begin{aligned}
u_1(z) &= \frac{\gamma_{13}}{2}\left(\frac{z^2e^{\beta z}}{\beta}-\frac{2ze^{\beta z}}{\beta^2}+\frac{2e^{\beta z}}{\beta^3}\right)+\gamma_7\left(\frac{ze^{\beta z}}{\beta}-\frac{e^{\beta z}}{\beta^2}\right)-\frac{\gamma_{10}}{6\beta^2}e^{3\beta z}-\frac{\gamma_{11}}{2\beta^2}e^{2\beta z} \\
&-\frac{\gamma_{12}}{\beta_1(\beta+\beta_1)}e^{(\beta+\beta_1)z}+\frac{\gamma_{14}}{\beta_1(\beta_1-\beta)}e^{\beta_1 z}-\frac{\gamma_{15}}{\beta_2(\beta_2-\beta)}e^{\beta_2 z}-\frac{\gamma_{12}}{\beta\beta_1}e^{-\beta z}-\frac{\gamma_8}{2\beta}z^2+ \\
&\frac{1}{\beta}\left(c_9+\frac{\gamma_8}{\beta^2}-\frac{\gamma_9}{\beta}+\frac{\gamma_{10}}{2\beta}+\frac{\gamma_{11}}{\beta}-\frac{\gamma_{14}}{(\beta_1-\beta)}+\frac{\gamma_{15}}{(\beta_2-\beta)}\right)e^{\beta z}+c_{10}
\end{aligned}
\tag{3.310}
$$

$$
\begin{aligned}
u_1(0)=0 \Rightarrow c_{10} &= \frac{\gamma_{10}}{6\beta^2}+\frac{\gamma_{11}}{2\beta^2}+\frac{\gamma_{12}}{\beta_1(\beta_1-\beta)}-\frac{\gamma_{13}}{\beta^3}+\frac{\gamma_7}{\beta^2}-\frac{\gamma_{14}}{\beta_1(\beta_1-\beta)}+ \\
&\frac{\gamma_{15}}{\beta_2(\beta_2-\beta)}+\frac{\gamma_{12}}{\beta\beta_1}-\left(\frac{c_9}{\beta}+\frac{\gamma_8}{\beta^3}-\frac{\gamma_9}{\beta^2}+\frac{\gamma_{10}}{2\beta^2}+\frac{\gamma_{11}}{\beta^2}-\frac{\gamma_{14}}{\beta(\beta_1-\beta)}+\frac{\gamma_{15}}{\beta(\beta_2-\beta)}\right)
\end{aligned}
\tag{3.311}
$$

$$
\begin{aligned}
&u_1(1)=0 \Rightarrow \frac{c_9}{\beta}\left(e^{\beta}-1\right)= \\
&\left(-\frac{\gamma_{13}}{2}\left(\frac{1}{\beta}-\frac{2}{\beta^2}+\frac{2}{\beta^3}\right)-\gamma_7\left(\frac{1}{\beta}-\frac{1}{\beta^2}\right)-\frac{1}{\beta}\left(\frac{\gamma_8}{\beta^2}-\frac{\gamma_9}{\beta}+\frac{\gamma_{10}}{2\beta}+\frac{\gamma_{11}}{\beta}-\frac{\gamma_{14}}{(\beta_1-\beta)}+\frac{\gamma_{15}}{(\beta_2-\beta)}\right)\right)e^{\beta}+ \\
&\frac{\gamma_{10}}{6\beta^2}e^{3\beta}+\frac{\gamma_{11}}{2\beta^2}e^{2\beta}+\frac{\gamma_{12}}{\beta_1(\beta_1+\beta)}e^{(\beta_1+\beta)}-\frac{\gamma_{14}}{\beta_1(\beta_1-\beta)}e^{\beta}+\frac{\gamma_{15}}{\beta_2(\beta_2-\beta)}e^{\beta_2}+\frac{\gamma_{12}}{\beta\beta_1}e^{-\beta}+ \\
&\frac{\gamma_8}{2\beta}-\frac{\gamma_{10}}{6\beta^2}-\frac{\gamma_{11}}{2\beta^2}-\frac{\gamma_{12}}{\beta_1(\beta_1+\beta)}+\frac{\gamma_{13}}{\beta^3}-\frac{\gamma_7}{\beta^2}+\frac{\gamma_{14}}{\beta_1(\beta_1-\beta)}-\frac{\gamma_{15}}{\beta_2(\beta_2-\beta)}-\frac{\gamma_{12}}{\beta\beta_1} \\
&+\frac{\gamma_8}{\beta^3}-\frac{\gamma_9}{\beta^2}+\frac{\gamma_{10}}{2\beta^2}+\frac{\gamma_{11}}{\beta^2}-\frac{\gamma_{14}}{\beta_1(\beta_1-\beta)}+\frac{\gamma_{15}}{\beta_2(\beta_2-\beta)}
\end{aligned}
$$

$$
c_9=\frac{\beta\left(\begin{aligned}&\left(\frac{\gamma_{13}}{\beta_2}-\frac{\gamma_{13}}{2\beta}-\frac{\gamma_7}{\beta}\right)e^{\beta}+\left(\frac{\gamma_7}{\beta^2}-\frac{\gamma_{13}}{\beta^3}-\frac{\gamma_8}{\beta^3}+\frac{\gamma_9}{\beta^3}-\frac{\gamma_{10}}{2\beta^2}-\frac{\gamma_{11}}{\beta^2}+\frac{\gamma_{14}}{\beta(\beta_1-\beta)}-\frac{\gamma_{15}}{\beta(\beta_2-\beta)}\right)\left(e^{\beta}-1\right)+\\&\frac{\gamma_{10}}{6\beta^2}\left(e^{3\beta}-1\right)+\frac{\gamma_{11}}{2\beta^2}\left(e^{2\beta}-1\right)+\frac{\gamma_{12}}{\beta_1(\beta_1+\beta)}\left(e^{(\beta_1+\beta)}-1\right)-\frac{\gamma_{14}}{\beta_1(\beta_1-\beta)}\left(e^{\beta_1}-1\right)+\\&\frac{\gamma_{15}}{\beta_2(\beta_2-\beta)}\left(e^{\beta_2}-1\right)+\frac{\gamma_{12}}{\beta\beta_1}\left(e^{-\beta}-1\right)+\frac{\gamma_8}{2\beta}\end{aligned}\right)}{\left(e^{\beta}-1\right)}
\tag{3.312}
$$

Therefore,

$$u_1(z) = A_8 z^2 e^{\beta z} + A_9 e^{\beta z} + A_{10} e^{\beta z} + A_{11} e^{3\beta z} - A_{12} e^{2\beta z} - A_{13} e^{(\beta+\beta_1)z} + A_{14} e^{\beta_1 z} - A_{15} e^{\beta_2 z} - A_{16} e^{-\beta z} - A_{17} z^2 + c_{10} \quad (3.313)$$

where

$$A_8 = \frac{\gamma_{13}}{2\beta}, A_9 = \left(\frac{\gamma_7}{\beta} - \frac{\gamma_{13}}{\beta^2}\right), A_{10} = \left(\frac{\gamma_{13}}{\beta^3} - \frac{\gamma_7}{\beta^2} + \frac{1}{\beta}\left(\begin{matrix} c_9 + \frac{\gamma_8}{\beta^2} - \frac{\gamma_9}{\beta} + \frac{\gamma_{10}}{2\beta} + \frac{\gamma_{11}}{\beta} - \\ \frac{\gamma_{14}}{(\beta_1 - \beta)} + \frac{\gamma_{15}}{(\beta_2 - \beta)} \end{matrix}\right)\right),$$

$$A_{11} = \frac{\gamma_{10}}{6\beta^2}, A_{12} = \frac{\gamma_{11}}{2\beta^2}, A_{13} = \frac{\gamma_{12}}{\beta_1(\beta + \beta_1)}, A_{14} = \frac{\gamma_{14}}{\beta_1(\beta_1 - \beta)},$$

$$A_{15} = \frac{\gamma_{15}}{\beta_2(\beta_2 - \beta)}, A_{16} = \frac{\gamma_{12}}{\beta\beta_1}, A_{17} = \frac{\gamma_8}{2\beta}$$

Consider equation (3.277) given as

$$\frac{d^2 w_1}{dz^2} - \beta \frac{dw_1}{dz} = \gamma_{16} + \gamma_{17} z - \gamma_{18} e^{\beta z}$$

where

$$\gamma_{16} = \frac{4dBe}{c\left[(1 + BiBe)^2 + Be^2\right]}, \gamma_{17} = \frac{4dBeB}{c\left[(1 + BiBe)^2 + Be^2\right]}, \gamma_{18} = \frac{4dBeA}{c\left[(1 + BiBe)^2 + Be^2\right]}$$

Let

$$q_5 = \frac{dw_1}{dz} \quad (3.314)$$

Then

$$\frac{dq_5}{dz} - \beta q_5 = \gamma_{16} + \gamma_{17} z - \gamma_{18} e^{\beta z} \quad (3.315)$$

So

$$q_5\left(z\right)=e^{\beta z}\int_0^z\left(\gamma_{16}e^{-\beta x}+\gamma_{17}xe^{-\beta x}-\gamma_{18}\right)dx+c_{11}e^{\beta z}$$

$$=e^{\beta z}\left(-\frac{\gamma_{16}}{\beta}e^{-\beta x}\Big|_0^z-\frac{\gamma_{17}x}{\beta}e^{-\beta x}\Big|_0^z+\frac{1}{\beta^2}e^{-\beta x}\Big|_0^z-\gamma_{18}x\Big|_0^z\right)+c_{11}e^{\beta z}$$

$$=\frac{\gamma_{16}}{\beta}\left(e^{\beta z}-1\right)-\frac{\gamma_{17}}{\beta}z+\frac{\gamma_{17}}{\beta^2}\left(e^{\beta z}-1\right)-\gamma_{18}ze^{\beta z}+c_{11}e^{\beta z}$$

$$q_5\left(z\right)=\left(c_{11}+\frac{\gamma_{16}}{\beta}+\frac{\gamma_{17}}{\beta^2}\right)e^{\beta z}-\gamma_{18}ze^{\beta z}-\frac{\gamma_{17}}{\beta}z-\left(\frac{\gamma_{16}}{\beta}+\frac{\gamma_{17}}{\beta^2}\right) \tag{3.316}$$

i.e

$$\frac{dw_1}{dz}=q_5\left(z\right)$$

i.e

$$w_1\left(z\right)=\left(\frac{c_{11}}{\beta}+\frac{\gamma_{16}}{\beta^2}+\frac{\gamma_{17}}{\beta^3}\right)e^{\beta z}-\gamma_{18}\left(\frac{ze^{\beta z}}{\beta}-\frac{e^{\beta z}}{\beta^2}\right)-\frac{\gamma_{17}}{2\beta}z^2-\left(\frac{\gamma_{16}}{\beta}+\frac{\gamma_{17}}{\beta^2}\right)z+c_{12} \tag{3.317}$$

$$w_1\left(0\right)=0\Rightarrow c_{12}=-\left(\frac{c_{11}}{\beta}+\frac{\gamma_{16}}{\beta^2}+\frac{\gamma_{17}}{\beta^3}+\frac{\gamma_{18}}{\beta^2}\right) \tag{3.318}$$

$$w_1\left(1\right)=0\Rightarrow\frac{c_{11}}{\beta}\left(e^{\beta z}-1\right)=\frac{\gamma_{16}}{\beta}+\frac{\gamma_{17}}{\beta^2}+\frac{\gamma_{17}}{2\beta}+\frac{\gamma_{18}}{\beta}e^{\beta}-\frac{\gamma_{18}}{\beta^2}e^{\beta}-\frac{\gamma_{16}}{\beta^2}e^{\beta}-\frac{\gamma_{17}}{\beta^3}e^{\beta}+\frac{\gamma_{16}}{\beta^2}+\frac{\gamma_{17}}{\beta^3}+\frac{\gamma_{18}}{\beta^2}$$

i.e

$$c_{11}=\frac{\beta\left(\left(\frac{\gamma_{16}}{\beta}+\frac{\gamma_{17}}{\beta^2}+\frac{\gamma_{17}}{2\beta}\right)+\frac{\gamma_{18}}{\beta}e^{\beta}-\left(\frac{\gamma_{18}}{\beta^2}+\frac{\gamma_{16}}{\beta^2}+\frac{\gamma_{17}}{\beta^3}\right)\left(e^{\beta}-1\right)\right)}{\left(e^{\beta z}-1\right)} \tag{3.319}$$

Therefore,

$$w_1(z) = A_{18}e^{\beta z} - A_{19}ze^{\beta z} - A_{20}z^2 - A_{21}z + c_{12} \tag{3.320}$$

where

$$A_{18} = \left(\frac{c_{11}}{\beta} + \frac{\gamma_{16}}{\beta^2} + \frac{\gamma_{17}}{\beta^3} + \frac{\gamma_{18}}{\beta^2}\right), A_{19} = \frac{\gamma_{18}}{\beta}, A_{20} = \frac{\gamma_{17}}{2\beta}, A_{21} = \left(\frac{\gamma_{16}}{\beta} + \frac{\gamma_{17}}{\beta^2}\right)$$

Also consider (3.278), given by

$$\left.\begin{array}{l}
\dfrac{d^2\theta_1}{dz^2} - \beta_1\dfrac{d\theta_1}{dz} = \gamma_{19}e^{2\beta z} + \gamma_{20}e^{\beta z} + \gamma_{21}e^{\beta_1 z} - \gamma_{22}z + \gamma_{23} + \gamma_{24}ze^{\beta z} - \gamma_{25}z^2 - \\
\gamma_{26}e^{4\beta z} - \gamma_{27}e^{3\beta z} - \gamma_{28}e^{(2\beta+\beta_1)z} + \gamma_{29}e^{(\beta+\beta_1)z} + \gamma_{30}ze^{2\beta z} - \gamma_{31}z^2e^{2\beta z} + \gamma_{32}e^{(\beta+\beta_2)z} + \\
\gamma_{33}z^2e^{\beta z} - \gamma_{34}e^{\beta_2 z} + \gamma_{35}e^{-\beta z} - \gamma_{36}e^{2\beta_1 z} + \gamma_{37}ze^{\beta_1 z} \\
\theta_1(0) = 0, \theta_1(1) = 0
\end{array}\right\}$$

where

$$\gamma_{19} = \begin{pmatrix} \frac{4\operatorname{Re}\Pr b_1 A_1}{c} - \frac{4\Pr dEcA^2}{c\left[(1+BiBe)^2 + Be^2\right]} - \Pr Ec\left(A_1 B^2 - 2ABA_2\beta + c_6 A^2\right) - \\ 2\Pr Ec\left(AA_9\beta + AA_{10}\beta^2 + 2BA_{12}\beta\right) - \left(2A_2^2\beta^2 + 4A_1\beta\left(c_6\beta - B\right)\right) \end{pmatrix},$$

$$\gamma_{20} = \begin{pmatrix} \frac{4\operatorname{Re}\Pr b_1 A_2}{c} + \frac{8\Pr dEcA^2}{c\left[(1+BiBe)^2 + Be^2\right]} - \Pr Ec\left(A_2 B^2 - 2ABc_6\beta\right) + \\ 2\Pr Ec\left(BA_9 + BA_{10}\beta\right) - A_2\beta\left(c_6\beta - 2B\right) \end{pmatrix},$$

$$\gamma_{21} = \left(\frac{4\operatorname{Re}\Pr b_1 A_3}{c} - \Pr EcA_3 B^2 + 2\Pr EcBA_{14}\beta_1 - A_3\beta_1\left(c_6\beta_1 - 2B\right)\right),$$

$$\gamma_{22} = \left(\frac{4\operatorname{Re}\Pr b_1 B_1}{c} + \frac{8\Pr dEcAB}{c\left[(1+BiBe)^2 + Be^2\right]} - \Pr EcB_1 B^2 + 4\Pr EcBA_{17}\right),$$

$$\gamma_{23} = \left(\frac{4\operatorname{Re}\Pr b_1 c_6}{c} - \frac{4\Pr dEcA^2}{c\left[(1+BiBe)^2 + Be^2\right]} - \Pr EcB^2 c_6 - 2\Pr EcAA_{16}\beta^2 - B^2\right),$$

$$\gamma_{24} = \left(\frac{8\Pr dEcAB}{c\left[(1+BiBe)^2 + Be^2\right]} - 2\Pr EcABB_1\beta + 2\Pr Ec\left(2AA_{17}\beta + B\left(2A_8 + A_9\beta\right)\right) + A_2\beta^2\beta_1\right),$$

$$\gamma_{25} = \frac{4\Pr dEcB^2}{c\left[(1+BiBe)^2 + Be^2\right]}, \gamma_{26} = \left(8A_1^2\beta^2 + \Pr EcA_1 A^2 + 6\Pr EcAA_{11}\beta^2\right),$$

$$\gamma_{27} = \left(\Pr Ec\left(A_2 A^2 - 2ABA_1\beta\right) - 2\Pr Ec\left(2AA_{12}\beta^2 - 3BA_{11}\beta\right) + 9A_1 A_2\beta^2\right),$$

$$\gamma_{28} = \left(\Pr EcA_3 A^2 - 2\Pr EcAA_{13}\beta\left(\beta + \beta_1\right) + A_1 A_3\left(\beta_1^2 + 4\beta^2 + 4\beta\beta_1\right)\right),$$

$$\gamma_{29} = \left(2\Pr EcABA_3\beta - 2\Pr Ec\left(AA_{14}\beta\beta_1 + BA_{13}\left(\beta + \beta_1\right)\right) + A_2 A_3\left(\beta_1^2 + \beta^2 + 2\beta\beta_1\right)\right),$$

$$\gamma_{30} = \left(\Pr EcB_1 A^2 - 2\Pr EcA\beta^2\left(A_8 + A_9\right) + 4A_1\beta^2\beta_1\right), \gamma_{31} = \Pr EcAA_8\beta, \gamma_{32} = 2\Pr EcAA_{15}\beta\beta_2,$$

$$\gamma_{33} = 2\Pr EcBA_8\beta, \gamma_{34} = 2\Pr EcBA_{15}\beta_2, \gamma_{35} = 2\Pr EcBA_{16}\beta, \gamma_{36} = 2A_3^2\beta_1^2, \gamma_{37} = A_3\beta_1^3$$

Let

$$q_6 = \frac{d\theta_1}{dz} \tag{3.321}$$

Then

$$\frac{dq_6}{dz}-\beta_1 q_6=\gamma_{19}e^{2\beta z}+\gamma_{20}e^{\beta z}+\gamma_{21}e^{\beta_1 z}-\gamma_{22}z+\gamma_{23}+\gamma_{24}ze^{\beta z}-\gamma_{25}z^2-\gamma_{26}e^{4\beta z}-$$
$$\gamma_{27}e^{3\beta z}-\gamma_{28}e^{(2\beta+\beta_1)z}+\gamma_{29}e^{(\beta+\beta_1)z}+\gamma_{30}ze^{2\beta z}-\gamma_{31}z^2e^{2\beta z}+\gamma_{32}e^{(\beta+\beta_2)z}+\gamma_{33}z^2e^{\beta z}- \quad (3.322)$$
$$\gamma_{34}e^{\beta_2 z}+\gamma_{35}e^{-\beta z}-\gamma_{36}e^{2\beta_1 z}+\gamma_{37}ze^{\beta_1 z}$$

So

$$q_6(z)=e^{\beta_1 z}\int_0^z\left(\begin{array}{l}\gamma_{19}e^{(2\beta-\beta_1)x}+\gamma_{20}e^{(\beta-\beta_1)x}+\gamma_{21}-\gamma_{22}xe^{-\beta_1 x}+\gamma_{23}e^{-\beta_1 x}+\gamma_{24}xe^{(\beta-\beta_1)x}-\\ \gamma_{25}x^2e^{-\beta_1 x}-\gamma_{26}e^{(4\beta-\beta_1)x}-\gamma_{27}e^{(3\beta-\beta_1)x}-\gamma_{28}e^{2\beta x}+\gamma_{29}e^{\beta x}+\\ \gamma_{30}xe^{(2\beta-\beta_1)x}-\gamma_{31}x^2e^{(2\beta-\beta_1)x}+\gamma_{32}e^{(\beta+\beta_2-\beta_1)x}+\gamma_{33}x^2e^{(\beta-\beta_1)x}-\\ \gamma_{34}e^{(\beta_2-\beta_1)x}+\gamma_{35}e^{-(\beta+\beta_1)x}-\gamma_{36}e^{\beta_1 x}+\gamma_{37}x\end{array}\right)dx+c_{13}e^{\beta_1 z}$$

$$=\frac{\gamma_{19}}{(2\beta-\beta_1)}\left(e^{2\beta z}-e^{\beta_1 z}\right)+\frac{\gamma_{20}}{(\beta-\beta_1)}\left(e^{\beta z}-e^{\beta_1 z}\right)+\gamma_{21}ze^{\beta_1 z}+\gamma_{22}\left(\frac{z}{\beta_1}+\frac{1}{\beta_1^2}\left(1-e^{\beta_1 z}\right)\right)-$$
$$\frac{\gamma_{23}}{\beta_1}\left(1-e^{\beta_1 z}\right)+\gamma_{24}\left(\frac{ze^{\beta z}}{(\beta-\beta_1)}-\frac{\left(e^{\beta z}-e^{\beta_1 z}\right)}{(\beta-\beta_1)^2}\right)+\gamma_{25}\left(\frac{z^2}{\beta_1}+\frac{2z}{\beta_1^2}+\frac{2}{\beta^3}\left(1-e^{\beta_1 z}\right)\right)-$$
$$\frac{\gamma_{26}}{(4\beta-\beta_1)}\left(e^{4\beta z}-e^{\beta_1 z}\right)-\frac{\gamma_{27}}{(3\beta-\beta_1)}\left(e^{3\beta z}-e^{\beta_1 z}\right)-\frac{\gamma_{28}}{2\beta}\left(e^{(2\beta+\beta_1)z}-e^{\beta_1 z}\right)+$$
$$\frac{\gamma_{29}}{\beta}\left(e^{(\beta+\beta_1)z}-e^{\beta_1 z}\right)+\gamma_{30}\left(\frac{ze^{2\beta z}}{(2\beta-\beta_1)}-\frac{\left(e^{2\beta z}-e^{\beta_1 z}\right)}{(2\beta-\beta_1)^2}\right)+\gamma_{31}\left(\begin{array}{l}\dfrac{z^2e^{2\beta z}}{(2\beta-\beta_1)}-\\ \dfrac{2ze^{2\beta z}}{(2\beta-\beta_1)^2}+\dfrac{2\left(e^{2\beta z}-e^{\beta_1 z}\right)}{(2\beta-\beta_1)^3}\end{array}\right)+$$
$$\frac{\gamma_{32}}{(\beta+\beta_2-\beta_1)}\left({}^{(\beta+\beta_2)z}-e^{\beta_1 z}\right)+\gamma_{33}\left(\frac{z^2e^{\beta z}}{(\beta-\beta_1)}-\frac{2ze^{\beta z}}{(\beta-\beta_1)^2}+\frac{2\left(e^{\beta z}-e^{\beta_1 z}\right)}{(\beta-\beta_1)^3}\right)-$$
$$\frac{\gamma_{34}}{(\beta_2-\beta_1)}\left(e^{\beta_2 z}-e^{\beta_1 z}\right)+\frac{\gamma_{35}}{(\beta+\beta_1)}\left(e^{-\beta z}-e^{\beta_1 z}\right)-\frac{\gamma_{36}}{\beta_1}\left(e^{2\beta_1 z}-e^{\beta_1 z}\right)+\frac{\gamma_{37}}{2}z^2e^{\beta_1 z}+c_{13}e^{\beta_1 z}$$

That is

$$q_6(z) = \left(\frac{\gamma_{19}}{(2\beta-\beta_1)} - \frac{\gamma_{30}}{(2\beta-\beta_1)^2} - \frac{2\gamma_{31}}{(2\beta-\beta_1)^3} \right) e^{2\beta z} -$$

$$\left(\begin{array}{l} \frac{\gamma_{19}}{(2\beta-\beta_1)} + \frac{\gamma_{20}}{(\beta-\beta_1)} + \frac{\gamma_{22}}{\beta_1^2} - \frac{\gamma_{23}}{\beta_1} - \frac{\gamma_{24}}{(\beta-\beta_1)^2} + \frac{2\gamma_{25}}{\beta^3} - \frac{\gamma_{26}}{(4\beta-\beta_1)} - \frac{\gamma_{27}}{(3\beta-\beta_1)} - \\ \frac{\gamma_{28}}{2\beta} + \frac{\gamma_{29}}{\beta} - \frac{\gamma_{30}}{(2\beta-\beta_1)^2} - \frac{2\gamma_{31}}{(2\beta-\beta_1)^3} + \frac{\gamma_{32}}{(\beta+\beta_2-\beta_1)} + \frac{2\gamma_{33}}{(\beta-\beta_1)^3} - \frac{\gamma_{34}}{(\beta_2-\beta_1)} + \\ \frac{\gamma_{35}}{(\beta_2+\beta_1)} - \frac{\gamma_{36}}{\beta_1} - c_{13} \end{array} \right) e^{\beta_1 z} +$$

$$\left(\frac{\gamma_{20}}{(\beta-\beta_1)} - \frac{\gamma_{24}}{(\beta-\beta_1)^2} + \frac{2\gamma_{33}}{(\beta-\beta_1)^3} \right) e^{\beta z} + \gamma_{21} z e^{\beta_1 z} + \left(\frac{\gamma_{22}}{\beta_1} + \frac{2\gamma_{25}}{\beta_1^2} \right) z + \left(\frac{\gamma_{22}}{\beta_1^2} - \frac{\gamma_{23}}{\beta_1} + \frac{2\gamma_{25}}{\beta^3} \right) +$$

$$\left(\frac{\gamma_{24}}{(\beta-\beta_1)} - \frac{2\gamma_{33}}{(\beta-\beta_1)^2} \right) z e^{\beta z} + \frac{\gamma_{25}}{\beta_1} z^2 - \frac{\gamma_{26}}{(4\beta-\beta_1)} e^{4\beta z} - \frac{\gamma_{27}}{(3\beta-\beta_1)} e^{3\beta z} - \frac{\gamma_{28}}{2\beta} e^{(2\beta+\beta_1)z} +$$

$$\frac{\gamma_{29}}{\beta} e^{(\beta+\beta_1)z} + \left(\frac{\gamma_{30}}{(2\beta-\beta_1)} - \frac{2\gamma_{31}}{(2\beta-\beta_1)^2} \right) z e^{2\beta z} - \frac{\gamma_{31}}{(2\beta-\beta_1)} z^2 e^{2\beta z} + \frac{\gamma_{32}}{(\beta+\beta_2-\beta_1)} e^{(\beta+\beta_2)z} +$$

$$\frac{\gamma_{33}}{(\beta-\beta_1)} z^2 e^{\beta z} - \frac{\gamma_{34}}{(\beta_2-\beta_1)} e^{\beta_2 z} + \frac{\gamma_{35}}{(\beta+\beta_1)} e^{-\beta z} - \frac{\gamma_{36}}{\beta_1} e^{2\beta_1 z} + \frac{\gamma_{37}}{2} z^2 e^{\beta_1 z} \tag{3.323}$$

That is

$$\frac{d\theta_1}{dz} = q_6(z)$$

$$
\begin{aligned}
\theta_1(z) &= \frac{1}{2\beta}\left(\frac{\gamma_{19}}{(2\beta-\beta_1)}-\frac{\gamma_{30}}{(2\beta-\beta_1)^2}-\frac{2\gamma_{31}}{(2\beta-\beta_1)^3}\right)e^{2\beta z}+\\
&\frac{1}{\beta_1}\left(\begin{array}{l} c_{13}-\frac{\gamma_{19}}{(2\beta-\beta_1)}-\frac{\gamma_{20}}{(\beta-\beta_1)}-\frac{\gamma_{22}}{\beta_1^2}+\frac{\gamma_{23}}{\beta_1}+\frac{\gamma_{24}}{(\beta-\beta_1)^2}-\frac{2\gamma_{25}}{\beta^3}+\frac{\gamma_{26}}{(4\beta-\beta_1)}+\frac{\gamma_{27}}{(3\beta-\beta_1)}+\frac{\gamma_{28}}{2\beta}-\frac{\gamma_{29}}{\beta}+\\ \frac{\gamma_{30}}{(2\beta-\beta_1)^2}+\frac{2\gamma_{31}}{(2\beta-\beta_1)^3}-\frac{\gamma_{32}}{(\beta+\beta_2-\beta_1)}-\frac{2\gamma_{33}}{(\beta-\beta_1)^3}+\frac{\gamma_{34}}{(\beta_2-\beta_1)}-\frac{\gamma_{35}}{(\beta_2+\beta_1)}+\frac{\gamma_{36}}{\beta_1}\end{array}\right)e^{\beta_1 z}+\\
&\frac{1}{\beta}\left(\frac{\gamma_{20}}{(\beta-\beta_1)}-\frac{\gamma_{24}}{(\beta-\beta_1)^2}+\frac{2\gamma_{33}}{(\beta-\beta_1)^3}\right)e^{\beta z}+\gamma_{21}\left(\frac{ze^{\beta_1 z}}{\beta_1}-\frac{e^{\beta_1 z}}{\beta_1^2}\right)+\frac{1}{2}\left(\frac{\gamma_{22}}{\beta_1}+\frac{2\gamma_{25}}{\beta_1^2}\right)z^2+\\
&\left(\frac{\gamma_{22}}{\beta_1^2}-\frac{\gamma_{23}}{\beta_1}+\frac{2\gamma_{25}}{\beta^3}\right)z+\left(\frac{\gamma_{24}}{(\beta-\beta_1)}-\frac{2\gamma_{33}}{(\beta-\beta_1)^2}\right)\cdot\left(\frac{ze^{\beta z}}{\beta}-\frac{e^{\beta z}}{\beta^2}\right)+\frac{\gamma_{25}}{3\beta_1}z^3-\frac{\gamma_{26}}{4\beta(4\beta-\beta_1)}e^{4\beta z}-\\
&\frac{\gamma_{27}}{3\beta(3\beta-\beta_1)}e^{3\beta z}-\frac{\gamma_{28}}{2\beta(2\beta+\beta_1)}e^{(2\beta+\beta_1)z}+\frac{\gamma_{29}}{\beta(\beta+\beta_1)}e^{(\beta+\beta_1)z}+\left(\frac{\gamma_{30}}{(2\beta-\beta_1)}-\frac{2\gamma_{31}}{(2\beta-\beta_1)^2}\right)\cdot\left(\frac{ze^{2\beta z}}{2\beta}-\frac{e^{2\beta z}}{4\beta^2}\right)-\\
&\frac{\gamma_{31}}{(2\beta-\beta_1)}\left(\frac{z^2e^{2\beta z}}{2\beta}-\frac{2ze^{2\beta z}}{4\beta^2}+\frac{2e^{2\beta z}}{8\beta^3}\right)+\frac{\gamma_{32}}{(\beta+\beta_2)(\beta+\beta_2-\beta_1)}e^{(\beta+\beta_2)z}+\\
&\frac{\gamma_{33}}{(\beta-\beta_1)}\left(\frac{z^2e^{\beta z}}{\beta}-\frac{2ze^{\beta z}}{\beta^2}+\frac{2e^{\beta z}}{\beta^3}\right)-\frac{\gamma_{34}}{\beta_2(\beta_2-\beta_1)}e^{\beta_2 z}-\frac{\gamma_{35}}{\beta(\beta+\beta_1)}e^{-\beta z}-\frac{\gamma_{36}}{2\beta_1^2}e^{2\beta_1 z}+\\
&\frac{\gamma_{37}}{2}\left(\frac{z^2e^{\beta_1 z}}{\beta_1}-\frac{2ze^{\beta_1 z}}{\beta_1^2}+\frac{2e^{\beta_1 z}}{\beta_1^3}\right)+c_{14}
\end{aligned}
\tag{3.324}
$$

$$
\begin{aligned}
&\theta_1(0)=0 \Rightarrow c_{14}=\frac{1}{2\beta}\left(\frac{\gamma_{30}}{(2\beta-\beta_1)^2}+\frac{2\gamma_{31}}{(2\beta-\beta_1)^3}-\frac{\gamma_{19}}{(2\beta-\beta_1)}\right)- \\
&\frac{1}{\beta_1}\left(\begin{array}{l}
c_{13}-\dfrac{\gamma_{19}}{(2\beta-\beta_1)}-\dfrac{\gamma_{20}}{(\beta-\beta_1)}-\dfrac{\gamma_{22}}{\beta_1^2}+\dfrac{\gamma_{23}}{\beta_1}+\dfrac{\gamma_{24}}{(\beta-\beta_1)^2}-\dfrac{2\gamma_{25}}{\beta^3}+\dfrac{\gamma_{26}}{(4\beta-\beta_1)}+ \\
\dfrac{\gamma_{27}}{(3\beta-\beta_1)}+\dfrac{\gamma_{28}}{2\beta}-\dfrac{\gamma_{29}}{\beta}+\dfrac{\gamma_{30}}{(2\beta-\beta_1)^2}+\dfrac{2\gamma_{31}}{(2\beta-\beta_1)^3}-\dfrac{\gamma_{32}}{(\beta+\beta_2-\beta_1)}- \\
\dfrac{2\gamma_{33}}{(\beta-\beta_1)^3}+\dfrac{\gamma_{34}}{(\beta_2-\beta_1)}-\dfrac{\gamma_{35}}{(\beta_2+\beta_1)}+\dfrac{\gamma_{36}}{\beta_1}
\end{array}\right)- \\
&\frac{1}{\beta}\left(\frac{\gamma_{20}}{(\beta-\beta_1)}-\frac{\gamma_{24}}{(\beta-\beta_1)^2}+\frac{2\gamma_{33}}{(\beta-\beta_1)^3}\right)+\frac{\gamma_{21}}{\beta_1^2}+\frac{1}{\beta^2}\left(\frac{\gamma_{24}}{(\beta-\beta_1)}-\frac{2\gamma_{33}}{(\beta-\beta_1)^2}\right)+ \\
&\frac{\gamma_{26}}{4\beta(4\beta-\beta_1)}+\frac{\gamma_{27}}{3\beta(3\beta-\beta_1)}+\frac{\gamma_{28}}{2\beta(2\beta+\beta_1)}-\frac{\gamma_{29}}{\beta(\beta+\beta_1)}+ \\
&\frac{1}{4\beta^2}\left(\frac{\gamma_{30}}{(2\beta-\beta_1)}-\frac{2\gamma_{31}}{(2\beta-\beta_1)^2}\right)+\frac{2\gamma_{31}}{8\beta^3(2\beta-\beta_1)}-\frac{\gamma_{32}}{(\beta+\beta_2)(\beta+\beta_2-\beta_1)}- \\
&\frac{2\gamma_{33}}{\beta^3(\beta-\beta_1)}+\frac{\gamma_{34}}{\beta_2(\beta_2-\beta_1)}+\frac{\gamma_{35}}{\beta(\beta+\beta_1)}+\frac{\gamma_{36}}{2\beta_1^2}-\frac{2\gamma_{37}}{2\beta_1^3}
\end{aligned}
\tag{3.325}
$$

$$\theta_1(1)=0 \Rightarrow c_{13}=$$

$$\beta_1\frac{\left(\begin{aligned}&\frac{1}{2\beta}\left(\frac{\gamma_{30}}{(2\beta-\beta_1)^2}+\frac{2\gamma_{31}}{(2\beta-\beta_1)^3}-\frac{\gamma_{19}}{(2\beta-\beta_1)}\right)\left(e^{2\beta}-1\right)-\\&\frac{\left(e^{\beta_1}-1\right)}{\beta_1}\left(\begin{aligned}&-\frac{\gamma_{19}}{(2\beta-\beta_1)}-\frac{\gamma_{20}}{(\beta-\beta_1)}-\frac{\gamma_{22}}{\beta_1^2}+\frac{\gamma_{23}}{\beta_1}+\frac{\gamma_{24}}{(\beta-\beta_1)^2}-\frac{2\gamma_{25}}{\beta^3}+\frac{\gamma_{26}}{(4\beta-\beta_1)}+\frac{\gamma_{27}}{(3\beta-\beta_1)}+\frac{\gamma_{28}}{2\beta}-\frac{\gamma_{29}}{\beta}+\\&\frac{\gamma_{30}}{(2\beta-\beta_1)^2}+\frac{2\gamma_{31}}{(2\beta-\beta_1)^3}-\frac{\gamma_{32}}{(\beta+\beta_2-\beta_1)}-\frac{2\gamma_{33}}{(\beta-\beta_1)^3}+\frac{\gamma_{34}}{(\beta_2-\beta_1)}-\frac{\gamma_{35}}{(\beta_2+\beta_1)}+\frac{\gamma_{36}}{\beta_1}\end{aligned}\right)-\\&\frac{1}{\beta}\left(\frac{\gamma_{20}}{(\beta-\beta_1)}-\frac{\gamma_{24}}{(\beta-\beta_1)^2}+\frac{2\gamma_{33}}{(\beta-\beta_1)^3}\right)\left(e^{\beta}-1\right)+\frac{\gamma_{21}\left(e^{\beta_1}-1\right)}{\beta_1^2}+\frac{\left(e^{\beta}-1\right)}{\beta^2}\left(\frac{\gamma_{24}}{(\beta-\beta_1)}-\frac{2\gamma_{33}}{(\beta-\beta_1)^2}\right)\\&\frac{\gamma_{26}\left(e^{4\beta}-1\right)}{4\beta(4\beta-\beta_1)}+\frac{\gamma_{27}\left(e^{3\beta}-1\right)}{3\beta(3\beta-\beta_1)}+\frac{\gamma_{28}\left(e^{(2\beta+\beta_1)}-1\right)}{2\beta(2\beta+\beta_1)}-\frac{\gamma_{29}\left(e^{(\beta+\beta_1)}-1\right)}{\beta(\beta+\beta_1)}+\\&\frac{1}{4\beta^2}\left(\frac{\gamma_{30}}{(2\beta-\beta_1)}-\frac{2\gamma_{31}}{(2\beta-\beta_1)^2}\right)\left(e^{2\beta}-1\right)+\frac{2\gamma_{31}}{8\beta^3(2\beta-\beta_1)}\left(e^{2\beta}-1\right)-\frac{\gamma_{32}\left(e^{(\beta+\beta_2)}-1\right)}{(\beta+\beta_2)(\beta+\beta_2-\beta_1)}-\\&\frac{2\gamma_{33}\left(e^{\beta}-1\right)}{\beta^3(\beta-\beta_1)}+\frac{\gamma_{34}\left(e^{\beta_2}-1\right)}{\beta_2(\beta_2-\beta_1)}+\frac{\gamma_{35}\left(e^{-\beta}-1\right)}{\beta(\beta+\beta_1)}+\frac{\gamma_{36}\left(e^{2\beta_1}-1\right)}{2\beta_1^2}-\frac{2\gamma_{37}\left(e^{\beta_1}-1\right)}{2\beta_1^3}-\left(\frac{\gamma_{21}}{\beta_1}+\frac{\gamma_{37}}{\beta_1}-\frac{2\gamma_{37}}{2\beta_1^2}\right)e^{\beta_1}-\\&\left(\frac{1}{2}\left(\frac{\gamma_{22}}{\beta_1}+\frac{2\gamma_{25}}{\beta_1^2}\right)+\frac{\gamma_{37}}{\beta_1^2}+\frac{\gamma_{23}}{\beta_1}+\frac{2\gamma_{25}}{\beta^3}+\frac{\gamma_{25}}{3\beta_1}\right)-\left(\frac{1}{\beta}\left(\frac{\gamma_{24}}{(\beta-\beta_1)}-\frac{2\gamma_{33}}{(\beta-\beta_1)^2}\right)+\frac{\gamma_{33}}{\beta(\beta-\beta_1)}-\frac{2\gamma_{33}}{\beta^2(\beta-\beta_1)}\right)e^{\beta}-\\&\left(\frac{1}{2\beta}\left(\frac{\gamma_{30}}{(2\beta-\beta_1)}-\frac{2\gamma_{31}}{(2\beta-\beta_1)^2}\right)-\frac{\gamma_{31}}{2\beta(2\beta-\beta_1)}+\frac{2\gamma_{31}}{4\beta^2(2\beta-\beta_1)}\right)e^{2\beta}\end{aligned}\right)}{\left(e^{\beta_1}-1\right)}$$

(3.326)

Therefore,

$$\begin{aligned}\theta_1(z)&=A_{22}e^{2\beta z}+A_{23}e^{\beta_1 z}+A_{24}e^{\beta z}+A_{25}ze^{\beta_1 z}+A_{26}z^2+A_{27}z+A_{28}ze^{\beta z}+A_{29}z^3-\\&A_{30}e^{4\beta z}-A_{31}e^{3\beta z}-A_{32}e^{(2\beta+\beta_1)z}+A_{33}e^{(\beta+\beta_1)z}+A_{34}ze^{2\beta z}-A_{35}z^2e^{2\beta z}+A_{36}e^{(\beta+\beta_1)z}+\\&A_{37}z^2e^{\beta z}-A_{38}e^{\beta_2 z}-A_{39}e^{-\beta z}-A_{40}e^{2\beta_1 z}+A_{41}z^2e^{\beta_1 z}+c_{14}\end{aligned}$$

(3.327)

where

$$A_{22}=\left(\frac{1}{2\beta}\left(\frac{\gamma_{19}}{(2\beta-\beta_1)}-\frac{\gamma_{30}}{(2\beta-\beta_1)^2}+\frac{2\gamma_{31}}{(2\beta-\beta_1)^3}\right)-\frac{1}{4\beta^2}\left(\frac{\gamma_{30}}{(2\beta-\beta_1)}+\frac{2\gamma_{31}}{(2\beta-\beta_1)^2}\right)-\frac{2\gamma_{31}}{8\beta^3(2\beta-\beta_1)}\right),$$

$$A_{23}=\frac{1}{\beta_1}\left(\begin{array}{l}c_{13}-\frac{\gamma_{19}}{(2\beta-\beta_1)}-\frac{\gamma_{20}}{(\beta-\beta_1)}-\frac{\gamma_{22}}{\beta_1^2}+\frac{\gamma_{23}}{\beta_1}+\frac{\gamma_{24}}{(\beta-\beta_1)^2}-\frac{2\gamma_{25}}{\beta^3}+\frac{\gamma_{26}}{(4\beta-\beta_1)}+\frac{\gamma_{27}}{(3\beta-\beta_1)}+\\ \frac{\gamma_{28}}{2\beta}-\frac{\gamma_{29}}{\beta}+\frac{\gamma_{30}}{(2\beta-\beta_1)^2}+\frac{2\gamma_{31}}{(2\beta-\beta_1)^3}-\frac{\gamma_{32}}{(\beta+\beta_2-\beta_1)}-\frac{2\gamma_{33}}{(\beta-\beta_1)^3}+\frac{\gamma_{34}}{(\beta_2-\beta_1)}-\\ \frac{\gamma_{35}}{(\beta_2+\beta_1)}+\frac{\gamma_{36}}{\beta_1}+\frac{\gamma_{37}}{\beta_1^2}-\frac{\gamma_{21}}{\beta_1}\end{array}\right),$$

$$A_{24}=\frac{1}{\beta}\left(\frac{\gamma_{20}}{(\beta-\beta_1)}-\frac{\gamma_{24}}{(\beta-\beta_1)^2}+\frac{2\gamma_{33}}{(\beta-\beta_1)^3}-\frac{1}{\beta}\left(\frac{\gamma_{24}}{(\beta-\beta_1)}-\frac{2\gamma_{33}}{(\beta-\beta_1)^2}\right)+\frac{2\gamma_{33}}{\beta^2(\beta-\beta_1)}\right),$$

$$A_{25}=\left(\frac{\gamma_{21}}{\beta_1}-\frac{\gamma_{37}}{\beta_1^2}\right),A_{26}=\frac{1}{2}\left(\frac{\gamma_{22}}{\beta_1}-\frac{\gamma_{25}}{\beta_1^2}\right),A_{27}=\left(\frac{\gamma_{22}}{\beta_1^2}-\frac{\gamma_{23}}{\beta_1}-\frac{2\gamma_{25}}{\beta^3}\right),$$

$$A_{28}=\frac{1}{\beta}\left(\frac{\gamma_{24}}{(\beta-\beta_1)}-\frac{2\gamma_{33}}{(\beta-\beta_1)^2}-\frac{2\gamma_{33}}{\beta(\beta-\beta_1)}\right),A_{29}=\frac{\gamma_{25}}{3\beta_1},A_{30}=\frac{\gamma_{26}}{4\beta(4\beta-\beta_1)},A_{31}=\frac{\gamma_{27}}{3\beta(3\beta-\beta_1)},$$

$$A_{32}=\frac{\gamma_{28}}{2\beta(2\beta-\beta_1)},A_{33}=\frac{\gamma_{29}}{\beta(\beta+\beta_1)},A_{34}=\frac{1}{2\beta}\left(\frac{\gamma_{30}}{(2\beta-\beta_1)}+\frac{2\gamma_{31}}{(2\beta-\beta_1)^2}+\frac{\gamma_{31}}{\beta(2\beta-\beta_1)}\right),$$

$$A_{35}=\frac{\gamma_{31}}{2\beta(2\beta-\beta_1)},A_{36}=\frac{\gamma_{32}}{(\beta+\beta_2)(\beta+\beta_2-\beta_1)},A_{37}=\frac{\gamma_{33}}{\beta(\beta-\beta_1)},A_{38}=\frac{\gamma_{34}}{\beta_2(\beta_2-\beta_1)},$$

$$A_{39}=\frac{\gamma_{35}}{\beta(\beta+\beta_1)},A_{40}=\frac{\gamma_{36}}{2\beta_1^2},A_{41}=\frac{\gamma_{37}}{2\beta_1}$$

Also consider equation (3.279)

$$\begin{array}{l}\frac{d^2\phi_1}{dz^2}-\beta_2\frac{d\phi_1}{dz}=\gamma_{38}e^{2\beta z}+\gamma_{39}e^{\beta z}+\gamma_{40}e^{\beta_1 z}+\gamma_{41}e^{\beta_2 z}+\gamma_{42}-\gamma_{43}ze^{\beta_1 z}-\gamma_{44}ze^{\beta z}-\\ \gamma_{45}z+\gamma_{46}e^{4\beta z}+\gamma_{47}e^{3\beta z}+\gamma_{48}e^{(2\beta+\beta_1)z}-\gamma_{49}e^{(\beta+\beta_1)z}+\gamma_{50}ze^{2\beta z}+\gamma_{51}z^2e^{2\beta z}-\gamma_{52}e^{(\beta_1+\beta_2)z}-\\ \gamma_{53}z^2e^{\beta z}+\gamma_{54}e^{-\beta z}+\gamma_{55}e^{2\beta_1 z}-\gamma_{56}z^2e^{\beta_1 z}-\gamma_{57}e^{(2\beta+\beta_1)z}-\gamma_{58}e^{(\beta_1+\beta_2)z}+\gamma_{59}ze^{\beta_2 z},\\ \phi_1(0)=0,\phi_1(1)=1\end{array}$$

where

$$\gamma_{38}=\left(\begin{array}{l}\frac{4g_1\operatorname{Re}ScA_4}{c}-\frac{\operatorname{Re}ScT_D\left(2\beta\left(2\beta A_{22}+A_{34}\right)+\left(2\beta A_{34}-2A_{35}\right)\right)}{c}-\\ \beta\left(A_2A_5+2A_4c_6\right)-\beta\left(\beta A_2A_5-2A_4B_1\right)\end{array}\right),$$

$$\gamma_{39}=\left(\frac{4g_1\operatorname{Re}ScA_5}{c}-\frac{\operatorname{Re}ScT_D\left(\beta A_{28}+2A_{37}\right)}{c}-\beta A_5c_6+\beta A_5B_1\right),$$

$$\gamma_{40}=\left(\frac{4g_1\operatorname{Re}ScA_6}{c}+\operatorname{Re}ScT_D\left(\beta_1\left(\beta_1A_{23}+A_{25}\right)+\left(\beta_1A_{25}+2A_{41}\right)\right)-\beta_1A_6c_6+\beta_1A_6B_1\right),$$

$$\gamma_{41}=\left(\frac{4g_1\operatorname{Re}ScA_7}{c}-\frac{\operatorname{Re}ScT_D\beta_2^2A_{38}}{c}-\beta_2A_7c_6+\beta_2A_7B_1\right),$$

$$\gamma_{42}=\left(\frac{4l\operatorname{Re}Scc_6}{c}-\frac{2\operatorname{Re}ScT_DA_{26}}{c}\right),\gamma_{43}=\left(\frac{\operatorname{Re}ScT_D\left(\beta_1\left(\beta_1A_{25}+2A_{41}\right)+2\beta_1A_{41}\right)}{c}-\beta_1B_1A_6\right),$$

$$\gamma_{44}=\left(\frac{\operatorname{Re}ScT_D\left(\beta\left(\beta A_{28}+2A_{37}\right)+2\beta A_{37}\right)}{c}-\beta B_1A_5\right),\gamma_{45}=\left(\frac{6\operatorname{Re}ScT_DA_{29}}{c}\right),$$

$$\gamma_{46}=\left(\frac{16\operatorname{Re}ScT_D\beta^2A_{30}}{c}-2\beta A_1A_4-4\beta^2A_1A_4\right),$$

$$\gamma_{47}=\left(\frac{9\operatorname{Re}ScT_D\beta^2A_{31}}{c}-2\beta A_2A_4\left(1+\beta\right)-\beta A_1A_5\left(1+2\beta\right)\right),$$

$$\gamma_{48}=\left(\frac{\operatorname{Re}ScT_D\left(2\beta+\beta_1\right)^2A_{32}}{c}-\beta_1A_1A_6-2\beta A_3A_4-\beta\beta_1\left(A_1A_6+A_3A_4\right)\right),$$

$$\gamma_{49}=\left(\frac{\operatorname{Re}ScT_D\left(\beta+\beta_1\right)^2A_{33}}{c}+\beta_1A_2A_6\left(1+\beta\right)-\beta A_3A_5\left(1+\beta_1\right)\right),$$

$$\gamma_{50}=\left(\frac{\operatorname{Re}ScT_D\left(4\beta A_{35}-2\beta\left(2\beta A_{34}-2A_{35}\right)\right)}{c}+2\beta B_1A_4\right),$$

$$\gamma_{51}=\frac{4\operatorname{Re}ScT_D\beta^2A_{35}}{c},\gamma_{52}=\left(\frac{\operatorname{Re}ScT_D\left(\beta+\beta_2\right)^2A_{36}}{c}+\beta_2A_2A_7+\beta\beta_2A_2A_7\right),$$

$$\gamma_{53}=\frac{\operatorname{Re}ScT_D\beta^2A_{37}}{c},\gamma_{54}=\frac{4\operatorname{Re}ScT_D\beta^2A_{39}}{c},\gamma_{55}=\left(\frac{4\operatorname{Re}ScT_D\beta_1^2A_{40}}{c}-\beta_1A_3A_6\left(1+\beta_1\right)\right),$$

$$\gamma_{56}=\frac{4\operatorname{Re}ScT_D\beta_1^2A_{41}}{c},\gamma_{57}=\beta_2A_1A_7\left(1+2\beta_2\right),\gamma_{58}=\beta_2A_3A_7\left(1+\beta_1\right),\gamma_{59}=\beta_2B_1A_7$$

Let

$$q_7=\frac{d\phi_1}{dz} \tag{3.328}$$

Then

$$\frac{d^2q_7}{dz^2}-\beta_2 q_7=\gamma_{38}e^{2\beta z}+\gamma_{39}e^{\beta z}+\gamma_{40}e^{\beta_1 z}+\gamma_{41}e^{\beta_2 z}+\gamma_{42}-\gamma_{43}ze^{\beta_1 z}-\gamma_{44}ze^{\beta z}-$$
$$\gamma_{45}z+\gamma_{46}e^{4\beta z}+\gamma_{47}e^{3\beta z}+\gamma_{48}e^{(2\beta+\beta_1)z}-\gamma_{49}e^{(\beta+\beta_1)z}+\gamma_{50}ze^{2\beta z}+\gamma_{51}z^2e^{2\beta z}-\gamma_{52}e^{(\beta_1+\beta_2)z}- \quad (3.329)$$
$$\gamma_{53}z^2e^{\beta z}+\gamma_{54}e^{-\beta z}+\gamma_{55}e^{2\beta_1 z}-\gamma_{56}z^2e^{\beta_1 z}-\gamma_{57}e^{(2\beta+\beta_1)z}-\gamma_{58}e^{(\beta_1+\beta_2)z}+\gamma_{59}ze^{\beta_2 z}$$

So

$$q_7(z)=e^{\beta_2 z}\int_0^z\begin{pmatrix}\gamma_{38}e^{(2\beta-\beta_2)x}+\gamma_{39}e^{(\beta-\beta_2)x}+\gamma_{40}e^{(\beta_1-\beta_2)x}+\gamma_{41}+\gamma_{42}e^{-\beta_2 x}-\gamma_{43}xe^{(\beta_1-\beta_2)x}-\\ \gamma_{44}xe^{(\beta-\beta_2)x}-\gamma_{45}xe^{-\beta_2 x}+\gamma_{46}e^{(4\beta-\beta_2)x}+\gamma_{47}e^{(3\beta-\beta_2)x}+\gamma_{48}e^{(2\beta+\beta_1-\beta_2)x}-\\ \gamma_{49}e^{(\beta+\beta_1-\beta_2)x}+\gamma_{50}ze^{(2\beta-\beta_2)x}+\gamma_{51}x^2e^{(2\beta-\beta_2)x}-\gamma_{52}e^{\beta x}-\gamma_{53}x^2e^{(\beta-\beta_2)x}+\\ \gamma_{54}e^{-(\beta+\beta_2)x}+\gamma_{55}e^{(2\beta_1-\beta_2)x}-\gamma_{56}x^2e^{(\beta_1-\beta_2)x}-\gamma_{57}e^{2\beta x}-\gamma_{58}e^{\beta_1 x}+\gamma_{59}x\end{pmatrix}dx+c_{15}e^{\beta_2 z}$$

$$=\frac{\gamma_{38}\left(e^{2\beta z}-e^{\beta_2 z}\right)}{(2\beta-\beta_2)}+\frac{\gamma_{39}\left(e^{\beta z}-e^{\beta_2 z}\right)}{(\beta-\beta_2)}+\frac{\gamma_{40}\left(e^{\beta_1 z}-e^{\beta_2 z}\right)}{(\beta_1-\beta_2)}+\gamma_{41}ze^{\beta_2 z}-\frac{\gamma_{42}\left(1-e^{\beta_2 z}\right)}{\beta}-$$
$$\gamma_{43}\left(\frac{ze^{\beta_1 z}}{(\beta_1-\beta_2)}-\frac{\left(e^{\beta_1 z}-e^{\beta_2 z}\right)}{(\beta_1-\beta_2)^2}\right)-\gamma_{44}\left(\frac{ze^{\beta z}}{(\beta-\beta_2)}-\frac{\left(e^{\beta z}-e^{\beta_2 z}\right)}{(\beta_1-\beta_2)^2}\right)+\gamma_{45}\left(\frac{z}{\beta_2}+\frac{\left(1-e^{\beta_2 z}\right)}{\beta_2^2}\right)+$$
$$\frac{\gamma_{46}\left(e^{4\beta z}-e^{\beta_2 z}\right)}{(4\beta-\beta_2)}+\frac{\gamma_{47}\left(e^{3\beta z}-e^{\beta_2 z}\right)}{(3\beta-\beta_2)}+\frac{\gamma_{48}\left(e^{(2\beta+\beta_1)z}-e^{\beta_2 z}\right)}{(2\beta+\beta_1-\beta_2)}-\frac{\gamma_{49}\left(e^{(\beta+\beta_1)z}-e^{\beta_2 z}\right)}{(\beta+\beta_1-\beta_2)}+$$
$$\gamma_{50}\left(\frac{ze^{2\beta z}}{(2\beta-\beta_2)}-\frac{\left(e^{2\beta z}-e^{\beta_2 z}\right)}{(2\beta-\beta_2)^2}\right)+\gamma_{51}\left(\frac{z^2e^{2\beta z}}{(2\beta-\beta_2)}-\frac{2ze^{2\beta z}}{(2\beta-\beta_2)^2}+\frac{2\left(e^{2\beta z}-e^{\beta_2 z}\right)}{(2\beta-\beta_2)^3}\right)-$$
$$\frac{\gamma_{52}\left(e^{(\beta+\beta_2)z}-e^{\beta_2 z}\right)}{\beta}-\gamma_{53}\left(\frac{z^2e^{2\beta z}}{(\beta-\beta_2)}-\frac{2ze^{\beta z}}{(\beta-\beta_2)^2}+\frac{2\left(e^{\beta z}-e^{\beta_2 z}\right)}{(\beta-\beta_2)^3}\right)-\frac{\gamma_{54}\left(e^{-\beta z}-e^{\beta_2 z}\right)}{(\beta+\beta_2)}+$$
$$\frac{\gamma_{55}\left(e^{2\beta_1 z}-e^{\beta_2 z}\right)}{(2\beta_1-\beta_2)}-\gamma_{56}\left(\frac{z^2e^{\beta_1 z}}{(\beta_1-\beta_2)}-\frac{2ze^{\beta_1 z}}{(\beta_1-\beta_2)^2}+\frac{2\left(e^{\beta_1 z}-e^{\beta_2 z}\right)}{(\beta_1-\beta_2)^3}\right)-\frac{\gamma_{57}\left(e^{(2\beta+\beta_2)z}-e^{\beta_2 z}\right)}{2\beta}-$$
$$\frac{\gamma_{58}\left(e^{(\beta+\beta_2)z}-e^{\beta_2 z}\right)}{\beta_1}+\frac{\gamma_{52}z^2e^{\beta_2 z}}{2}+c_{15}e^{\beta_2 z}$$

That is

$$q_7(z)=\left(\frac{\gamma_{38}}{(2\beta-\beta_2)}-\frac{\gamma_{50}}{(2\beta-\beta_2)^2}+\frac{2\gamma_{51}}{(2\beta-\beta_2)^3}\right)e^{2\beta z}-$$

$$\left(\begin{array}{l}\frac{\gamma_{38}}{(2\beta-\beta_2)}+\frac{\gamma_{39}}{(\beta-\beta_2)}+\frac{\gamma_{40}}{(\beta_1-\beta_2)}-\frac{\gamma_{42}}{\beta_2}+\frac{\gamma_{43}}{(\beta_1-\beta_2)^2}+\frac{\gamma_{44}}{(\beta-\beta_2)^2}+\frac{\gamma_{45}}{\beta_2^2}+\\ \frac{\gamma_{46}}{(4\beta-\beta_2)}+\frac{\gamma_{47}}{(3\beta-\beta_2)}+\frac{\gamma_{48}}{(2\beta+\beta_1-\beta_2)}-\frac{\gamma_{49}}{(\beta+\beta_1-\beta_2)}-\frac{\gamma_{50}}{(2\beta-\beta_2)^2}+\\ \frac{2\gamma_{51}}{(2\beta-\beta_2)^3}-\frac{\gamma_{52}}{\beta}-\frac{2\gamma_{53}}{(\beta-\beta_2)^3}-\frac{\gamma_{54}}{(\beta+\beta_2)}+\frac{\gamma_{55}}{(2\beta_1-\beta_2)}-\frac{2\gamma_{56}}{(\beta_1-\beta_2)^2}-\frac{\gamma_{57}}{2\beta}-\frac{\gamma_{58}}{\beta_1}-c_{15}\end{array}\right)e^{\beta_2 z}+$$

$$\left(\frac{\gamma_{39}}{(\beta-\beta_2)}+\frac{\gamma_{44}}{(\beta-\beta_2)^2}-\frac{2\gamma_{53}}{(\beta-\beta_2)^3}\right)e^{\beta z}+\left(\frac{\gamma_{40}}{(\beta_1-\beta_2)}+\frac{\gamma_{43}}{(\beta_1-\beta_2)^2}-\frac{2\gamma_{56}}{(\beta_1-\beta_2)^3}\right)e^{\beta_1 z}+\gamma_{41}ze^{\beta_2 z}-$$

$$\left(\frac{\gamma_{42}}{\beta_2}-\frac{\gamma_{45}}{\beta_2^2}\right)-\left(\frac{\gamma_{43}}{(\beta_1-\beta_2)}+\frac{2\gamma_{56}}{(\beta_1-\beta_2)^2}\right)ze^{\beta_1 z}-\left(\frac{\gamma_{44}}{(\beta_1-\beta_2)}+\frac{2\gamma_{53}}{(\beta_1-\beta_2)^2}\right)ze^{\beta z}+\frac{\gamma_{45}}{\beta_2}z+\frac{\gamma_{46}}{(4\beta-\beta_2)}e^{4\beta z}+$$

$$\frac{\gamma_{47}}{(3\beta-\beta_2)}e^{3\beta z}+\frac{\gamma_{48}e^{(2\beta+\beta_1)z}}{(2\beta+\beta_1-\beta_2)}-\frac{\gamma_{49}e^{(\beta+\beta_1)z}}{(\beta+\beta_1-\beta_2)}+\left(\frac{\gamma_{50}}{(2\beta-\beta_2)}+\frac{2\gamma_{56}}{(2\beta-\beta_2)^2}\right)ze^{2\beta z}+\frac{\gamma_{51}z^2e^{2\beta z}}{(2\beta-\beta_2)}-$$

$$\frac{\gamma_{52}e^{(\beta+\beta_2)z}}{\beta}-\frac{\gamma_{53}z^2e^{\beta z}}{(\beta-\beta_2)}-\frac{\gamma_{54}e^{-\beta z}}{(\beta+\beta_2)}+\frac{\gamma_{55}e^{2\beta_1 z}}{(2\beta_1-\beta_2)}-\frac{\gamma_{56}z^2e^{\beta_1 z}}{(\beta_1-\beta_2)}-\frac{\gamma_{57}e^{(2\beta+\beta_2)z}}{2\beta}-\frac{\gamma_{58}e^{(\beta+\beta_2)z}}{\beta_1}+\frac{\gamma_{59}z^2e^{\beta_2 z}}{2}$$

(3.330)

That is

$$\frac{d\phi_1}{dz}=q_7(z)$$

Integrating $q_7(z)$ with respect to z we obtain

$$\phi_1(z)=\frac{1}{2\beta}\left(\frac{\gamma_{38}}{(2\beta-\beta_2)}-\frac{\gamma_{50}}{(2\beta-\beta_2)^2}+\frac{2\gamma_{51}}{(2\beta-\beta_2)^3}\right)e^{2\beta z}-$$

$$\frac{1}{\beta_2}\left(\begin{array}{l}\frac{\gamma_{38}}{(2\beta-\beta_2)}+\frac{\gamma_{39}}{(\beta-\beta_2)}+\frac{\gamma_{40}}{(\beta_1-\beta_2)}-\frac{\gamma_{42}}{\beta_2}+\frac{\gamma_{43}}{(\beta_1-\beta_2)^2}+\frac{\gamma_{44}}{(\beta-\beta_2)^2}+\frac{\gamma_{45}}{\beta_2^2}+\\ \frac{\gamma_{46}}{(4\beta-\beta_2)}+\frac{\gamma_{47}}{(3\beta-\beta_2)}+\frac{\gamma_{48}}{(2\beta+\beta_1-\beta_2)}-\frac{\gamma_{49}}{(\beta+\beta_1-\beta_2)}-\frac{\gamma_{50}}{(2\beta-\beta_2)^2}+\\ \frac{2\gamma_{51}}{(2\beta-\beta_2)^3}-\frac{\gamma_{52}}{\beta}-\frac{2\gamma_{53}}{(\beta-\beta_2)^3}-\frac{\gamma_{54}}{(\beta+\beta_2)}+\frac{\gamma_{55}}{(2\beta_1-\beta_2)}-\frac{2\gamma_{56}}{(\beta_1-\beta_2)^2}-\frac{\gamma_{57}}{2\beta}-\frac{\gamma_{58}}{\beta_1}-c_{15}\end{array}\right)e^{\beta_2 z}+$$

$$\frac{1}{\beta}\left(\frac{\gamma_{39}}{(\beta-\beta_2)}+\frac{\gamma_{44}}{(\beta-\beta_2)^2}-\frac{2\gamma_{53}}{(\beta-\beta_2)^3}\right)e^{\beta z}+\frac{1}{\beta_1}\left(\frac{\gamma_{40}}{(\beta_1-\beta_2)}+\frac{\gamma_{43}}{(\beta_1-\beta_2)^2}-\frac{2\gamma_{56}}{(\beta_1-\beta_2)^3}\right)e^{\beta_1 z}+$$

$$\gamma_{41}\left(\frac{ze^{\beta_2 z}}{\beta_2}-\frac{e^{\beta_2 z}}{\beta_2^2}\right)-\left(\frac{\gamma_{42}}{\beta_2}-\frac{\gamma_{45}}{\beta_2^2}\right)z-\left(\frac{\gamma_{43}}{(\beta_1-\beta_2)}+\frac{2\gamma_{56}}{(\beta_1-\beta_2)^2}\right)\left(\frac{ze^{\beta_1 z}}{\beta_1}-\frac{e^{\beta_1 z}}{\beta_1^2}\right)-$$

$$\left(\frac{\gamma_{44}}{(\beta_1-\beta_2)}+\frac{2\gamma_{53}}{(\beta_1-\beta_2)^2}\right)\left(\frac{ze^{\beta z}}{\beta}-\frac{e^{\beta z}}{\beta^2}\right)+\frac{\gamma_{45}}{\beta_2}z^2+\frac{\gamma_{46}}{4\beta(4\beta-\beta_2)}e^{4\beta z}+\frac{\gamma_{47}}{3\beta(3\beta-\beta_2)}e^{3\beta z}+$$

$$\frac{\gamma_{48}e^{(2\beta+\beta_1)z}}{(2\beta+\beta)(2\beta+\beta_1-\beta_2)}-\frac{\gamma_{49}e^{(\beta+\beta_1)z}}{(\beta+\beta_1)(\beta+\beta_1-\beta_2)}+\left(\frac{\gamma_{50}}{(2\beta-\beta_2)}+\frac{2\gamma_{51}}{(2\beta-\beta_2)^2}\right)\left(\frac{ze^{2\beta z}}{2\beta}-\frac{e^{2\beta z}}{4\beta^2}\right)+$$

$$\frac{\gamma_{51}}{(2\beta-\beta_2)}\left(\frac{z^2e^{2\beta z}}{2\beta}-\frac{2ze^{2\beta z}}{4\beta^2}+\frac{2e^{2\beta z}}{8\beta^3}\right)-\frac{\gamma_{52}e^{(\beta+\beta_2)z}}{\beta(\beta+\beta_2)}-\frac{\gamma_{53}}{(\beta-\beta_2)}\left(\frac{z^2e^{\beta z}}{\beta}-\frac{2ze^{\beta z}}{\beta^2}+\frac{2e^{\beta z}}{\beta^3}\right)-$$

$$\frac{\gamma_{54}e^{-\beta z}}{\beta(\beta+\beta_2)}+\frac{\gamma_{55}e^{2\beta_1 z}}{2\beta_1(2\beta_1-\beta_2)}-\frac{\gamma_{56}}{(\beta_1-\beta_2)}\left(\frac{z^2e^{\beta_1 z}}{\beta_1}-\frac{2ze^{\beta_1 z}}{\beta_1^2}+\frac{2e^{\beta_1 z}}{\beta_1^3}\right)-\frac{\gamma_{57}e^{(2\beta+\beta_2)z}}{2\beta(2\beta+\beta_2)}-\frac{\gamma_{58}e^{(\beta+\beta_2)z}}{\beta_1(\beta_1+\beta_2)}+$$

$$\frac{\gamma_{59}}{2}\left(\frac{z^2e^{\beta_2 z}}{\beta_2}-\frac{2ze^{\beta_2 z}}{\beta_2^2}+\frac{2e^{\beta_2 z}}{\beta_2^3}\right)+c_{16}$$

(3.331)

$$
\begin{aligned}
&\phi_1(0)=0 \Rightarrow c_{16} = -\frac{1}{2\beta}\left(\frac{\gamma_{38}}{(2\beta-\beta_2)} - \frac{\gamma_{50}}{(2\beta-\beta_2)^2} + \frac{2\gamma_{51}}{(2\beta-\beta_2)^3}\right) + \\
&\frac{1}{\beta_2}\left(\begin{array}{l}
\frac{\gamma_{38}}{(2\beta-\beta_2)} + \frac{\gamma_{39}}{(\beta-\beta_2)} + \frac{\gamma_{40}}{(\beta_1-\beta_2)} - \frac{\gamma_{42}}{\beta_2} + \frac{\gamma_{43}}{(\beta_1-\beta_2)^2} + \frac{\gamma_{44}}{(\beta-\beta_2)^2} + \frac{\gamma_{45}}{\beta_2^2} + \\
\frac{\gamma_{46}}{(4\beta-\beta_2)} + \frac{\gamma_{47}}{(3\beta-\beta_2)} + \frac{\gamma_{48}}{(2\beta+\beta_1-\beta_2)} - \frac{\gamma_{49}}{(\beta+\beta_1-\beta_2)} - \frac{\gamma_{50}}{(2\beta-\beta_2)^2} + \\
\frac{2\gamma_{51}}{(2\beta-\beta_2)^3} - \frac{\gamma_{52}}{\beta} - \frac{2\gamma_{53}}{(\beta-\beta_2)^3} - \frac{\gamma_{54}}{(\beta+\beta_2)} + \frac{\gamma_{55}}{(2\beta_1-\beta_2)} - \frac{2\gamma_{56}}{(\beta_1-\beta_2)^2} - \frac{\gamma_{57}}{2\beta} - \frac{\gamma_{58}}{\beta_1} - c_{15}
\end{array}\right) - \\
&\frac{1}{\beta}\left(\frac{\gamma_{39}}{(\beta-\beta_2)} + \frac{\gamma_{44}}{(\beta-\beta_2)^2} - \frac{2\gamma_{53}}{(\beta-\beta_2)^3}\right) - \frac{1}{\beta_1}\left(\frac{\gamma_{40}}{(\beta_1-\beta_2)} + \frac{\gamma_{43}}{(\beta_1-\beta_2)^2} - \frac{2\gamma_{56}}{(\beta_1-\beta_2)^3}\right) + \\
&\frac{\gamma_{41}}{\beta_2^2} - \frac{1}{\beta_1^2}\left(\frac{\gamma_{43}}{(\beta_1-\beta_2)} + \frac{2\gamma_{56}}{(\beta_1-\beta_2)^2}\right) - \frac{1}{\beta^2}\left(\frac{\gamma_{44}}{(\beta_1-\beta_2)} - \frac{2\gamma_{53}}{(\beta_1-\beta_2)^2}\right) - \frac{\gamma_{46}}{4\beta(4\beta-\beta_2)} - \frac{\gamma_{47}}{3\beta(3\beta-\beta_2)} - \\
&\frac{\gamma_{48}}{(2\beta+\beta)(2\beta+\beta_1-\beta_2)} + \frac{\gamma_{49}}{(\beta+\beta_1)(\beta+\beta_1-\beta_2)} + \frac{1}{4\beta^2}\left(\frac{\gamma_{50}}{(2\beta-\beta_2)} - \frac{2\gamma_{51}}{(2\beta-\beta_2)^2}\right) - \frac{2\gamma_{51}}{8\beta^3(2\beta-\beta_2)} + \\
&\frac{\gamma_{52}}{\beta(\beta+\beta_2)} + \frac{2\gamma_{53}}{\beta^3(\beta-\beta_2)} - \frac{\gamma_{54}}{\beta(\beta+\beta_2)} - \frac{\gamma_{55}}{2\beta_1(2\beta_1-\beta_2)} - \frac{2\gamma_{56}}{\beta_1^3(\beta_1-\beta_2)} + \frac{\gamma_{57}}{2\beta(2\beta+\beta_2)} + \frac{\gamma_{58}}{\beta_1(\beta_1+\beta_2)} - \frac{\gamma_{59}}{\beta_2^3}
\end{aligned}
\tag{3.332}
$$

$$\phi_1(1)=0 \Rightarrow c_{15}\frac{1}{\beta_2} =$$

$$\frac{\begin{gathered}\frac{1}{\beta_2}\left(\begin{array}{l} \frac{\gamma_{38}}{(2\beta-\beta_2)}+\frac{\gamma_{39}}{(\beta-\beta_2)}+\frac{\gamma_{40}}{(\beta_1-\beta_2)}-\frac{\gamma_{42}}{\beta_2}+\frac{\gamma_{43}}{(\beta_1-\beta_2)^2}+\frac{\gamma_{44}}{(\beta-\beta_2)^2}+\frac{\gamma_{45}}{\beta_2^2}+ \\ \frac{\gamma_{46}}{(4\beta-\beta_2)}+\frac{\gamma_{47}}{(3\beta-\beta_2)}+\frac{\gamma_{48}}{(2\beta+\beta_1-\beta_2)}-\frac{\gamma_{49}}{(\beta+\beta_1-\beta_2)}-\frac{\gamma_{50}}{(2\beta-\beta_2)^2}+ \\ \frac{2\gamma_{51}}{(2\beta-\beta_2)^3}-\frac{\gamma_{52}}{\beta}-\frac{2\gamma_{53}}{(\beta-\beta_2)^3}-\frac{\gamma_{54}}{(\beta+\beta_2)}+\frac{\gamma_{55}}{(2\beta_1-\beta_2)}-\frac{2\gamma_{56}}{(\beta_1-\beta_2)^2}-\frac{\gamma_{57}}{2\beta}-\frac{\gamma_{58}}{\beta_1}\end{array}\right)(e^{\beta_2}-1)- \\
-\frac{1}{2\beta}\left(\frac{\gamma_{38}}{(2\beta-\beta_2)}-\frac{\gamma_{50}}{(2\beta-\beta_2)^2}+\frac{2\gamma_{51}}{(2\beta-\beta_2)^3}\right)(e^{2\beta}-1)- \\
\frac{1}{\beta}\left(\frac{\gamma_{39}}{(\beta-\beta_2)}+\frac{\gamma_{44}}{(\beta-\beta_2)^2}-\frac{2\gamma_{53}}{(\beta-\beta_2)^3}\right)(e^{\beta}-1)- \\
\frac{1}{\beta_1}\left(\frac{\gamma_{40}}{(\beta_1-\beta_2)}+\frac{\gamma_{43}}{(\beta_1-\beta_2)^2}-\frac{2\gamma_{56}}{(\beta_1-\beta_2)^3}\right)(e^{\beta_1}-1)+\frac{\gamma_{41}}{\beta_2^2}(e^{\beta_2}-1)- \\
\frac{1}{\beta_1^2}\left(\frac{\gamma_{43}}{(\beta_1-\beta_2)}-\frac{2\gamma_{56}}{(\beta_1-\beta_2)^2}\right)(e^{\beta_1}-1)- \\
\frac{1}{\beta^2}\left(\frac{\gamma_{44}}{(\beta_1-\beta_2)}-\frac{2\gamma_{53}}{(\beta_1-\beta_2)^2}\right)(e^{\beta}-1)-\frac{\gamma_{46}(e^{4\beta}-1)}{4\beta(4\beta-\beta_2)}-\frac{\gamma_{47}(e^{3\beta}-1)}{3\beta(3\beta-\beta_2)}-\frac{\gamma_{48}(e^{(2\beta+\beta_1)}-1)}{(2\beta+\beta)(2\beta+\beta_1-\beta_2)}+ \\
\frac{\gamma_{49}(e^{(\beta+\beta_1)}-1)}{(\beta+\beta_1)(\beta+\beta_1-\beta_2)}+\frac{1}{4\beta^2}\left(\frac{\gamma_{50}}{(2\beta-\beta_2)}-\frac{2\gamma_{51}}{(2\beta-\beta_2)^2}\right)(e^{2\beta}-1)-\frac{2\gamma_{51}(e^{2\beta}-1)}{(2\beta-\beta_2)}+ \\
\frac{\gamma_{52}(e^{(\beta+\beta_2)}-1)}{\beta(\beta+\beta_2)}+\frac{2\gamma_{53}(e^{\beta}-1)}{\beta^3(\beta-\beta_2)}-\frac{\gamma_{54}(e^{-\beta}-1)}{\beta(\beta+\beta_2)}-\frac{\gamma_{55}(e^{2\beta_1}-1)}{2\beta_1(2\beta_1-\beta_2)}-\frac{2\gamma_{56}(e^{\beta_1}-1)}{\beta_1^3(\beta_1-\beta_2)}+\frac{\gamma_{57}(e^{(2\beta+\beta_2)}-1)}{2\beta(2\beta+\beta_2)}+ \\
\frac{\gamma_{58}(e^{(\beta_1+\beta_2)}-1)}{\beta_1(\beta_1+\beta_2)}-\frac{\gamma_{59}(e^{\beta_2}-1)}{\beta_2^3}-\left(\frac{\gamma_{41}}{\beta_2}+\frac{\gamma_{59}}{2\beta_2}-\frac{\gamma_{59}}{\beta_2^2}\right)e^{\beta_2}+\left(\frac{\gamma_{42}}{\beta_2}-\frac{\gamma_{45}}{\beta_2^2}-\frac{\gamma_{45}}{2\beta_2}\right)+ \\
\frac{1}{\beta_1}\left(\frac{\gamma_{43}}{(\beta_1-\beta_2)}-\frac{2\gamma_{56}}{(\beta_1-\beta_2)^2}+\frac{\gamma_{56}}{(\beta_1-\beta_2)}-\frac{2\gamma_{56}}{\beta_1(\beta_1-\beta_2)}\right)e^{\beta_1}+ \\
\frac{1}{\beta}\left(\frac{\gamma_{44}}{(\beta_1-\beta_2)}-\frac{2\gamma_{53}}{(\beta_1-\beta_2)^2}+\frac{\gamma_{53}}{(\beta_1-\beta_2)}-\frac{2\gamma_{53}}{\beta_1(\beta_1-\beta_2)}\right)e^{\beta}- \\
\frac{1}{2\beta}\left(\frac{\gamma_{50}}{(2\beta-\beta_2)}-\frac{2\gamma_{51}}{(2\beta-\beta_2)^2}+\frac{\gamma_{51}}{(2\beta-\beta_2)}-\frac{2\gamma_{51}}{2\beta(2\beta-\beta_2)}\right)e^{2\beta}\end{gathered}}{(e^{\beta_2}-1)} \tag{3.333}$$

$$\phi_1(z) = A_{42}e^{2\beta z} - A_{43}e^{\beta_2 z} + A_{44}e^{\beta z} + A_{45}e^{\beta_1 z} + A_{46}ze^{\beta_2 z} - A_{47}z - A_{48}ze^{\beta_1 z} - A_{49}ze^{\beta z} + A_{50}z^2 + A_{51}e^{4\beta z} + A_{52}e^{3\beta z} + A_{53}e^{(2\beta+\beta_1)z} - A_{54}e^{(\beta+\beta_1)z} + A_{55}ze^{2\beta z} + A_{56}z^2e^{2\beta z} - A_{57}e^{(\beta+\beta_2)z} - A_{58}z^2e^{\beta z} + A_{59}e^{-\beta z} + A_{60}e^{2\beta_1 z} - A_{61}z^2e^{\beta_1 z} - A_{62}e^{(2\beta+\beta_2)z} - A_{63}e^{(\beta_1+\beta_2)z} + A_{64}z^2e^{\beta_2 z} + c_{16} \tag{3.334}$$

where

$$A_{42} = \left(\frac{1}{2\beta}\left(\frac{\gamma_{38}}{(2\beta-\beta_2)} - \frac{\gamma_{50}}{(2\beta-\beta_2)^2} + \frac{2\gamma_{51}}{(2\beta-\beta_2)^3} \right) - \frac{1}{4\beta^2}\left(\frac{\gamma_{50}}{(2\beta-\beta_2)} - \frac{2\gamma_{51}}{(2\beta-\beta_2)^2} \right) + \frac{\gamma_{51}}{4\beta^3(2\beta-\beta_2)} \right),$$

$$A_{43} = \frac{1}{\beta_2}\left(\begin{aligned} &\frac{\gamma_{38}}{(2\beta-\beta_2)} + \frac{\gamma_{39}}{(\beta-\beta_2)} + \frac{\gamma_{40}}{(\beta_1-\beta_2)} - \frac{\gamma_{42}}{\beta_2} + \frac{\gamma_{43}}{(\beta_1-\beta_2)^2} + \frac{\gamma_{44}}{(\beta-\beta_2)^2} + \frac{\gamma_{45}}{\beta_2^2} + \\ &\frac{\gamma_{46}}{(4\beta-\beta_2)} + \frac{\gamma_{47}}{(3\beta-\beta_2)} + \frac{\gamma_{48}}{(2\beta+\beta_1-\beta_2)} - \frac{\gamma_{49}}{(\beta+\beta_1-\beta_2)} - \frac{\gamma_{50}}{(2\beta-\beta_2)^2} + \\ &\frac{2\gamma_{51}}{(2\beta-\beta_2)^3} - \frac{\gamma_{52}}{\beta} - \frac{2\gamma_{53}}{(\beta-\beta_2)^3} - \frac{\gamma_{54}}{(\beta+\beta_2)} + \frac{\gamma_{55}}{(2\beta_1-\beta_2)} - \frac{2\gamma_{56}}{(\beta_1-\beta_2)^2} - \frac{\gamma_{57}}{2\beta} - \frac{\gamma_{58}}{\beta_1} + \\ &\frac{\gamma_{41}}{\beta_2} - \frac{\gamma_{59}}{\beta_2^2} \end{aligned} \right)$$

$$A_{44} = \frac{1}{\beta}\left(\frac{\gamma_{39}}{(\beta-\beta_2)} + \frac{\gamma_{44}}{(\beta-\beta_2)^2} - \frac{2\gamma_{53}}{(\beta-\beta_2)^3} + \frac{1}{\beta}\left(\frac{\gamma_{44}}{(\beta_1-\beta_2)} - \frac{2\gamma_{53}}{(\beta_1-\beta_2)^2} \right) - \frac{2\gamma_{53}}{\beta^2(\beta_1-\beta_2)} \right),$$

$$A_{45} = \frac{1}{\beta_1}\left(\frac{\gamma_{40}}{(\beta_1-\beta_2)} + \frac{\gamma_{43}}{(\beta_1-\beta_2)^2} - \frac{2\gamma_{56}}{(\beta_1-\beta_2)^3} + \frac{1}{\beta_1}\left(\frac{\gamma_{43}}{(\beta_1-\beta_2)} - \frac{2\gamma_{56}}{(\beta_1-\beta_2)^2} \right) - \frac{2\gamma_{56}}{\beta_1^2(\beta_1-\beta_2)} \right),$$

$$A_{46} = \left(\frac{\gamma_{41}}{\beta_2} - \frac{\gamma_{59}}{\beta_2^2} \right), A_{47} = \left(\frac{\gamma_{42}}{\beta_2} - \frac{\gamma_{45}}{\beta_2^2} \right), A_{48} = \frac{1}{\beta_1}\left(\frac{\gamma_{43}}{(\beta_1-\beta_2)} - \frac{2\gamma_{56}}{(\beta_1-\beta_2)^2} - \frac{2\gamma_{56}}{\beta_1(\beta_1-\beta_2)} \right),$$

$$A_{49} = \frac{1}{\beta}\left(\frac{\gamma_{44}}{(\beta_1-\beta_2)} - \frac{2\gamma_{53}}{(\beta_1-\beta_2)^2} - \frac{2\gamma_{53}}{\beta(\beta_1-\beta_2)} \right), A_{50} = \frac{\gamma_{45}}{2\beta_2}, A_{51} = \frac{\gamma_{46}}{4\beta(4\beta-\beta_2)},$$

$$A_{52} = \frac{\gamma_{47}}{3\beta(3\beta-\beta_2)}, \quad A_{53} = \frac{\gamma_{48}}{(2\beta+\beta_1)(2\beta+\beta_1-\beta_2)}, A_{54} = \frac{\gamma_{49}}{(\beta+\beta_1)(\beta+\beta_1-\beta_2)},$$

$$A_{55} = \left(\frac{\gamma_{50}}{(2\beta-\beta_2)} - \frac{2\gamma_{51}}{(2\beta-\beta_2)^2} + \frac{\gamma_{51}}{2\beta^2(2\beta-\beta_2)} \right), A_{56} = \frac{\gamma_{51}}{2\beta(2\beta-\beta_2)}, A_{57} = \frac{\gamma_{52}}{\beta(\beta+\beta_2)},$$

$$A_{58} = \frac{\gamma_{53}}{\beta(\beta-\beta_2)}, A_{59} = \frac{\gamma_{54}}{\beta(\beta+\beta_2)}, A_{60} = \frac{\gamma_{55}}{2\beta_1(2\beta_1-\beta_2)}, A_{61} = \frac{\gamma_{56}}{\beta_1(\beta_1-\beta_2)}, A_{62} = \frac{\gamma_{57}}{2\beta(2\beta+\beta_2)},$$

$$A_{63} = \frac{\gamma_{58}}{\beta_1(\beta_1+\beta_2)}, A_{64} = \frac{\gamma_{59}}{2\beta_2}$$

Therefore the solutions to case 3 are:

$$u(z) = A\left(e^{\beta z} - 1\right) - Bz + \alpha\begin{pmatrix} A_8 z^2 e^{\beta z} + A_9 e^{\beta z} + A_{10} e^{\beta z} + A_{11} e^{3\beta z} - A_{12} e^{2\beta z} - A_{13} e^{(\beta+\beta_1)z} + A_{14} e^{\beta_1 z} - A_{15} e^{\beta_2 z} - \\ A_{16} e^{-\beta z} - A_{17} z^2 + c_{10} \end{pmatrix} \tag{3.335}$$

$$w(z) = A_{18} e^{\beta z} - A_{19} z e^{\beta z} - A_{20} z^2 - A_{21} z + c_{12} \tag{3.336}$$

$$\theta(z) = A_1 e^{2\beta z} + A_2 e^{\beta z} + A_3 e^{\beta_1 z} - B_1 z + c_6 + \alpha\begin{pmatrix} A_{22} e^{2\beta z} + A_{23} e^{\beta_1 z} + A_{24} e^{\beta z} + A_{25} z e^{\beta_1 z} + A_{26} z^2 + A_{27} z + A_{28} z e^{\beta z} + A_{29} z^3 - A_{30} e^{4\beta z} - \\ A_{31} e^{3\beta z} - A_{32} e^{(2\beta+\beta_1)z} + A_{33} e^{(\beta+\beta_1)z} + A_{34} z e^{2\beta z} - A_{35} z^2 e^{2\beta z} + A_{36} e^{(\beta+\beta_1)z} + A_{37} z^2 e^{\beta z} - \\ A_{38} e^{\beta_2 z} - A_{39} e^{-\beta z} - A_{40} e^{2\beta_1 z} + A_{41} z^2 e^{\beta_1 z} + c_{14} \end{pmatrix} \tag{3.337}$$

$$\phi(z) = A_4 e^{2\beta z} + A_5 e^{\beta z} + A_6 e^{\beta_1 z} + A_7 e^{\beta_2 z} + c_8 + \alpha\begin{pmatrix} A_{42} e^{2\beta z} - A_{43} e^{\beta_2 z} + A_{44} e^{\beta z} + A_{45} e^{\beta_1 z} + A_{46} z e^{\beta_2 z} - A_{47} z - A_{48} z e^{\beta_1 z} - A_{49} z e^{\beta z} + A_{50} z^2 + \\ A_{51} e^{4\beta z} + A_{52} e^{3\beta z} + A_{53} e^{(2\beta+\beta_1)z} - A_{54} e^{(\beta+\beta_1)z} + A_{55} z e^{2\beta z} + A_{56} z^2 e^{2\beta z} - A_{57} e^{(\beta+\beta_2)z} - \\ A_{58} z^2 e^{\beta z} + A_{59} e^{-\beta z} + A_{60} e^{2\beta_1 z} - A_{61} z^2 e^{\beta_1 z} - A_{62} e^{(2\beta+\beta_2)z} - A_{63} e^{(\beta_1+\beta_2)z} + A_{64} z^2 e^{\beta_2 z} + c_{16} \end{pmatrix} \tag{3.338}$$

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

4.1 Analysis of Results

In this analysis, we established the criteria for existence of unique solutions of the model. This is to show the solution of the model formulated depends continuously on the initial and boundary conditions; that is the model is well posed. Also, we examined the properties of solution of the model formulated, this is to show the behaviour of the solution when values are assigned to some key parameters of the model. We solved the equations analytically using parameter expanding method and eigenfunction expansion technique. This is to see the effect of parameters involved on the concentration, temperature, primary and secondary velocities.

Finally, we examined the effect of the Reynolds number (Re), Radiation parameter (Ra), suction parameter (S), porosity parameter (P), thermal diffusion parameter (T_D), constant pressure gradient (λ), coefficient of viscosity (α), time dependent pressure gradient (ε), Prandtl number (Pr), Hartman number (Ha), chemical reaction parameter (K_r), Eckert number (Ec) , ion slip parameter (Bi) and Hall parameter (Be) on the steady and unsteady state problems. Analytical solutions of the model equations were computed using computer symbolic algebraic package MAPLE 17. The results obtained are shown in Figures 4.1 to 4.16 for case 1, Figures 4.17 to 4.27 for case 2 and Figures 4.27 to 4.45 for case 3.

The relationships between primary velocity along distance and with time for different values of radiation parameter and suction are displayed in Figures 4.1 to 4.3. Relationship between secondary velocity with time and distance for different values of

Reynolds number are displayed in Figures 4.4 to 4.5. The relationship of other controlling parameters with the concentration, temperature, primary and secondary velocities is depicted in Figures 4.6 to 4.45.

4.1.1 Graphs of case 1

In this section, the results of case 1 problem are presented as shown below

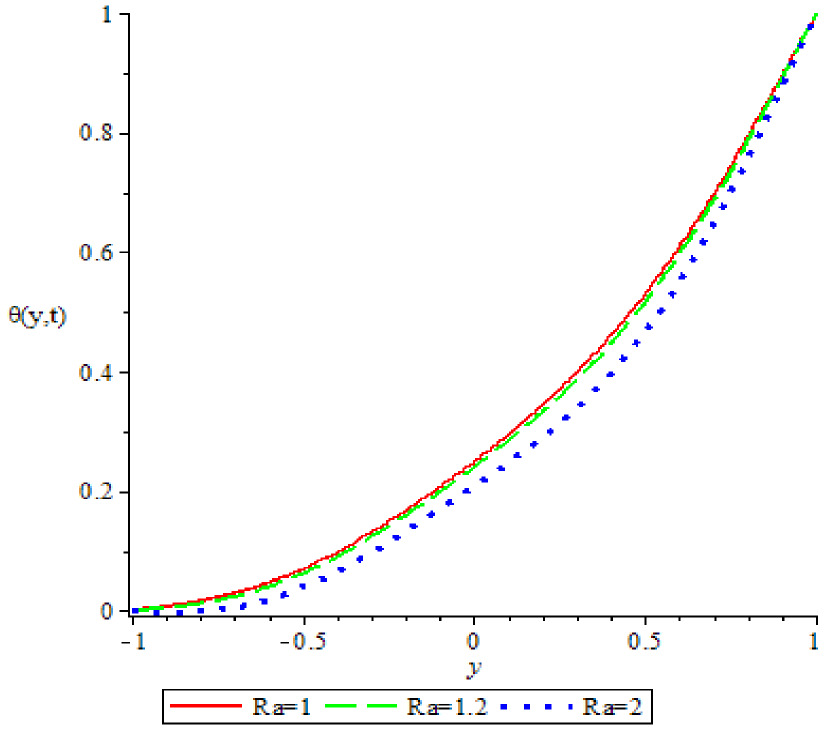

Figure 4.1: Effect of radiation parameter (Ra) on temperature profile $\theta(y,t)$ along distance

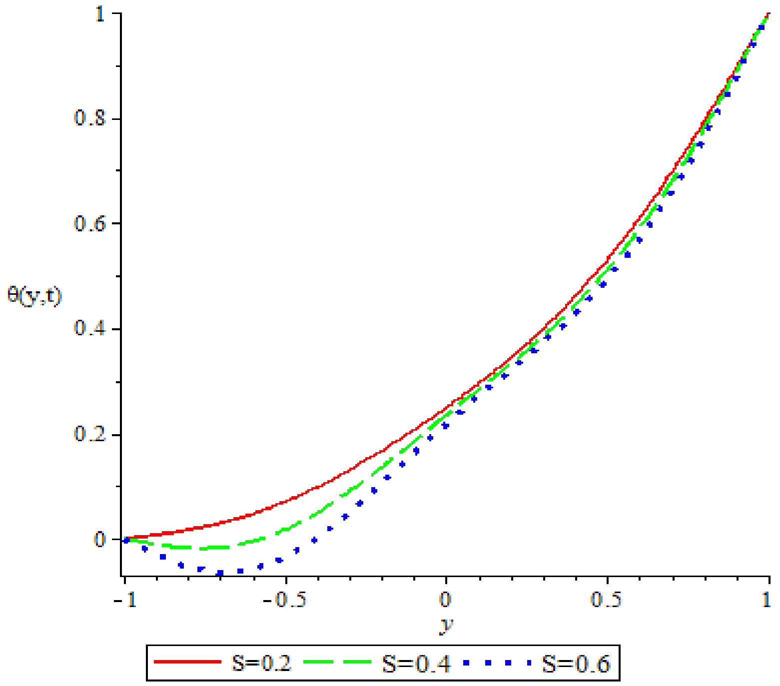

Figure 4.2: Effect of suction parameter (S) on temperature profile $\theta(y,t)$ along distance

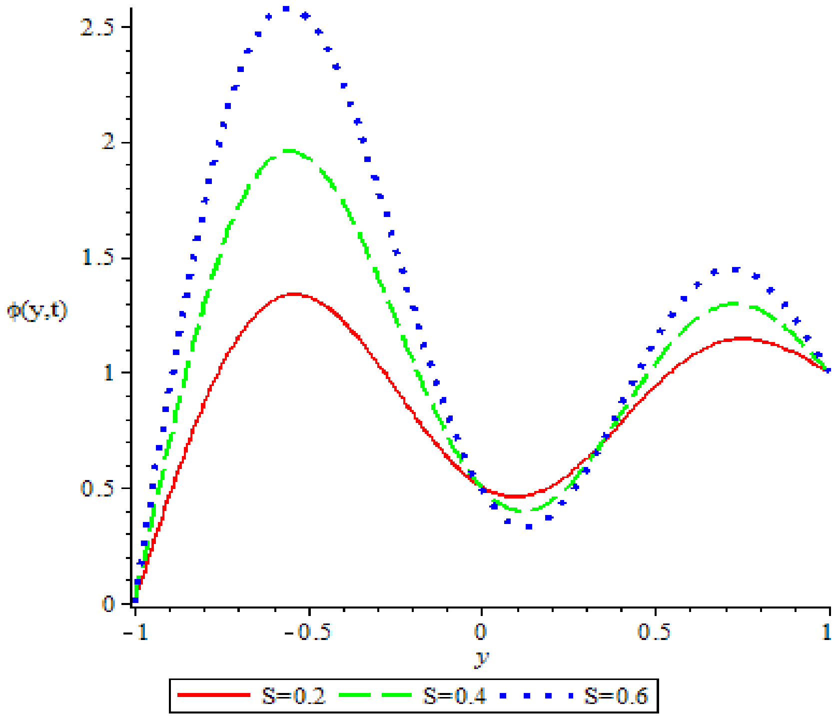

Figure 4.3: Effect of suction parameter (S) on concentration profile $\phi(y,t)$

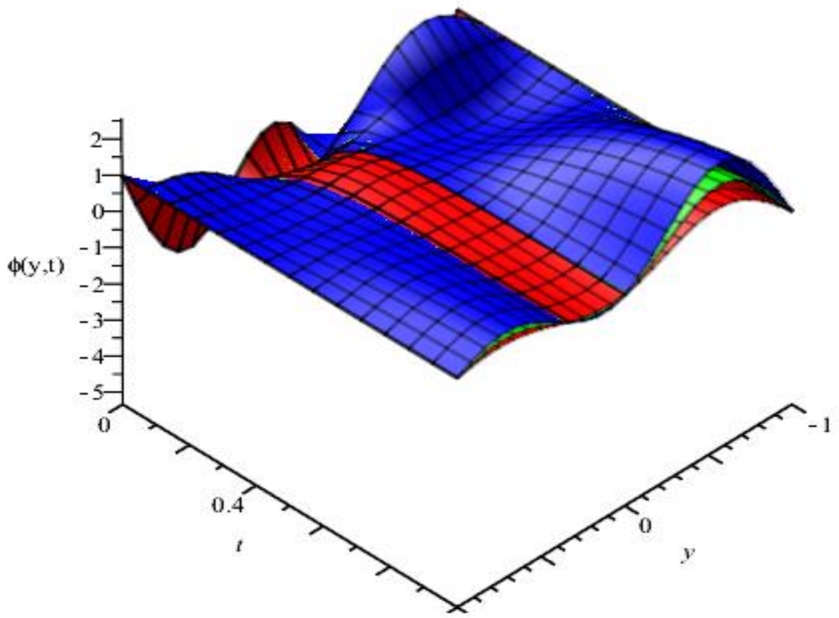

Figure 4.4: Effect of suction parameter (S) on concentration profile $\phi(y,t)$ along distance y and time t. S=0.2(red), S=0.4(green) and S=0.6(blue)

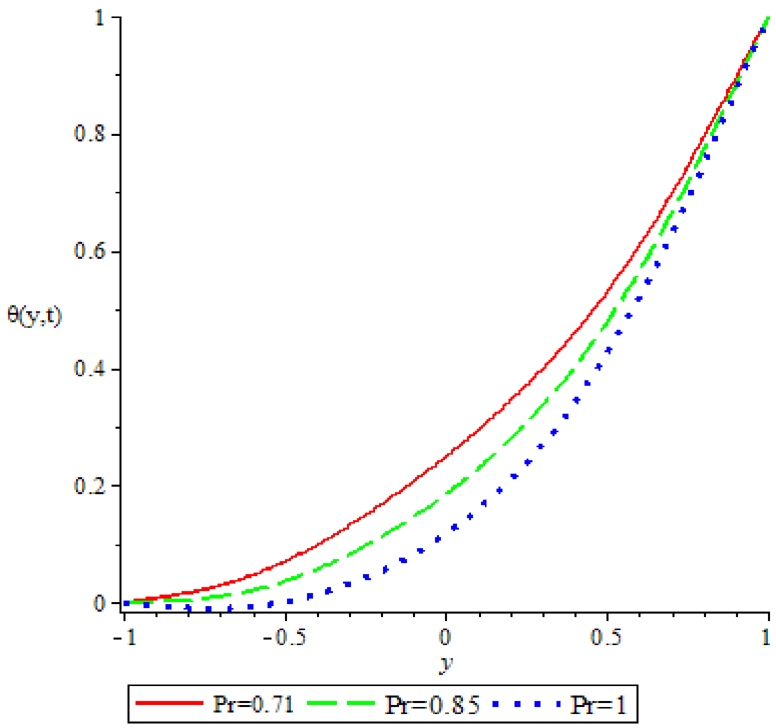

Figure 4.5: Effect of Prandtl number (Pr) on temperature profile $\theta(y,t)$

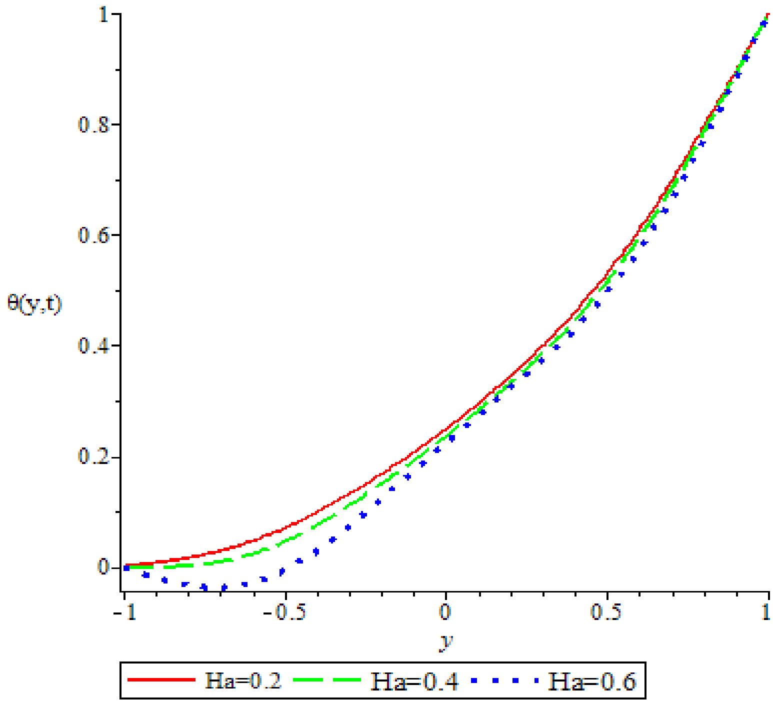

Figure 4.6: Effect of Hartman number (Ha) on temperature profile $\theta(y,t)$ along distance

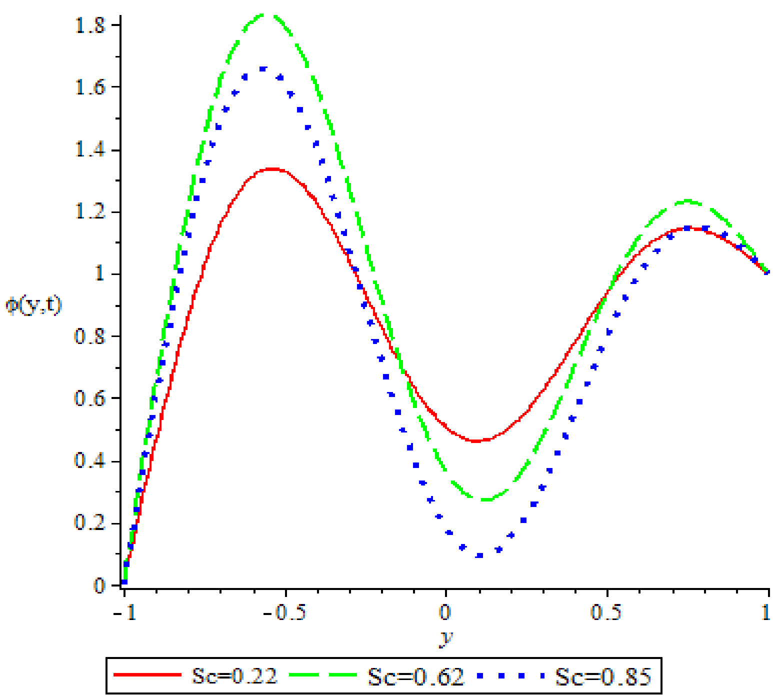

Figure 4.7: Effect of Schimdt number (Sc) on concentration profile $\phi(y,t)$

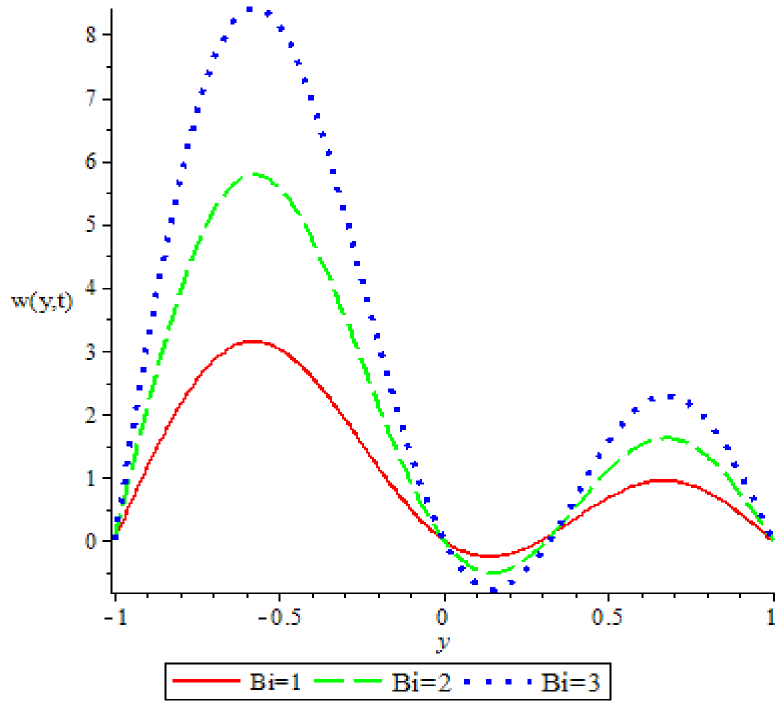

Figure 4.8: Effect of ion slip parameter (Bi) on secondary velocity profile $w(y,t)$

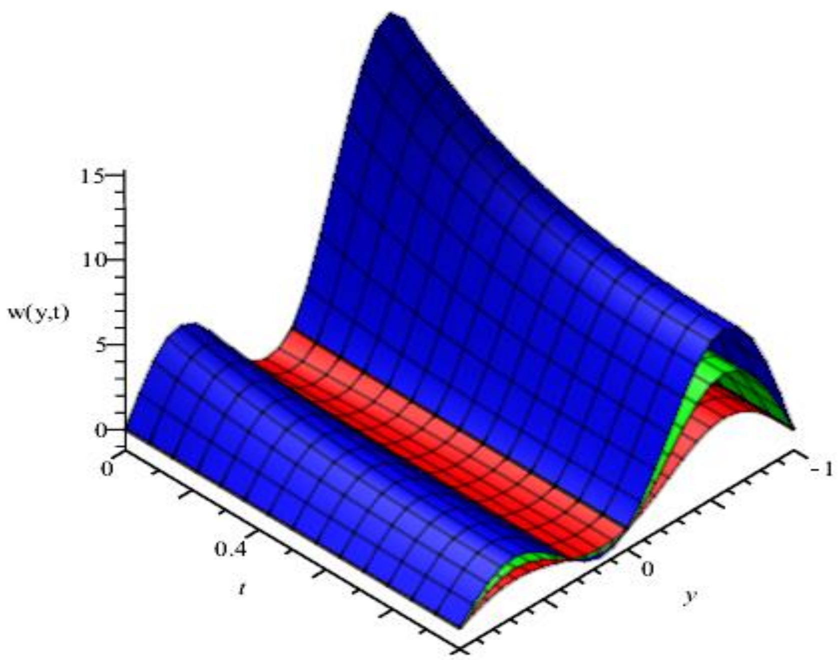

Figure 4.9: Effect of ion slip parameter (Bi) on secondary velocity profile $w(y,t)$ along distance y and time t. Bi=1(red), Bi=2(green) and Bi=3(blue)

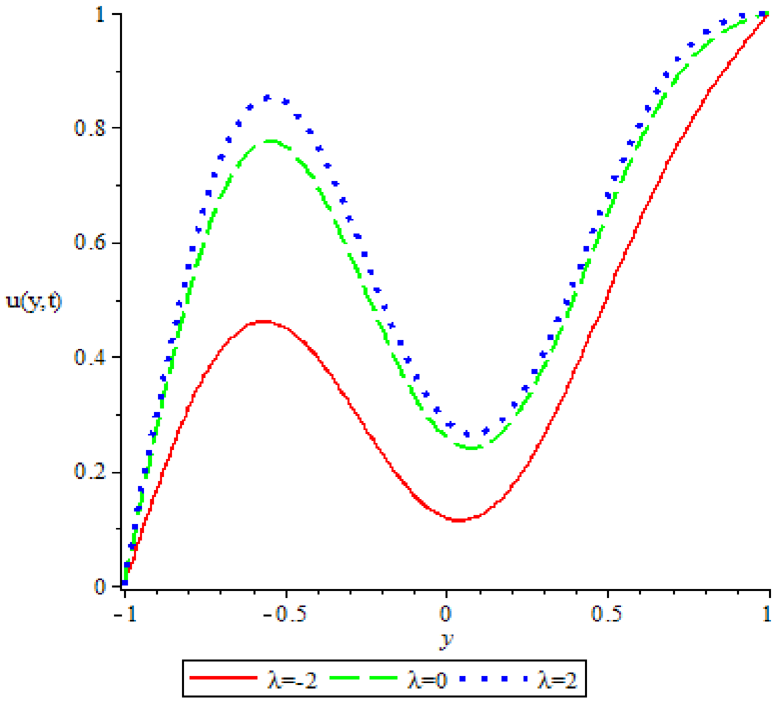

Figure 4.10: Effect of pressure gradient (λ) on primary velocity profile $u(y,t)$

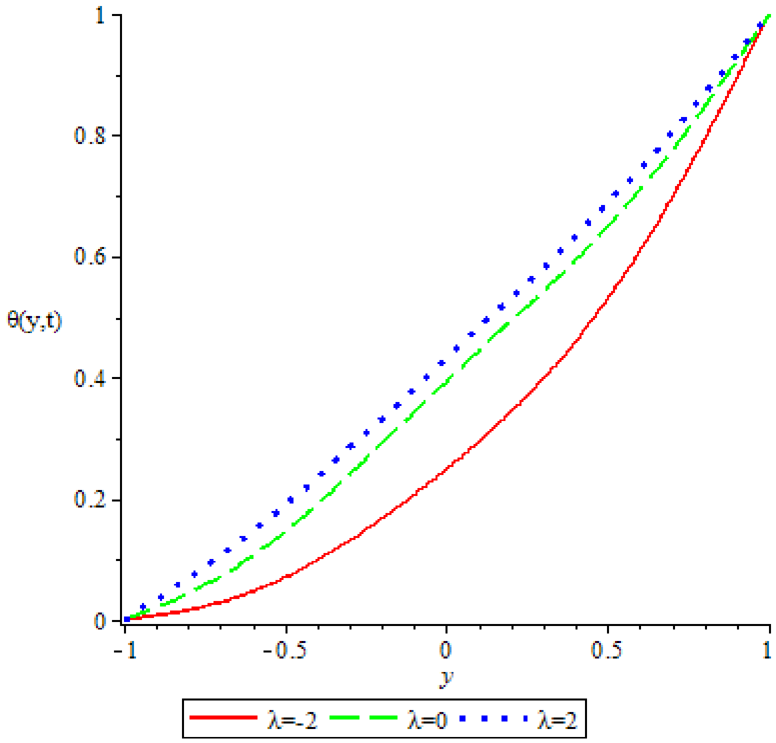

Figure 4.11: Effect of pressure gradient (λ) on temperature profile $\theta(y,t)$

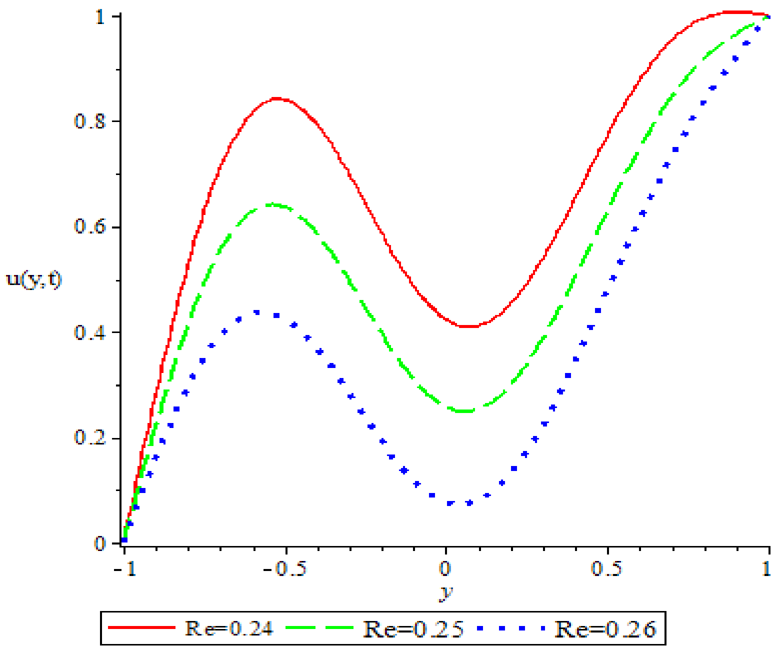

Figure 4.12: Effect of Reynolds number (Re) on primary velocity profile $u(y,t)$

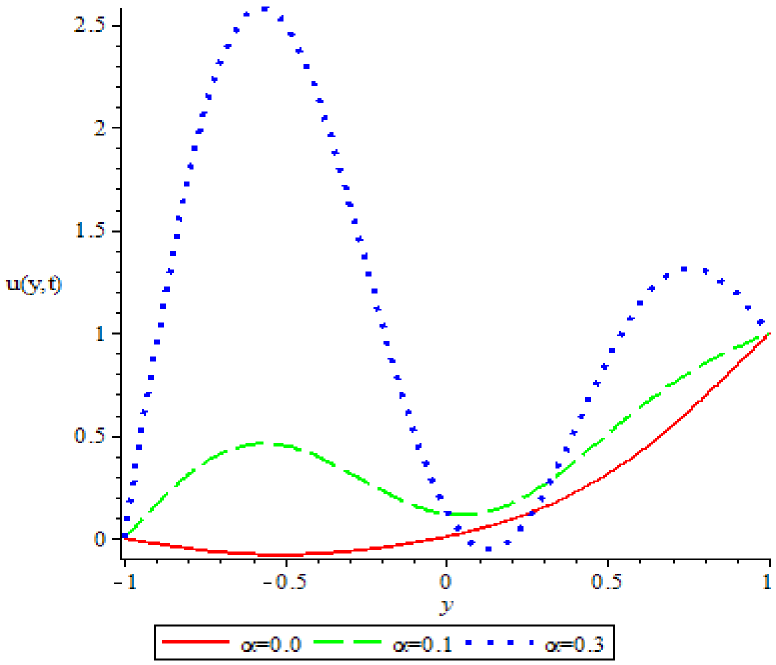

Figure 4.13: Effect of temperature dependent viscosity (α) on primary velocity profile $u(y,t)$

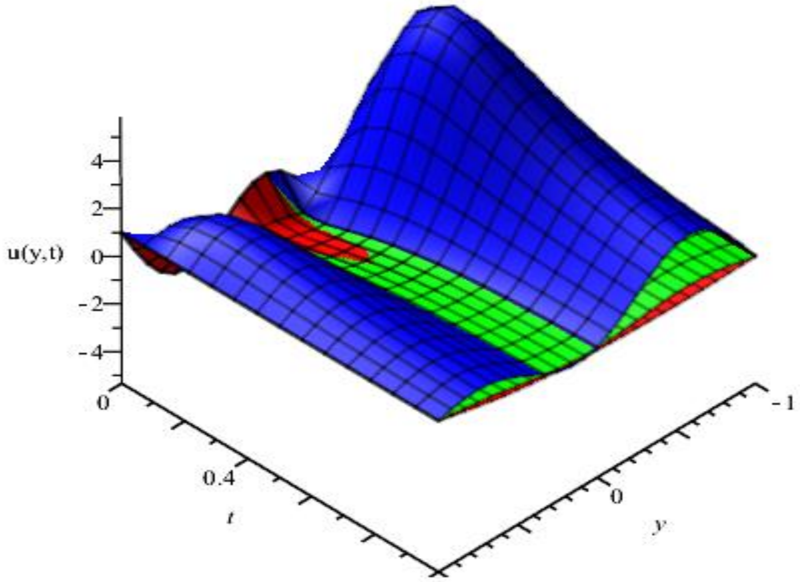

Figure 4.14: Effect of temperature dependent viscosity (α) on primary velocity profile $u(y,t)$ along distance y and time t. (α)=0.0(red), (α)=0.1(green) and (α) =0.3(blue)

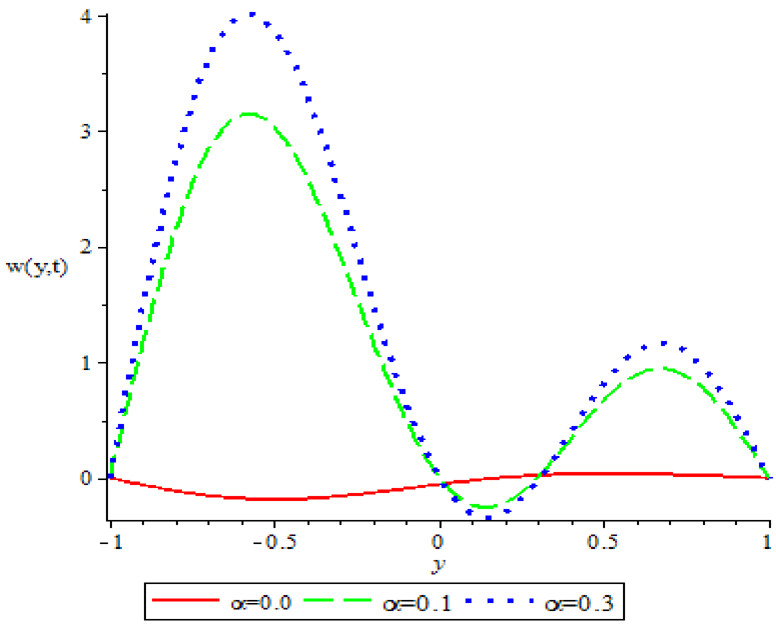

Figure 4.15: Effect of temperature dependent viscosity (α) on secondary velocity profile $w(y,t)$

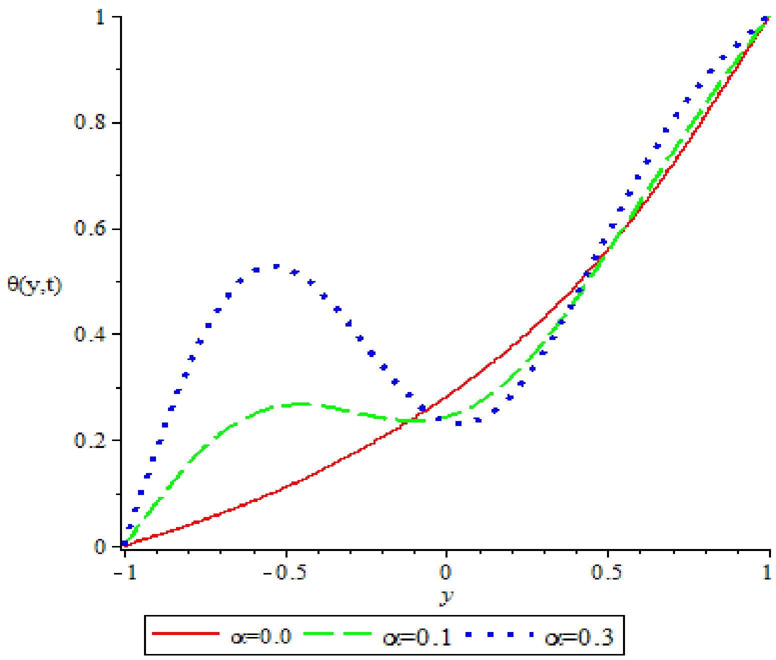

Figure 4.16: Effect of temperature dependent viscosity (α) on temperature profile $\theta(y,t)$

4.1.1.1 Discussion of results for case 1

Figure 4.1 displays the graph of temperature profile $\theta(y,t)$ for different values of radiation parameter (Ra). It is observed that temperature decreases as radiation parameter increases. Also, the temperature profile is observed to increase along distance y.

Figure 4.2 shows the graph of temperature $\theta(y,t)$ for different values of suction parameter (S). It is evident that increase in suction parameter leads to decrease in temperature. It is also seen that temperature increases along distance y.

Figure 4.3 presents the graph of concentration $\phi(y,t)$ for different values of suction parameter (S). It is observed that concentration oscillates along y while increase in suction parameter leads to increase in concentration.

Figure 4.4 presents the graph of concentration $\phi(y,t)$ for different values of suction parameter (S) along distance y and with time t. It is observed that concentration oscillates along y and decreases with time while increase in suction parameter leads to increase in concentration.

Figure 4.5 depicts the graph of temperature $\theta(y,t)$ for different values of Prandtl number. It is observed that temperature increases along distance while increase in Prandtl number leads to decrease in temperature.

Figure 4.6 depicts the graph of temperature $\theta(y,t)$ for different values of Hartman number (Ha). It is observed that temperature increases along distance while increase in Hartman number leads to decrease in temperature.

Figure 4.7 illustrates the effect of Schmidt number (Sc) on concentration $\phi(y,t)$. It is observed that concentration oscillates along distance. Also, increase in Schmidt number leads to increase in fluid concentration.

Figure 4.8 presents the graph of secondary velocity $w(y,t)$ for different values of ion slip parameter (Bi). It is observed that secondary velocity oscillates along distance y while increase in ion slip parameter leads to increase in secondary velocity.

Figure 4.9 presents the graph of secondary velocity $w(y,t)$ for different values of ion slip parameter (Bi) along distance and with time t. It is observed that secondary velocity oscillates along distance y and increases with time while increase in ion slip parameter leads to increase in secondary velocity.

Figure 4.10 shows the graph of primary velocity $u(y,t)$ for different values of pressure gradient (σ). It is observed that primary velocity oscillates along distance y. Also, increase in pressure gradient leads to increase in primary velocity.

Figure 4.11 displays the graph of temperature $\theta(y,t)$ for different values of pressure gradient (σ). It is observed that fluid temperature increases along distance y. Also, increase in pressure gradient leads to increase in temperature.

Figure 4.12 depicts the graph of primary velocity $u(y,t)$ for different values of Reynolds number (Re). It is observed that primary velocity oscillates along distance y. Also, increase in Reynolds number leads to decrease in primary velocity

Figure 4.13 depicts the graph of primary velocity $u(y,t)$ for different values of temperature dependent viscosity (α). It is observed that primary velocity is maximum when viscosity is temperature dependent as compared to when it is independent on temperature. Also, increase in temperature dependent viscosity leads to oscillation in primary velocity along distance y.

Figure 4.14 presents the graph of primary velocity $u(y,t)$ for different values of temperature viscosity parameter (α) along distance and with time t. It is observed that primary velocity oscillates along distance y and increases with time.

Figure 4.15 depicts the graph of secondary velocity $w(y,t)$ for different values of temperature dependent viscosity (α). It is observed that primary velocity is maximum when viscosity is temperature dependent as compared to when it is independent on temperature. Also, increase in temperature dependent viscosity leads to oscillation in secondary velocity along distance y.

Figure 4.16 depicts the graph of temperature profile $\theta(y,t)$ for different values of temperature dependent viscosity (α). It is observed that temperature increases with increase in viscosity. Also, increase in temperature dependent viscosity leads to increase in temperature along distance y.

4.1.2 Graphs of case 2

In this section the results of case 2 problem are presented as shown below

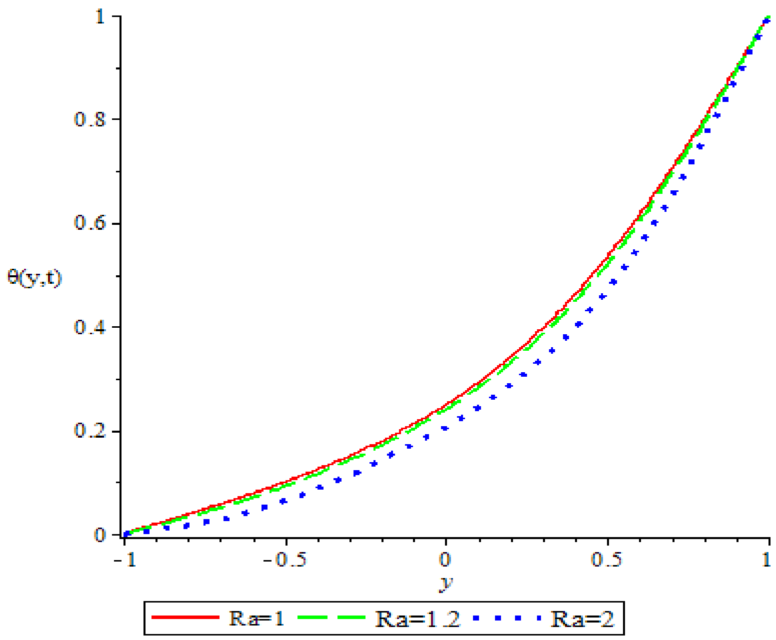

Figure 4.17: Effect of radiation parameter (Ra) on temperature profile $\theta(y,t)$

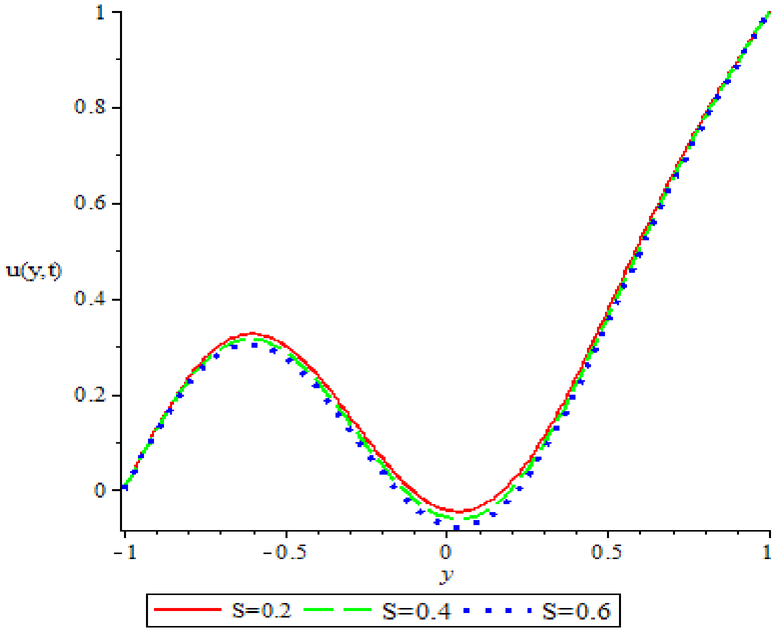

Figure 4.18: Effect of suction parameter (S) on primary velocity profile $u(y,t)$

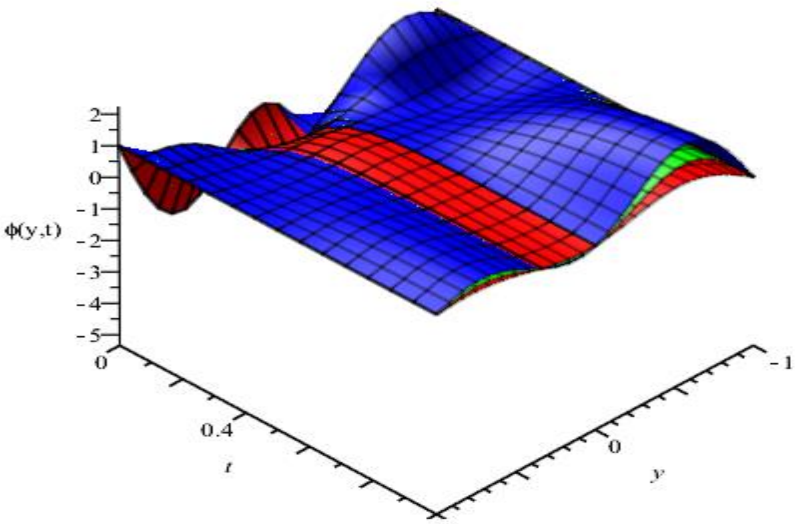

Figure 4.19: Effect of suction parameter (S) on concentration profile $\phi(y,t)$ along distance y and time t. S=0.2(red), S=0.4(green) and S=0.6(blue)

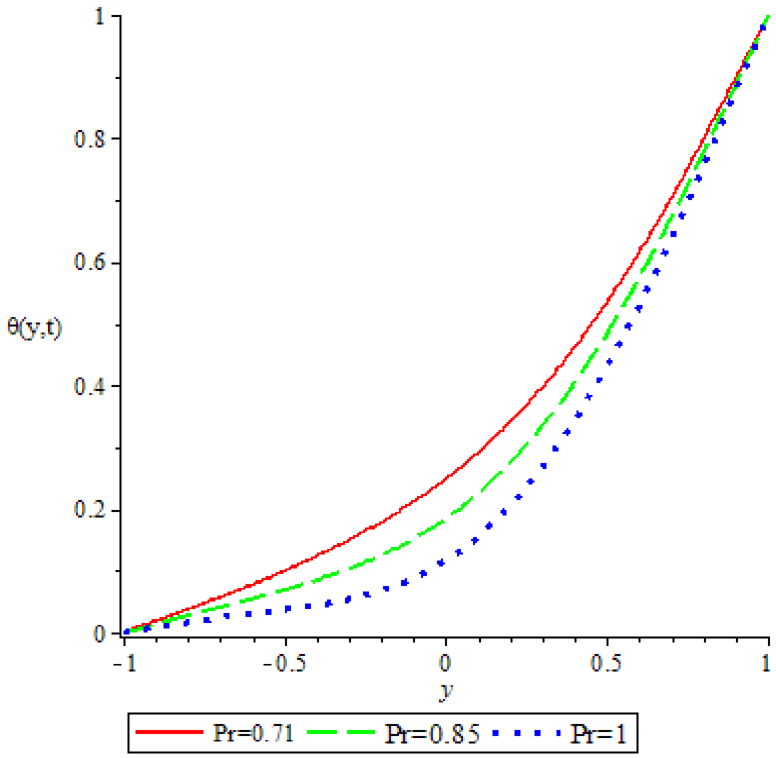

Figure 4.20: Effect of Prandtl number (Pr) on temperature profile $\theta(y,t)$

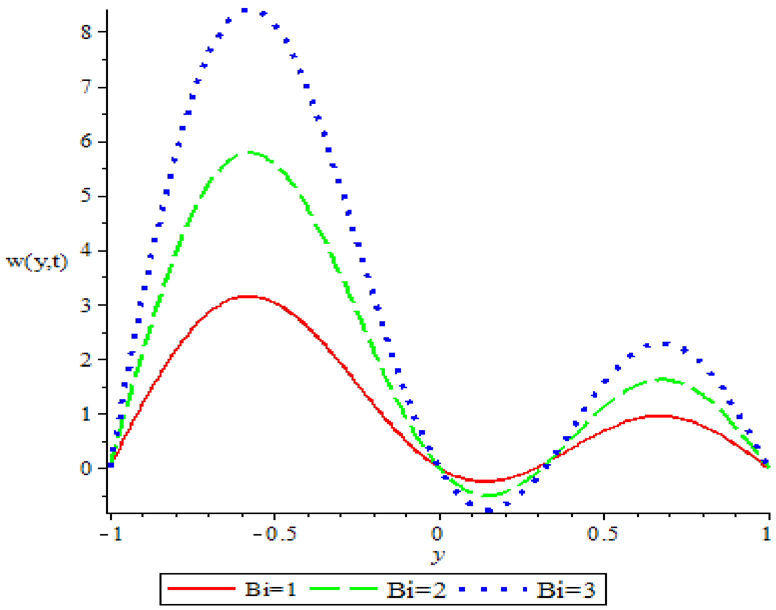

Figure 4.21: Effect of ion slip parameter (Bi) on secondary velocity profile $w(y,t)$

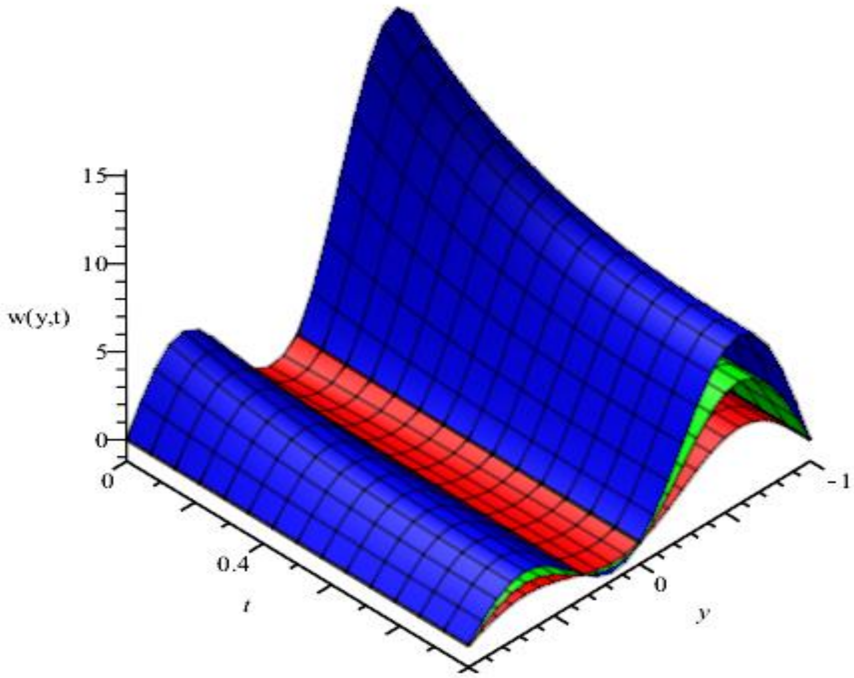

Figure 4.22: Effect of ion slip parameter (Bi) on secondary velocity profile $w(y,t)$ along distance y and time t. Bi=1(red), Bi=2(green) and Bi=3(blue)

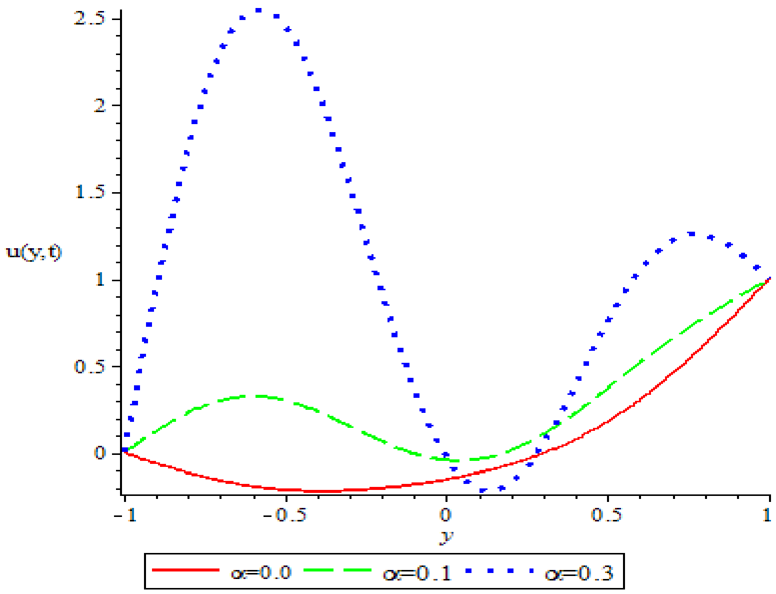

Figure 4.23: Effect of temperature dependent viscosity (α) on primary velocity profile $u(y,t)$

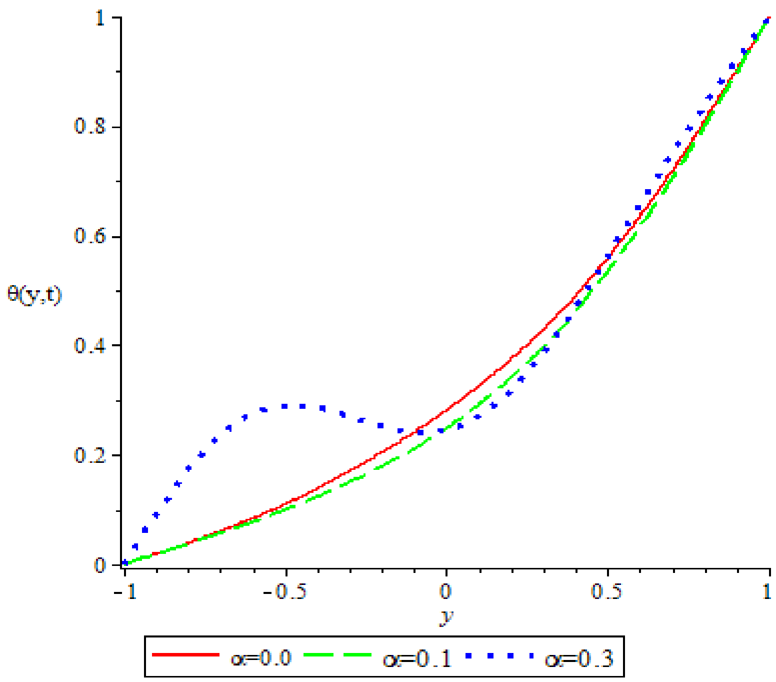

Figure 4.24: Effect of temperature dependent viscosity (α) on temperature profile $\theta(y,t)$

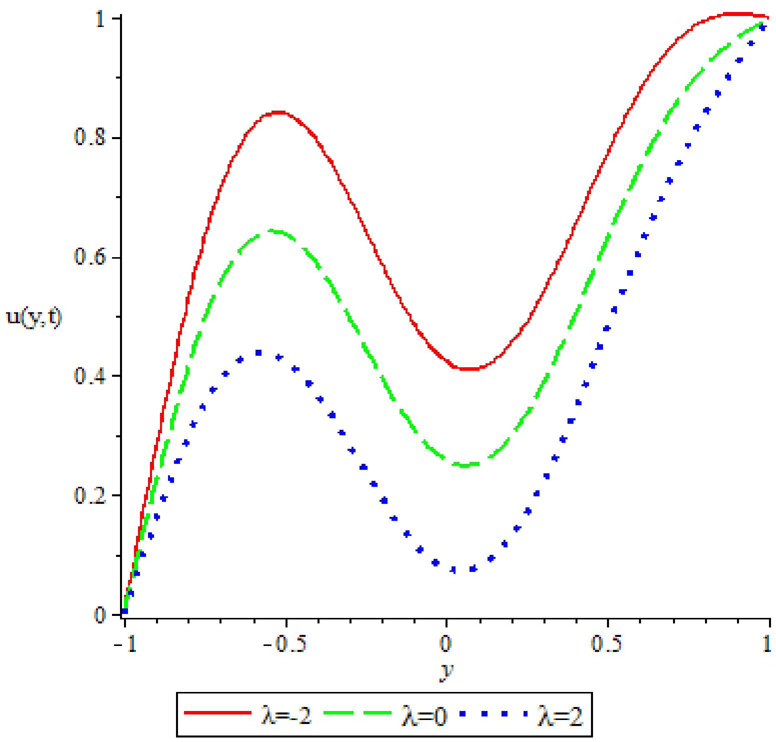

Figure 4.25: Effect of pressure gradient (λ) on primary velocity profile $u(y,t)$

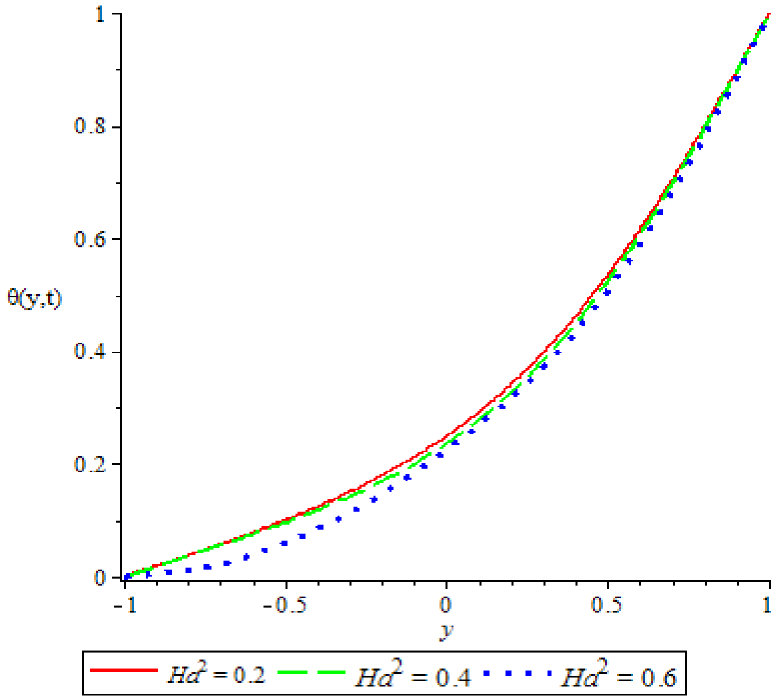

Figure 4.26: Effect of Hartman number (Ha^2) on temperature profile $\theta(y,t)$

4.1.2.1 Discussion of results of case 2

Figure 4.17 displays the graph of temperature profile $\theta(y,t)$ for different values of radiation parameter (Ra). It is observed that temperature decreases as radiation parameter increases. Also, the temperature profile is observed to increase along distance.

Figure 4.18 shows the graph of primary velocity $u(y,t)$ for different values of suction parameter (S). It is evident that increase in suction parameter leads to decrease in primary velocity. It is also seen that primary velocity increases along distance y.

Figure 4.19 illustrates the effect of suction parameter (S) on the concentration profile $\phi(y,t)$ of the flow along distance and with time t. It is observed that concentration increases with time while increase in suction parameter leads to decrease in the fluid concentration.

Figure 4.20 presents the graph of temperature $\theta(y,t)$ for different values of Prandtl number $(\Pr)$. It is observed temperature increases along distance y and increase in Prandtl number leads to decrease in temperature.

Figure 4.21 depicts the effect of ion slip parameter (Bi) on secondary velocity $w(y,t)$ along y. it is seen that the secondary velocity oscillates along distance y and increase in ion slip parameter leads to increase in secondary velocity.

Figure 4.22 displays the graph of secondary velocity $w(y,t)$ along distance and time for different values of ion slip parameter (Bi). It is observed that secondary velocity increases with time and oscillates along distance while increase in ion slip parameter leads to increase in secondary velocity.

Figure 4.23 depicts the graph of primary velocity $u(y,t)$ for different values of temperature dependent viscosity (α). It is observed that primary velocity is maximum when viscosity is temperature dependent as compared to when it is independent on temperature. Also, increase in temperature dependent viscosity leads to oscillation in primary velocity along distance y.

Figure 4.24 depicts the graph of temperature profile $\theta(y,t)$ for different values of temperature dependent viscosity (α). It is observed that temperature increases with increase in viscosity. Also, increase in temperature dependent viscosity leads to increase in temperature along distance y.

Figure 4.25 illustrates the effects pressure gradient (σ) on primary velocity profile $u(y,t)$. It is observed that primary velocity oscillates along distance y while increase in pressure gradient leads to decrease in primary velocity.

Figure 4.26 shows the effect of Hartman number (Ha^2) on temperature profile $\theta(y,t)$ along distance. It is observed that the temperature profile increases along distance y while increase in Hartman number leads to decrease in fluid temperature.

4.1.3 Graphs of Case 3

In this section, the result of case 3 problem are presented as shown below

Table 4.1: Comparison between analytical and numerical results

y	$\phi(y)$ **Perturbation Results**	$\phi(y)$ **NumericalResults**	$\lvert\phi_{numer} - \phi_{pertu}\rvert$
-1.0	0	0	0
-0.9	0.0422823619	0.0532818991	$1.100 \text{ x}10^{-2}$
-0.8	0.0858952491	0.1065003323	$2.061 \text{ x}10^{-2}$
-0.7	0.1307244697	0.1596163670	$2.889 \text{ x}10^{-2}$
-0.6	0.1767410393	0.2125895971	$3.585 \text{ x}10^{-2}$
-0.5	0.2239020289	0.2653779698	$4.148 \text{ x}10^{-2}$
-0.4	0.2721479199	0.3179376254	$4.579 \text{ x}10^{-2}$
-0.3	0.3213995380	0.3702227504	$4.882 \text{ x}10^{-2}$
-0.2	0.3715545147	0.4221854464	$5.063 \text{ x}10^{-2}$
-0.1	0.4224832010	0.4737756156	$5.129 \text{ x}10^{-2}$
0	0.4740239398	0.5249408660	$5.092 \text{ x}10^{-2}$
0.1	0.5259776533	0.5756264394	$4.965 \text{ x}10^{-2}$
0.2	0.5781016126	0.6257751639	$4.767 \text{ x}10^{-2}$
0.3	0.6301023141	0.6753274366	$4.523 \text{ x}10^{-2}$
0.4	0.6816273601	0.7242212372	$4.259 \text{ x}10^{-2}$
0.5	0.7322562239	0.7723921791	$4.014 \text{ x}10^{-2}$
0.6	0.7814897943	0.8197735995	$3.828 \text{ x}10^{-2}$
0.7	0.8287385948	0.8662966938	$3.756 \text{ x}10^{-2}$
0.8	0.8733095464	0.9118906966	$3.858 \text{ x}10^{-2}$
0.9	0.9143911649	0.9564831129	$4.209 \text{ x}10^{-2}$
1.0	0.9510366377	1.0000000000	$4.896 \text{ x}10^{-2}$

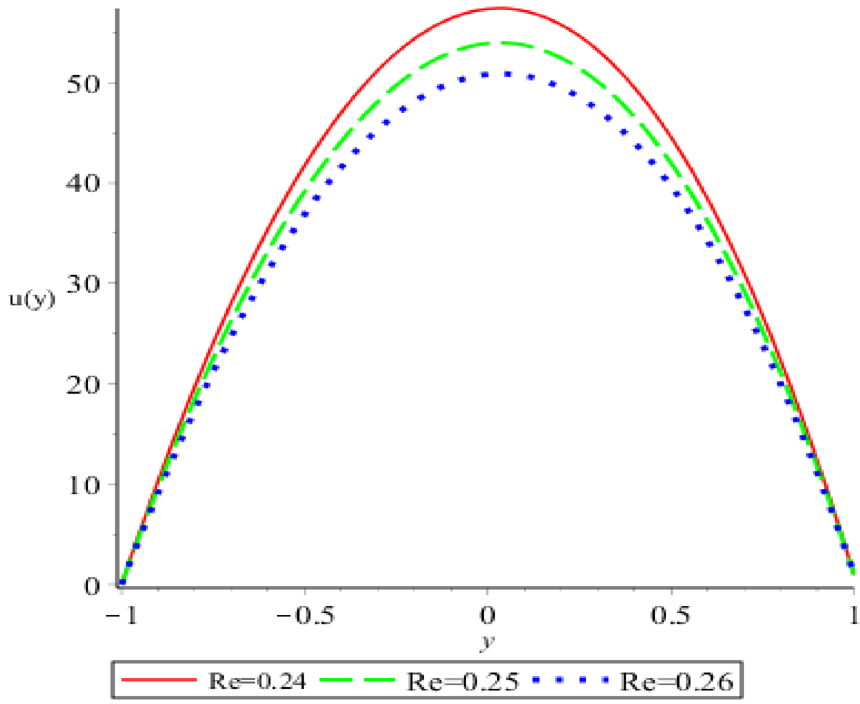

Figure 4.27: Effect of Reynolds number (Re) on primary velocity profile $u(y,t)$

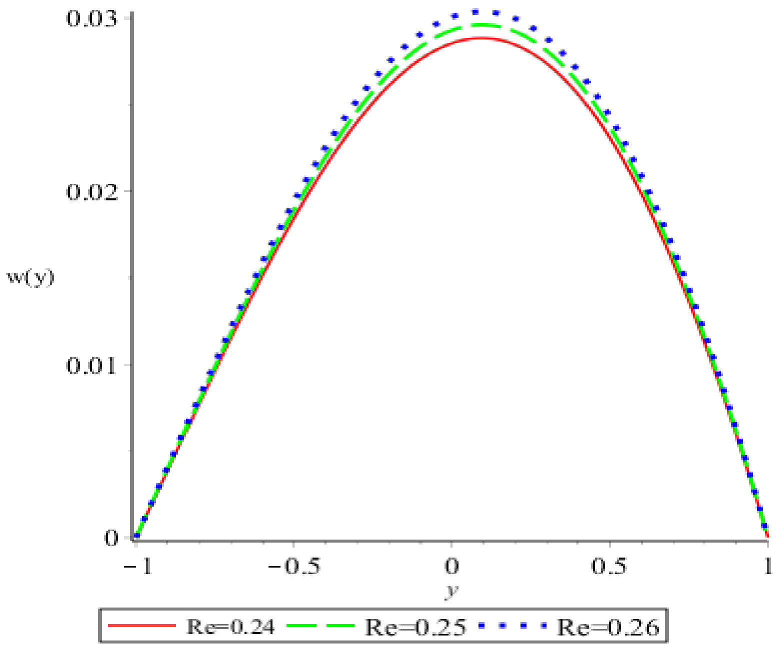

Figure 4.28: Effect of Reynolds number (Re) on secondary velocity profile $w(y,t)$

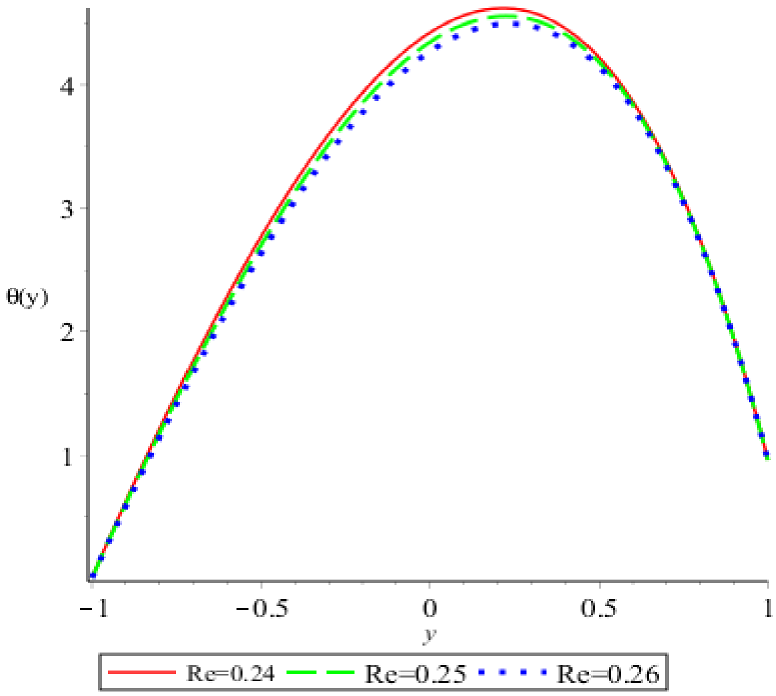

Figure 4.29: Effect of Reynolds number (Re) **on temperature profile** $\theta(y,t)$

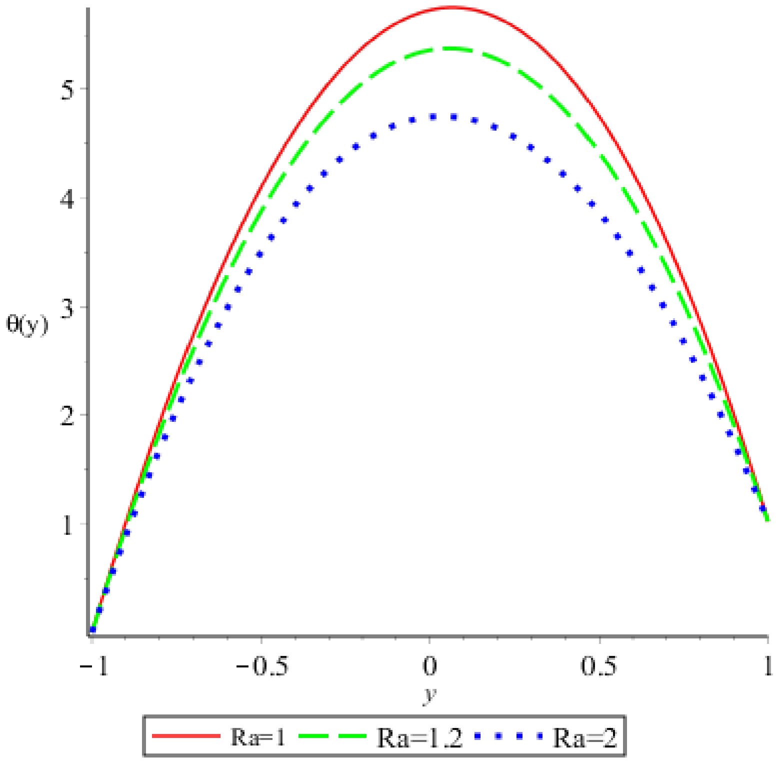

Figure 4.30: Effect of radiation parameter $(\mathrm{R}\,a)$ **on temperature profile** $\theta(y,t)$

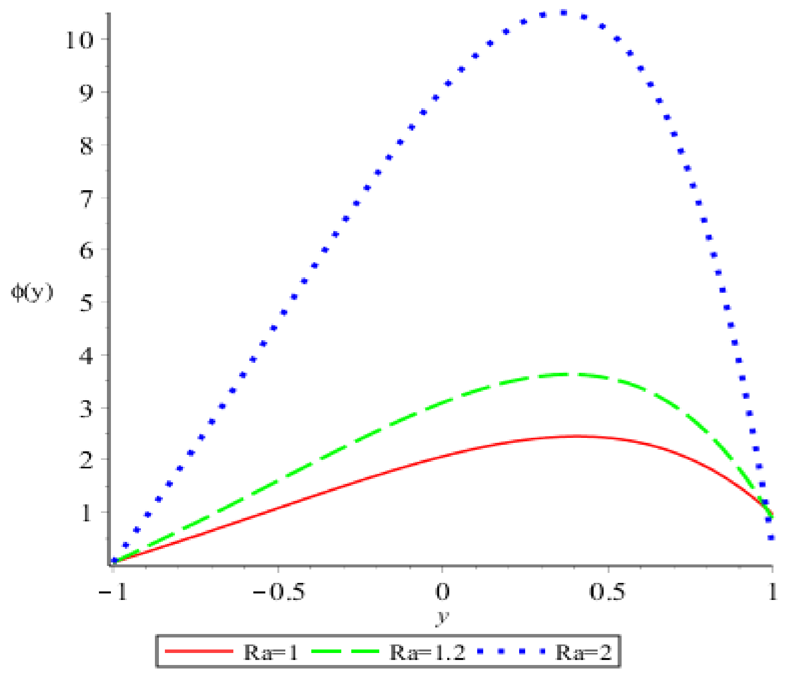

Figure 4.31: Effect of radiation parameter (Ra) **on concentration profile** $\phi(y,t)$

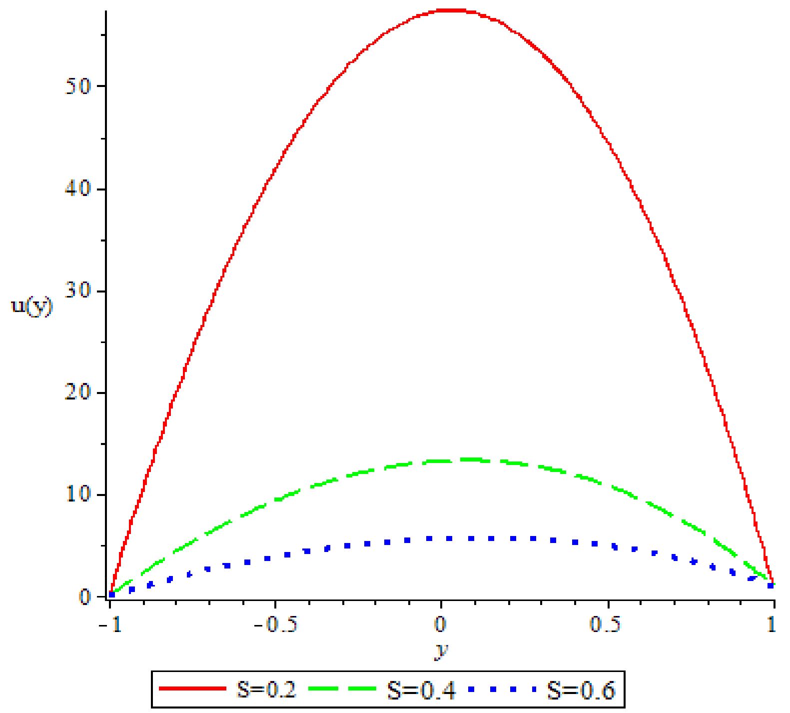

Figure 4.32: Effect of suction parameter (S) **on primary velocity profile** $u(y,t)$

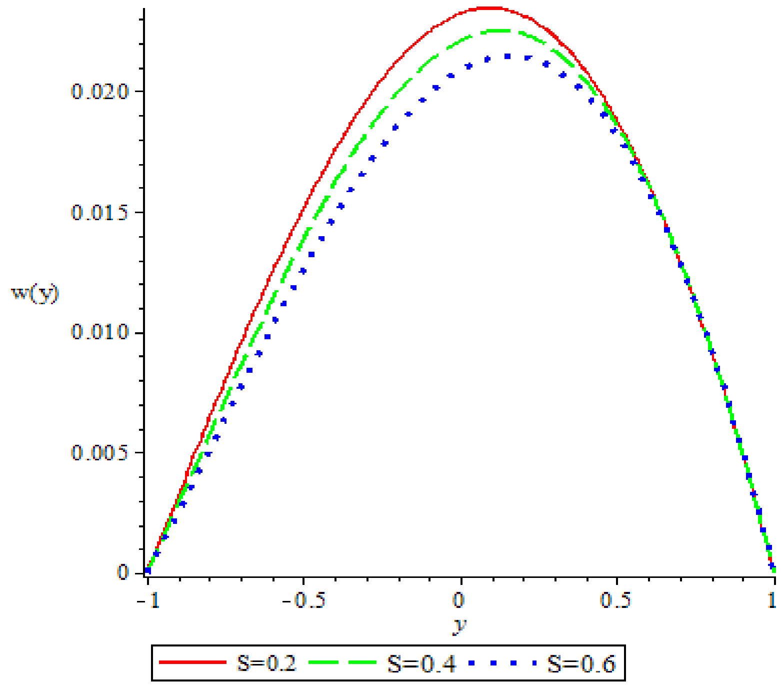

Figure 4.33: Effect of suction parameter (S) on secondary velocity profile $w(y,t)$

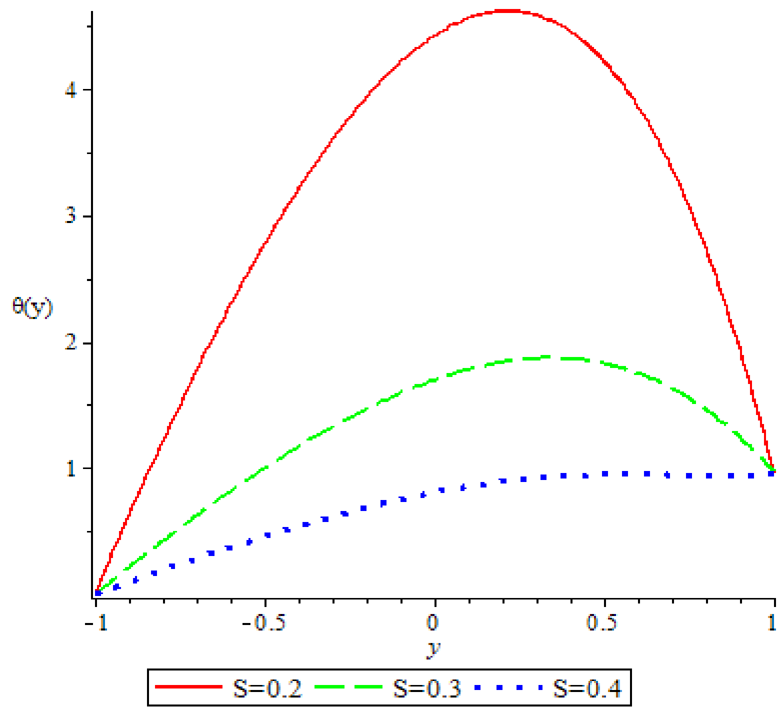

Figure 4.34: Effect of suction parameter (S) on temperature profile $\theta(y,t)$

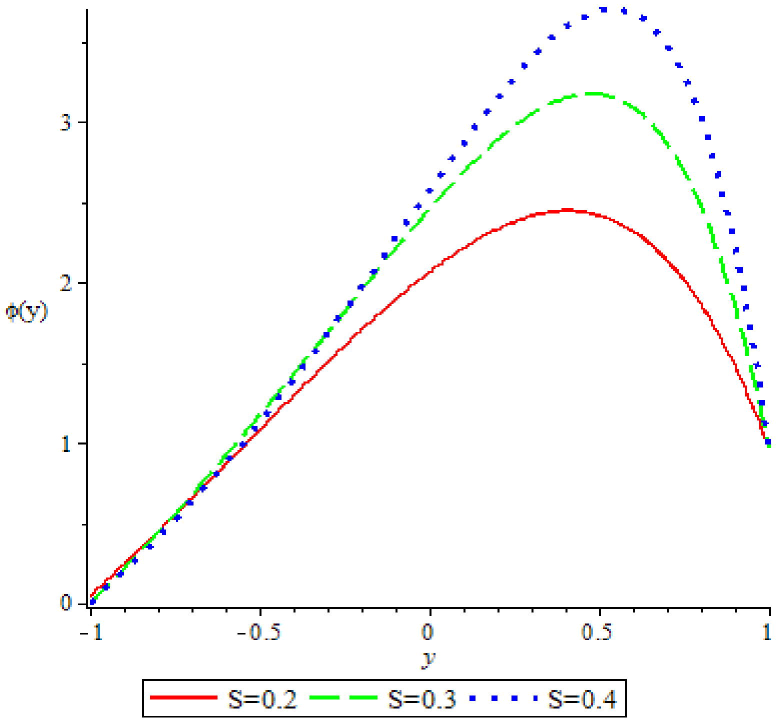

Figure 4.35: Effect of suction parameter (S) **on concentration profile** $\phi(y,t)$

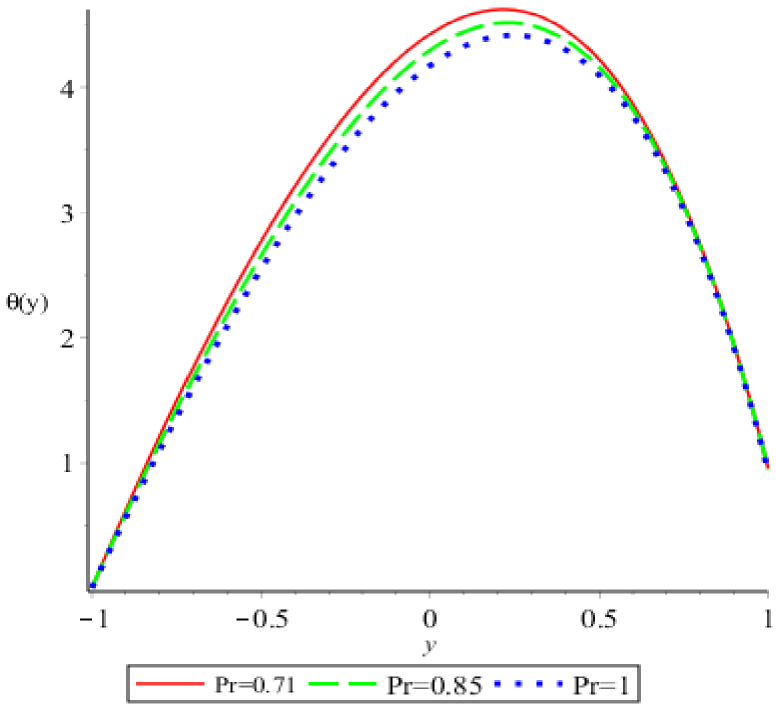

Figure 4.36: Effect of Prandtl number $(\Pr)$ **on temperature profile** $\theta(y,t)$

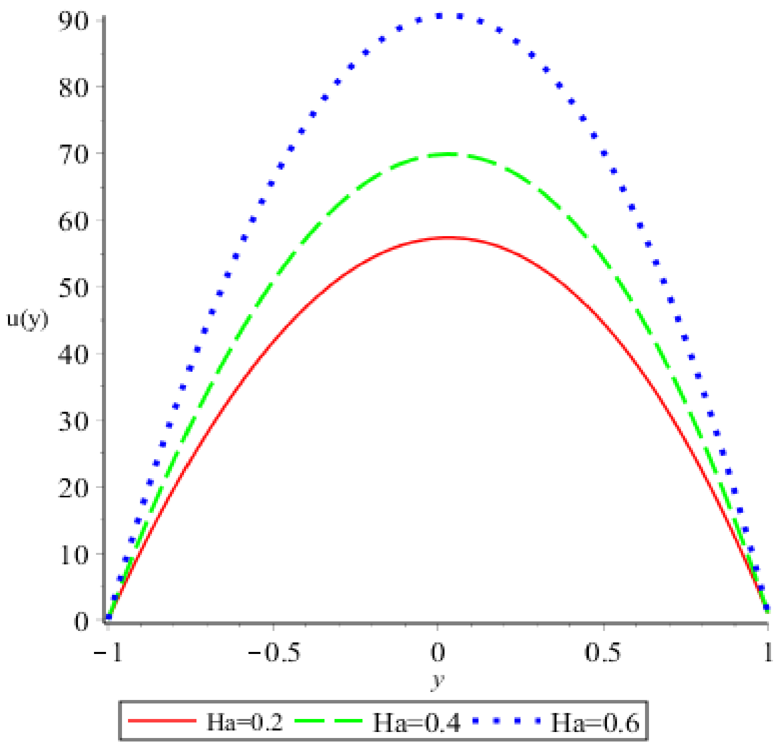

Figure 4.37: Effect of Hartman number (Ha) on primary velocity profile $u(y,t)$

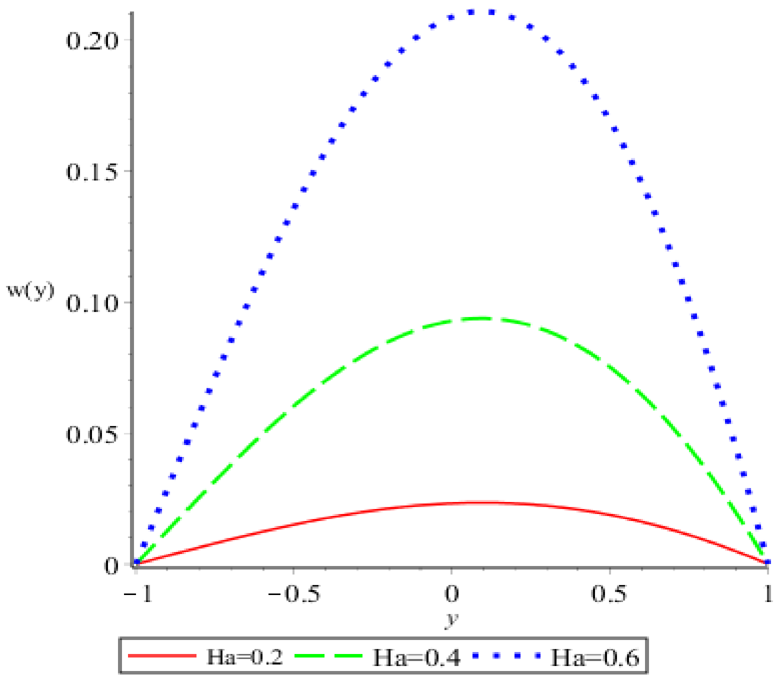

Figure 4.38: Effect of Hartman number (Ha) on secondary velocity profile $w(y,t)$

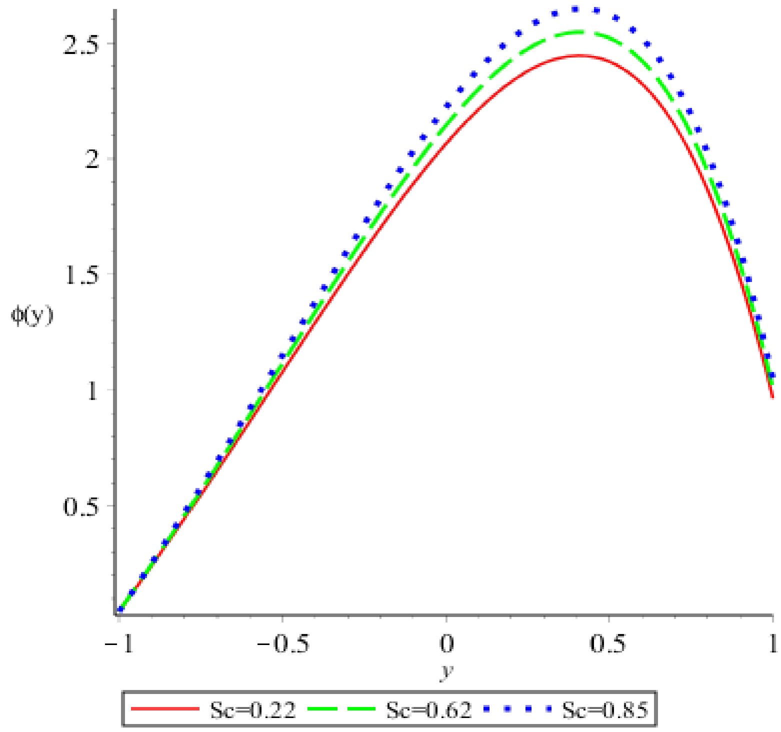

Figure 4.39: Effect of Schimdt number (Sc) **on concentration profile** $\phi(y,t)$

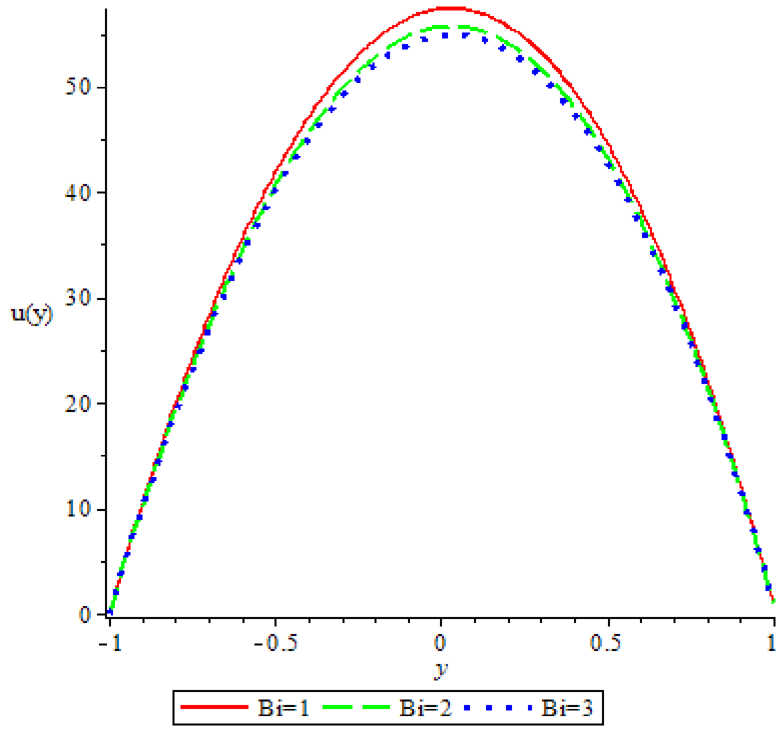

Figure 4.40: Effect of ion slip parameter (Bi) **on primary velocity profile** $u(y)$

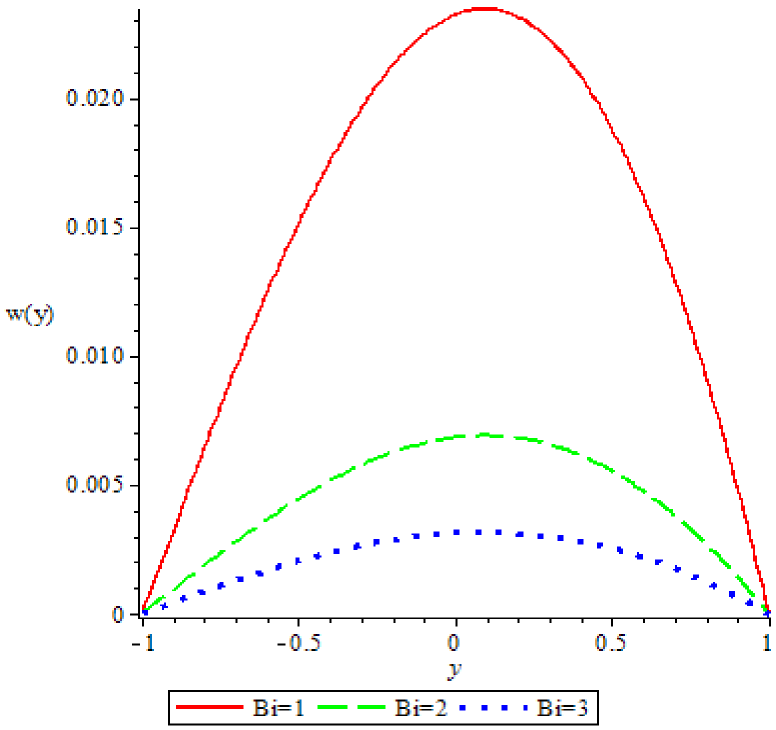

Figure 4.41: Effect of ion slip parameter (Bi) on secondary velocity profile $w(y)$

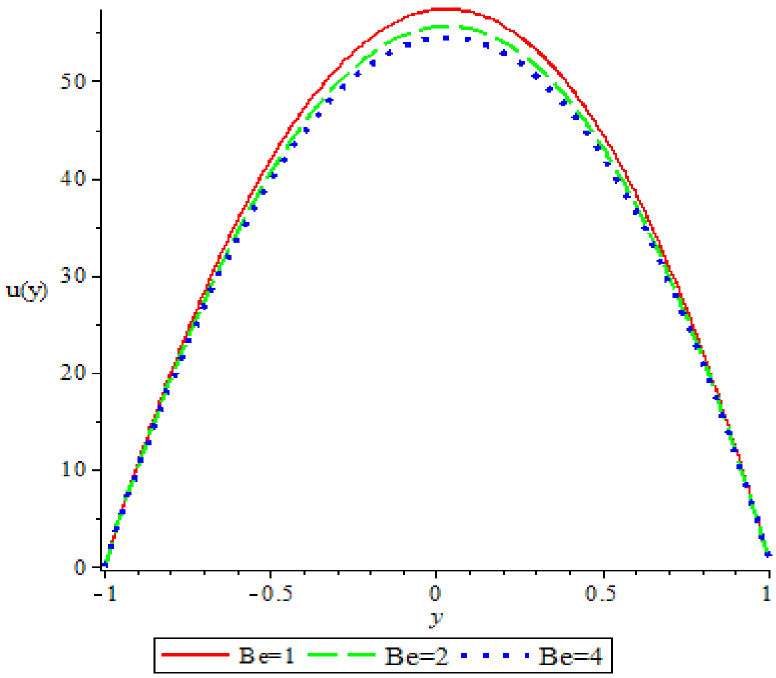

Figure 4.42: Effect of Hall parameter (Be) on primary velocity profile $u(y)$

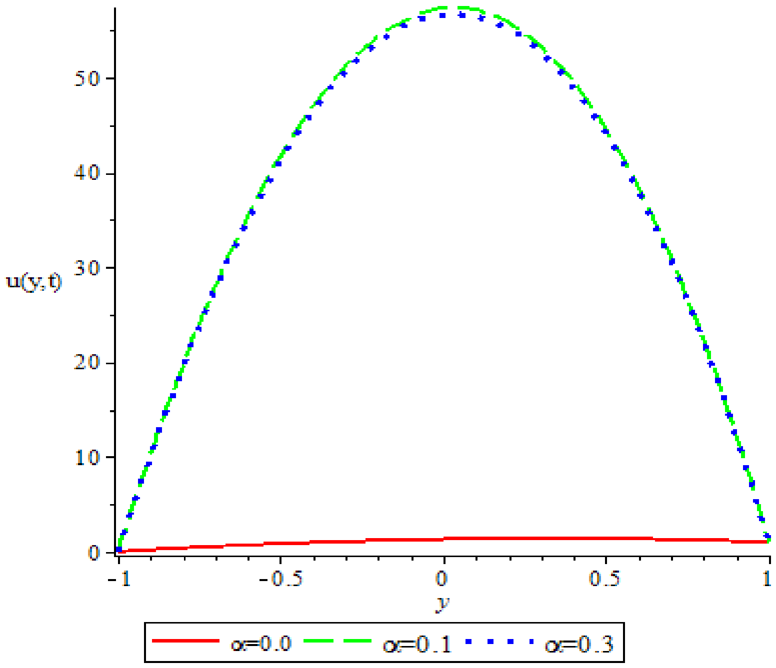

Figure 4.43: Effect of temperature dependent viscosity (α) on primary velocity profile $u(y)$

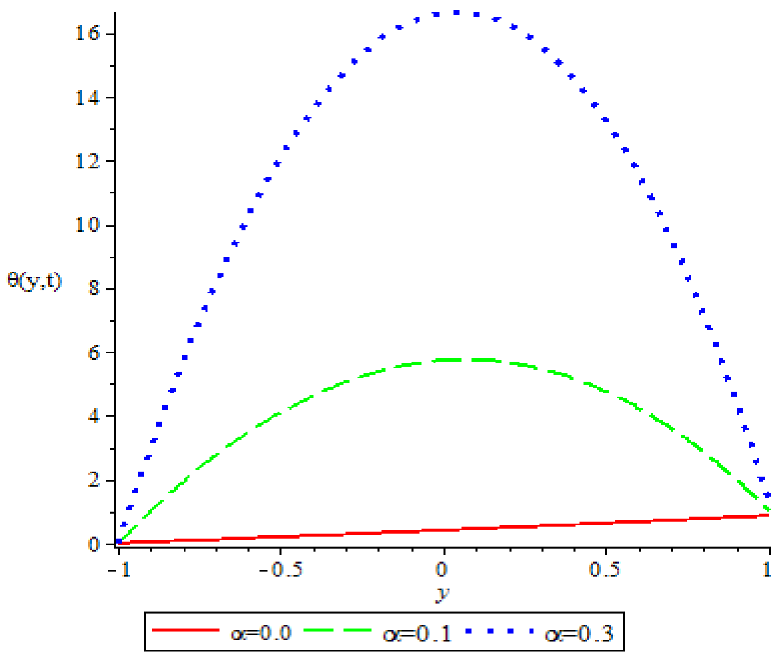

Figure 4.44: Effect of temperature dependent viscosity (α) on temperature profile $\theta(y)$

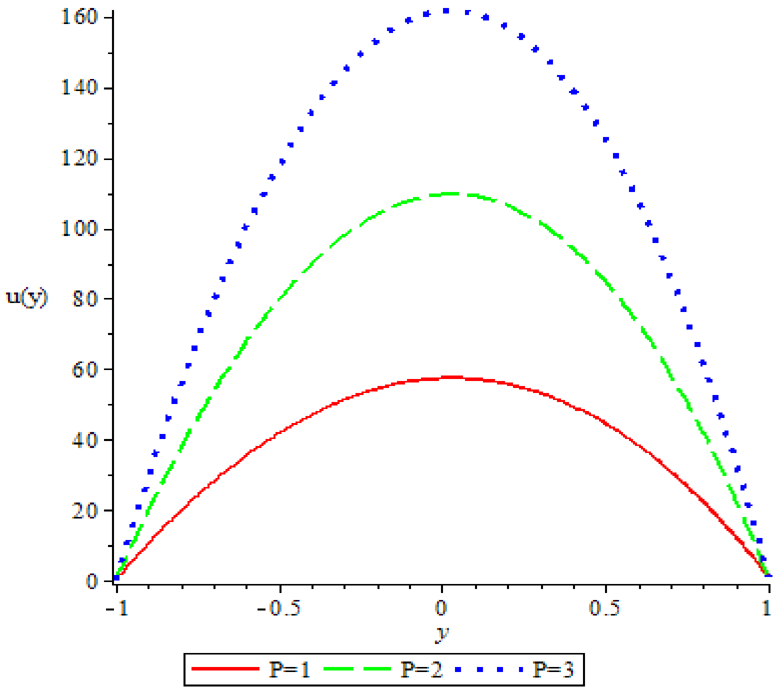

Figure 4.45: Effect of porosity parameter (P) on primary velocity profile $u(y)$

4.1.3.1 Discussion of the results of case 3

Table 4.1 above demonstrates agreement between the results obtained using perturbation technique and purely fourth-order Runge Kutta numerical integration approach coupled with shooting method at small and moderate parameter values. Generally, the difference is of order 10^{-2}.

Figure 4.27 depicts the graph of primary velocity for different values of Reynolds number. It is observed that primary velocity increases and then decreases along distance y. Also, increase in Reynolds number leads to decrease in primary velocity

Figure 4.28 displays the graph of secondary velocity for different values of Reynolds number. It is observed that secondary velocity increases and then decreases along distance y. Also, increase in Reynolds number leads to increase in secondary velocity

Figure 4.29 shows the graph of temperature for different values of Reynolds number. It is observed that fluid temperature increases and then decreases along distance y. Also, increase in Reynolds number leads to decrease in temperature

Figure 4.30 shows the graph of temperature for different values of radiation parameter. It is observed that fluid temperature increases and then decreases along distance y. Also, increase in radiation parameter leads to decrease in temperature

Figure 4.31 shows the graph of concentration for different values of radiation parameter. It is observed that fluid concentration increases and then decreases along distance y. Also, increase in radiation parameter leads to increase in concentration

Figure 4.32 depicts the graph of primary velocity for different values of suction parameter. It is observed that primary velocity increases and then decreases along distance y. Also, increase in suction parameter leads to decrease in primary velocity.

Figure 4.33 depicts the graph of secondary velocity for different values of suction parameter. It is observed that secondary velocity increases and then decreases along distance y. Also, increase in suction parameter leads to decrease in secondary velocity.

Figure 4.34 depicts the graph of temperature for different values of suction parameter. It is observed that temperature increases and then decreases along distance y. Also, increase in suction parameter leads to decrease in fluid temperature.

Figure 4.35 depicts the graph of fluid concentration for different values of suction parameter. It is observed that concentration increases and then decreases along distance y. Also, increase in suction parameter leads to increase in concentration

Figure 4.36 depicts the graph of temperature for different values of Prandtl number. It is observed that temperature increases and then decreases along distance y. Also, increase in Prandtl number leads to decrease in temperature.

Figure 4.37 depicts the graph of primary velocity for different values of Hartman number. It is observed that primary velocity increases and then decreases along distance y. Also, increase in Hartman number leads to increase in primary velocity.

Figure 4.38 depicts the graph of secondary velocity for different values of Hartman number. It is observed that secondary velocity increases and then decreases along distance y. Also, increase in Hartman number leads to increase in secondary velocity.

Figure 4.39 depicts the graph of concentration for different values of Schmidt number. It is observed that concentration increases and then decreases along distance y. Also, increase in Schmidt number leads to increase in concentration.

Figure 4.40 depicts the graph of primary velocity for different values of Hall parameter. It is observed that primary velocity increases and then decreases along distance y. Also, increase in Hall parameter leads to decrease in primary velocity.

Figure 4.41 depicts the graph of secondary velocity for different values of Hall parameter. It is observed that secondary velocity increases and then decreases along distance y. Also, increase in Hall parameter leads to decrease in secondary velocity.

Figure 4.42 depicts the graph of primary velocity for different values of Hall factor. It is observed that primary velocity increases and then decreases along distance y. Also, increase in Hall decrease leads to decrease in primary velocity.

Figure 4.43 depicts the graph of primary velocity $u(y,t)$ for different values of temperature dependent viscosity (α). It is observed that primary velocity is maximum when viscosity is temperature dependent as compared to when it is independent on temperature. Also, increase in temperature dependent viscosity leads to oscillation in primary velocity along distance y.

Figure 4.44depicts the graph of temperature profile $\theta(y,t)$ for different values of temperature dependent viscosity (α). It is observed that temperature increases with increase in viscosity. Also, increase in temperature dependent viscosity leads to increase in temperature along distance y.

Figure 4.45 shows the effect of porosity parameter on primary velocity profile along distance y. it is observed that the primary velocity increases and then decreases along y while increase in porosity parameter leads to increase in primary velocity.

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATIONS

5.1 CONCLUSION

For constant and variable pressure gradient $(\varepsilon = 0 \, and \, \varepsilon \neq 0)$ respectively, we have solved the equations governing the unsteady Couette flow of an electrically conducting incompressible fluid bounded by two parallel non conducting porous plates using the parameter expansion method and eigenfunction expansion technique. Also, we examined the steady state reaction of the flow using the parameter expansion technique. The effects of the dimensionless parameters as shown on the graphs were analyzed. From the results obtained, all the parameters have appreciable impact on the system since the

I. Radiation parameter reduced the temperature and primary velocity.

II. Suction parameter decreases primary velocity, secondary velocity and temperature while it enhances concentration.

III. Radiation parameter reduced the temperature and primary velocity.

IV. Prandtl number is observed to reduce temperature.

V. Schmidt number enhanced concentration and primary velocity.

VI. Hartman number enhance both primary and secondary velocities.

VII. Reynolds number reduced primary velocity and temperature while secondary velocity is enhanced.

VIII. Hall parameter reduced both primary and secondary velocities for steady state flow while it enhances secondary velocity for unsteady state flow.

IX. Constant pressure gradient enhances both temperature and primary velocity while variable pressure gradient is observed to reduce both velocities.

5.2 Contributions to Knowledge

From our findings, we achieve the following:

1. Model formulation by incorporating thermal radiation, chemical reaction equation and temperature dependent viscosity.
2. Existence of unique solution of the model by Lipschitz continuity approach
3. Properties of solution of transient state by

   - method of upper and lower solution.

   - Kolodner and Pederson Lemma

4. Properties of solution for steady case by
   - method of upper and lower solution.
5. Analytical solution.
6. The fluid concentration is at maximum value $\phi(y,t) = 2.5$ when $y = -0.5$ while the secondary velocity is at maximum value $w(y,t) = 8.0$ when $y = -0.5$.
7. Fluid flow attained maximum velocity $u(y) = 55$ when $y = 0$.
8. Graphical representation of the solution via MAPLE 17.

5.3 Recommendations

We also recommend this work for scientific and industrial use and also recommend for further study the flow of a viscous fluid through a cylinder or annulus with slip boundary conditions. Flows through a wavy microchannel can also be investigated under same boundary conditions as used in this research work.

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