MATHEMATICAL MODELLING OF MAGNETOHYDRODYNAMIC BOUNDARY LAYER FLOW OF NANOFLUID OVER A PERMEABLE EXPONENTIALY SHRINKING SHEET WITH ARRHENIUS CHEMICAL REACTION

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A THESIS SUBMITTED TO THE POSTGRADUATE SCHOOL FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA, NIGERIA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF DOCTOR OF PHILOSOPHY (PhD) IN MATHEMATICS

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ABSTRACT

This thesis investigates the dynamics of magnetohydrodynamics boundary layer flow of nanofluid over exponentially shrinking sheet with Arrhenius chemical reaction in the presence of large magnetic field. The ordinary differential equations were obtained from the partial differential equations governing the system by applying similarity parameters. The existence and uniqueness of solution of the dimensional and the transformed equations were examined by actual solution method, Derick and Grossman approach. The Properties of Solution were investigated using upper and lower solution method.The transformed equations were considered in four forms: Transient state with Arrhenius chemical reaction, steady state with Arrhenius chemical reaction, transient state with chemical reaction of constant reaction rate and steady state with chemical reaction of constant reaction rate. The equations for each form were solved using iteration perturbation technique. The physical effect of various emerging flow parameters on the fluid velocity, temperature and concentration are presented graphically and discussed. From the results obtained, it was observed that the Magnetic parameter enhanced both thermal boundary layer thickness and fluid flow along x and y direction. Also, velocity parameter and thermophoresis parameter enhanced both thermal boundary layer thickness and species concentration. The fluid temperature is at maximum value $\theta(\eta) = 4.3$ when $\eta = 10$. It was also discovered that increasing values of local Reynolds number, velocity ratio and unsteadiness decreases the primary velocity while permeability, magnetic effect, thermal grashof number and activation energy increases the velocity. The secondary velocity is increased with increasing values of local Reynolds number, permeability, magnetic effect and unsteadiness while velocity ratio and activation energy decrease the velocity. Temperature is enhanced with increase in local Reynolds number, Prandtl, magnetic effect, heat source, velocity ratio, Brownian diffusion, thermophoresis, Eckert number, Frank-kamenetskii and unsteadiness parameters, though was decreased by Radiation and activation energy. Concentration appreciates with increase in Prandtl, velocity ratio, thermophoresis and activation energy and decreases with local Reynolds number, Schmidt number, chemical reaction parameter and unsteadiness respectively. The outcome from this research work is of importance to engineering and industries especially in packaging of bulk products where shrink wrapping of products like foods, paper production, textile and even high temperature environment such as geothermal engineering where reactions rates are dependent on temperature. The result from this research work is of importance to industries that produce domestic consumables like toothpaste and food industries in production of tomato paste and fruit juice.

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Abbreviation, Glossaries and Symbols

u, v and w Velocity components along x, y and z directions

υ Kinematic viscosity

t Time

α Thermal diffusivity

Q Heat generation coefficient

σ_e Electrical conductivity

ρ_f Density of base fluid

ρ_p Density of nanoparticle

C_f Specific heat of the base fluid

C_p Specific heat of the nanoparticle

$\tau = \left(\frac{(\rho C)_p}{(\rho C)_f} \right)$ Ratio of nanoparticle heat capacity to the base fluid heat capacity

D_B Brownian diffusion coefficient

D_τ Thermophoresis diffusion coefficient

T Temperature of the fluid

C Concentration of the fluid

T_∞ Uniform temperature far from the sheet

C_∞ Uniform concentration far from the sheet

β_T Thermal expansion coefficient

β_C Volumetric expansion coefficient of concentration

β	Exothermic/Endothermic parameter
g_v	Acceleration due to gravity
k	Energy constant
k_r	Chemical reaction constant
E_a	Activation energy of the reaction
ω	Fitted rate constant
q_r	Radiative heat flux
a	Unsteadiness parameter
R_e	Local Reynolds number
γ	Permeability
E_c	Eckert number
P_r	Prandtl number
M	Magnetic parameter
δ	Frank-kemenetskii parameter
R	Radiation parameter
S_c	Schmidt number
N_b	Brownian motion parameter
N_t	Thermophoretic parameter
σ	Chemical reaction parameter
$G_{r\theta}$	Thermal Grashof number
$G_{r\phi}$	Concentration Grashof number

η Dimensionless fluid distance

f' Dimensionless fluid velocity along x - axis

g' Dimensionless fluid velocity along y – axis

θ Dimensionless fluid temperature

ϕ Dimensionless fluid concentration

ψ velocity ratio parameter

$\in$ Activation energy parameter

Ω Porosity parameter

b constant

$\in^0$ Artificial parameter order zero

$\in^1$ Artificial parameter of order one

Γ Porosity

CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

The human race has always been interested in discovering nature and some fundamental factors affecting source of life. Life as we know it would not exist without fluids and without the behaviour that fluids exhibit. The air and water we take which make most of our body mass are fluids. Motion of air keeps us comfortable in warm room, and air provides the oxygen we need to sustain life. Similarly, most of our body fluids are water based. And proper motion of these fluids within our body system, even down to the cellular level, is essential to good health. It is clear that fluids are completely necessary for the support of carbon-based life forms. The knowledge and understanding of the basic principles and concept of fluid mechanics are essential to analyse any system in which fluid is the working medium. The design of almost all means of transportation requires application of fluid mechanics. Fluids occur, and often dominate physical phenomena. In a more practical setting, we easily see that fluids greatly influence our comfort (or lack thereof); they are involved in our transportation systems in many ways we stated above; they have an effect on our recreation (e.g. basket ball's and footballs are inflated with air) and entertainment (the sound from the speakers of a TV would not reach our ears in the absence of air) and even on our sleep (water beds). (McDonough, 2009)

From this it is easy to see that engineers must have at least a working knowledge of fluid behaviour to accurately analyze many, if not most, of the systems they will encounter.

1.1.1 Magnetohydrodynamics

In recent year, a great deal of interest has been evidenced in the study of magnetohydrodynamics boundary layer flow because of its much industrial application. Magnetohydrodynamics (MHD) is a discipline concerned with the dynamics of electrically conducting fluids in a magnetic field. These fluids include salt water, liquid metals (such as mercury, gallium, and molten iron) and ionized gases or plasma (such as solar atmosphere). The term MHD is made up of the words magneto-meaning magnetic, hydro-meaning fluids and dynamics-meaning movement. The field of MHD was initiated by the Swedish physicist Hanes Alfven (1908-1995), who received the Nobel Prize in physics in 1970 for fundamental work and discoveries in magnetohydrodynamics with fruitful applications in different parts of plasma physics. MHD covers those phenomena, where, in an electrically conducting fluid, the velocity V and the magnetic field B are coupled. The magnetic field induces an electric current of density j in the moving conducting fluid (electromagnetism). The induced current creates forces on the liquid and also changes the magnetic field. Each unit volume of the fluid having magnetic field B experiences an MHD force $j \times B$ known as Lorentz force. The set of equations which describe MHD flows are combination of Navier-strokes equation of fluid dynamics and Maxwell's equation of electromagnetism. (Winifred, 2014)

1.1.2 Nanofluid

In spite of considerable previous research and development on heat transfer enhancement, major improvements in cooling capabilities have been constrained because of low thermal conductivity of conventional heat transfer fluids. Traditional heat transfer fluids such as water, lubricant, ethylene-glycol, engine oil, etc have a limitation in heat transfer capabilities because of lower thermal conductivities and do not meet modern cooling requirement. On the other hand, metals possess higher thermal conductivity in contrast to conventional heat transfer fluids(khan *et al.*, 2015). For instance, at room temperature the thermal conductivity of copper is approximately 700 times greater than that of water and 3000 times greater that of engine oil (Choi *et al.*, 1995).

Metal oxide and metallic particles have higher thermal conductivity than those of the conventional heat transfer fluids, and thus opined by different researcher's that inclusion of such highly conductive particles can increase thermal conductivity of heat transfer fluids. The necessity of improving the thermal conductivity and enhancing the heat transfer has led to the utilization of nanoparticles in the fluid. Nanofluids are engineered colloidal suspensions of nanoparticles in the base fluid, that is, suspended nanoparticles in conventional fluids are called nanofluids. This new heat transfer coolant can be considered to be the next generation heat transfer fluids because they offer exciting new possibilities to improve heat transfer compared to pure liquids (Nandy *et al.*, 2014)

The term nanoparticle comes from the latin prefix'nano'. It prefix is used to denote the 10^{-9} particle of a unit. In this context, nano-particles have a size between 100nm-2500nm. Particles smaller than 100nm are termed ultrafine. These objects are being extensively explored due to their possible application in medical, optical and electronics field. The most popular nanoparticles that use to produce nanofluids are: Aluminium oxide (AL_2O_3), Copper (II) oxide (CuO), copper (Cu). Water, oil, decene, acetone and ethylene glycol are the most common base fluids being used in producing nanofluids (Kostic, 2004).

Nanofluids are advantageous as they provide:

- More heat transfer between particles and fluids due to high specific surface area.
- High dispersion stability with predominant Brownian motion of particles.
- Adquate heat transfer intensification because of reduction in pumping power as compared to pure liquid.
- System miniaturization because particle clogging is reduce as compared to conventional slurries.
- Adjustable properties, including thermal conductivity and surface wet ability, by varying particle concentration to suit different applications.

Some of their limitations however include:

High cost of nanofluids.

Difficulties in production process. (Han, 2008).

1.1.3 Heat Transfer

Heat Transfer is the study of the exchange of thermal energy through a body or between bodies which occurs when there is temperature gradient. When two bodies are at different temperature, their thermal energy transfers from one with higher temperature to the one with lower temperature. Thermal energy is related to the temperature of matter for a given material and mass, the higher the temperature the greater it's thermal energy.

Heat is typically given the symbol Q, it is expressed in joules (J) as its S.I units. The time rate of heat transfer (power) is measured in watts (W), equal to joules per second and is denoted by q. Thermal flux occurs through one of these modes or combination of them.

Heat transfer due to convection involves the energy exchange between a solid surface and an adjacent fluid. Convection is the term applied to the heat transfer mechanism which occurs in a fluid by mixing of one portion of the fluid with another portion due to gross movements of the mass of fluid. Convection of heat transfer is classified as forced convection and free convection. If heat transfer between the fluid and the solid surface occurs by fluid motion induced by external forces then the mode of heat transfer is termed "Forced Convection". Heat transfer in all types of heat exchangers, nuclear reactors, air conditioning apparatus are example of devices functioning based on forced convection. If heat transfer between the fluid and solid surface occurs by the fluid motion due to the density differences caused by the buoyancy between the surface and fluid, then the mode of the transfer is termed as free convection or natural convection (as in sea and land breeze). The circulation of water in a vessel heated on stove, heat flows from a heated metal plate to the atmosphere are examples of free convection.

The process in which heat is transferred between those object that are in physical contact is called conduction, while radiation does not require a medium to pass through and it is the only form of heat transfer present in vacuum. It uses electromagnetic radiation known as photons which travels at the speed of light and is emitted by any matter with temperature above 0 degree Kelvin (-27 ^{0}C). We all experience radiative heat transfer every day, solar radiation absorbed by our skin is why we feel warmer in the sun than in the shade. (Cengel,2003)

Some of the problems studied by many researchers are limited to steady state flow. In many engineering problems, such as helicopter rotor, the ship propeller, the cascades of blades of turbo-machinery unsteady environment occurs. However, the flow of heat transfer problems in reality has unsteady nature owing to the sudden stretching/shrinking of the sheet. Hence it is very much important to investigate the simultaneous effects of thermal radiation, magnetic field and unsteadiness.

1.2 Statement of the Research Problem

There is increasing need for advance heat transfer fluids with reasonably higher thermal conductivities than are presently available because major improvement in cooling capability have been limited due to low thermal conductivities of ordinary heat transfer fluids. The frequent demand and interest by researchers to enhance the flow over shrinking/stretching surface because of many engineering process and applications has motivated us to seek for higher thermal conductivity of heat conveyance over a shrinking sheet. Based on these scenarios, there is need for this study to broaden the scope of what is already known about the boundary layer flow of nanofluid over permeable shrinking sheet.

1.3 Aim and Objectives of the Research

1.3.1 Aim of the Research

The aim of this research is to develop a mathematical model and study the dynamics of magnetohydrodynamics boundary layer flow of a nanofluid over a permeable shrinking sheet in the presence of thermal radiation with Arrhenius chemical reaction.

1.3.2 Objectives of the Study

The objectives of the study are to:

(i) Establish the criteria for the existence and the uniqueness of solution of the model formulated

(ii) Examine the properties of the solution of the model

(iii) Solve the resulting equations describing the MHD phenomenon over the shrinking sheet using the Iteration Perturbation Technique (IPT)

(iv) Provide the graphical representation of the system responses

1.4 Justification of the Study

Understanding fluid and the behaviour of fluid flow is important aspect of life and technology. In spite of recent development in heat transfer, ordinary heat transfer fluids has a limitation due to their low heat transfer strength. In contrast metallic particles known as nano particles gives new possibilities on improving heat transfer capacity, because the quality of a final product depends on the rate of heat transfer. It is hoped therefore that the outcome of this study will assist in identifying these sensitive parameters affecting the flow and suggest a framework that will improve it. Research institution, stakeholders in government and non-government bodies would find the study beneficial in creating innovative methods and techniques in improving the flow concept over the shrinking sheet.

1.5 Scope and Limitation of the Study

The focus of the work will be basically on modeling and simulation of magnetohydrodynamics boundary layer flow of a nanofluid over a permeable shrinking sheet in the presence of thermal radiation with Arrhenius chemical reaction. We shall establish the criteria for the existence and the uniqueness of solution of the model and also examine the features of solution of the model and then solve the equations by Iteration perturbation method. The work will be restricted to dynamics of incompressible nanofluid.

1.6 Definition of Terms

Nanofluid: Nanofluid are engineered colloidal suspensions of nanoparticles in the base fluid (Manca *et al*., 2010; Elena et al., 2011; Wang *et al*., 2020; Roberto *et al*., 2015).

Incompressible Flow: A flow in which the density of flow remains approximately constant throughout the flow field is called incompressible flow (e.g. liquid flow) (Ronald, 2013).

Thermal Conductivity: The property of the material which is related to the capacity of transmitting heat is called thermal conductivity (Yang, 2004).

Permeability: The property that allow fluids or gases to pass or diffuse through (Rahmouni et al., 2013).

Hydrodynamic Boundary Layer: The region near solid surface where the flow configuration is achieved by viscous drag directly from surface wall (Nikolov, 2021).

Steady Flow: The flow that are independent on time, that is if the properties at any point in the flow field do not change with time. Mathematically, it can be written as $\frac{d\varphi}{dt} = 0$, where φ is the fluid property (Swain, 2016).

Thermal Diffusivity: This is the ratio of thermal conductivity to the product of density and specific heat capacity at constant pressure (Salazar, 2003).

Unsteady Flow: This is a flow that changes with time, that is the rate of change with time is non zero. Mathematically, it can be written as $\frac{d\varphi}{dt} \neq 0$, where φ is the fluid property.

Convection: A process whereby heat is transferred through fluids (gases or liquids).

Radiation: A process in which heat is transferred directly by electromagnetic waves and it occurs when two bodies of different temperature are aligned (Fermi,1932).

Grashof Number (G_r)**:** A dimensionless quantity which approximates the ratio of the buoyancy to viscous force acting on the fluid. Mathematically it is expressed as $G_r = \frac{g\beta(T_s - T_\infty)L^3}{\upsilon^2}$ for thermal grashof and $G_{r\phi} = \frac{g\beta_c(C_s - C_a)L^3}{\upsilon^2}$ for solutal grashof.

Prandtl Number (P_r)**:** This is the ratio momentum diffusivity upon thermal diffusivity which is given by the relation $\Pr = \frac{\upsilon}{\alpha}$.

Reynold Number (R_e)**:** is the ratio of inertial forces to viscous forces within a fluid. The region where these forces change behavior is referred to as boundary layer. It is exoressed as $R_e = \frac{\rho L u}{\mu}$ where ρ density, u is the velocity, L is the length scale and μ is the viscosity of the fluid.

Schmidt Number S_c **:** is the ratio momentum diffusivity (kinematic viscosity) and mass diffusivity and mathematically given by $S_c = \frac{\upsilon}{D}$.

Eckert Number E_c **:** is a dimensionless number defining the ratio between kinetic of the flow and enthalpy. Mathematically, $E_c = \frac{u_w^2}{C_p(\Delta T)}$ where u_w^2 is the flow velocity, C_p the specific heat and ΔT is the temperature.

IPM: Iteration Perturbation Method.

Mathematical Modelling: is a description of a system using mathematical concepts and language.

Definitions:

Definition 1 (Olayiwola, 2011): A smooth function $\underline{u}$ is said to be a lower solution of the problem

$$Lu = f\left(x,t,u\right)$$

where

$$L = \frac{\partial}{\partial t} + a\left(x,t\right)\frac{\partial^2}{\partial x^2} + b\left(x,t\right)\frac{\partial}{\partial x} + c\left(x,t\right)$$

If $\underline{u}$ satisfies

$$L\underline{u} \le f\left(x,t,\underline{u}\right)$$

$$\underline{u}\left(x,0\right) \le f\left(x\right), \quad \underline{u}\left(0,\mathrm{t}\right) \le h_1\left(t\right), \quad \underline{u}\left(\mathrm{L},\mathrm{t}\right) \le h_2\left(t\right)$$

Definition 2 (Olayiwola, 2011): A smooth function $\bar{u}$ is said to be an upper solution of the problem

$$Lu = f\left(x,t,u\right)$$

where

$$L=\frac{\partial}{\partial t}+a(x,t)\frac{\partial^2}{\partial x^2}+b(x,t)\frac{\partial}{\partial x}+c(x,t)$$

If $\bar{u}$ satisfies

$$L\bar{u}\geq f\left(x,t,\bar{u}\right)$$

$$\bar{u}(x,0)\geq f\left(x\right),\quad \bar{u}(0,\mathrm{t})\geq h_1(t),\quad \bar{u}(\mathrm{L},\mathrm{t})\geq h_2(t)$$

Definition 3 (Olayiwola, 2011): A smooth function $\underline{u}$ is said to be a lower solution of the problem

$$Lu=f\left(x,u\right)$$

where

$$L=a(x)\frac{d^2}{dx^2}+b(x)\frac{d}{dx}+c(x)$$

If $\underline{u}$ satisfies

$$L\underline{u}\geq f\left(x,\underline{u}\right)$$

$$\underline{u}(0)\leq h_1,\quad \underline{u}(\mathrm{L})\leq h_2$$

Definition 4 (Olayiwola, 2011): A smooth function $\bar{u}$ is said to be an upper solution of the problem

$$Lu=f\left(x,u\right)$$

where

$$L=a(x)\frac{d^2}{dx^2}+b(x)\frac{d}{dx}+c(x)$$

If $\bar{u}$ satisfies

$$L\bar{u}\leq f\left(x,\bar{u}\right)\ \bar{u}(0)\geq h_1,\quad \bar{u}(\mathrm{L})\geq h_2$$

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Reviews of some related work

Many researchers have developed and extensively studied the transport properties of nanofluid over the last few decades. The first nanofluid model was proposed by Buongiorno (2006) and Recently, a lot of work have used mathematical nanofluid model developed by Buongiorno to examine different problems concerning the behavior of nanofluids. Abu-Nadal *et al*., (2008), Nady *et al*., (2014), Khan *et al*., (2015), Aiyesimi *et al*., (2015), Yusuf *et al*., (2016) to list few among others have worked on nanofluid under various application and different situation.

Buongiorno (2006) worked on convective transport in nanofluids. He considerd thermophoresis and Brownian motion effects, and highlighted that even though there are different elements which affects nanofluid flow such as gravity, diffusiophoresis, magnus effects, fluid drainage, Brownian diffusion and inertia, only thermophoresis and Brownian diffusion have significant effect on nanofluid. Recently, a lot of works with reference to Buongiorno idea have examined different problem concerning the behavior of nanofluid and their various applications under different situations.

Rajesh *et al.* (2015) examined Transient MHD Nanofluid Flow and Heat Transfer due to a Moving Vertical plate with Thermal Radiation and Temperature Oscillation Effects. The different physical parameters on the nanofluid flow and heat transfer characteristics were numerically examined and also that the heat transfer rate reduce with increase in Eckert number and magnetic parameter and heat transfer rate increase with nano particle volume fraction.

Nady *et al.* (2014) investigated unsteady MHD boundary layer flow and heat transfer of nanofluid over a permeable shrinking sheet in the presence of thermal radiation. They found out dual existence for the flow over the shrinking sheet. Magnetic and suction parameter, temperature, nanoparticle volume fraction, effective prandtl parameter, lewis number, nusselt and Sherwood numbers all have reasonable effects on the flow. It was discovered that magnetic field and wall mass suction widen the range of unsteadiness parameter for which the solution exists. The skin friction coefficient, local nusselt and sherwood numbers increase for the first solution and decrease for the the second solution with increase in magnetic parameter.

Yusuf *et al*. (2016) discussed the Analysis of couette flow of a nanofluid in an inclined channel with soret and dafour effects. They concluded that decrease in temperature and nanofraction profile resulted from increase in soret number. They also added that increase in thermal conductivity of the fluid due to larger values of prandtl number enable heat to diffuse away from the heated surface more rapidly, this imply that there is decrease in temperature due to reduction in prandtl number.

Khan and Khan (2015) studied the MHD boundary layer flow of a power-law nanofluid with new mass flux condition. Among their findings, Temperature increases with increase in thermophoresis and Brownian motion parameter, increase in stretching parameter values result to decrease in both temperature and concentration profiles.

Three-Dimensional Flow of a Nanofluid Induced by an Exponentially Stretching Sheet was investigate by (Khan *et al.* 2015). They derived the governing equations to be

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (2.1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\upsilon\frac{\partial^2 u}{\partial z^2} \qquad (2.2)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=\upsilon\frac{\partial^2 v}{\partial z^2} \qquad (2.3)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\alpha\frac{\partial^2 T}{\partial z^2}+\tau\left[D_B\left(\frac{\partial C}{\partial Z}\frac{\partial T}{\partial Z}\right)+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial Z}\right)^2\right] \qquad (2.4)$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=\left[D_B\left(\frac{\partial^2 C}{\partial Z^2}\right)+\frac{D_T}{T_\infty}\left(\frac{\partial^2 T}{\partial Z^2}\right)\right] \qquad (2.5)$$

and applied the implicit finite difference scheme known as Keller-box method for local similarity solution. They concluded that increase in thermophoresis and Brownian motion parameter increase the temperature. This means that heat transfer rate from the sheet reduces when the effects of Brownian motion and thermophoresis strengths are increased. They added that increase in nanoparticle fraction and mass transfer rate of the sheet decreases when thermophoresis parameter is increased.

Significant rise of attention by researchers in recent years is been given to coupled heat and mass transfer problems in the presence of chemical reaction due to its importance in many process. From the application point of view, the radiative transfer of heat in boundary layer flow is very important, because the quality of the final product is very much dependent on the rate of heat transfer of the ambient fluid particles (Nield and Bejan 2006).

Chemical reactions can occur in processes such as drying, distribution of temperature and moisture over agricultural fields and groves of fruit trees, damage of crops due to freezing, evaporation at the surface of the water body, energy transfer in wet cooling tower and flow in a desert cooler. Analysis of the transport processes and their interaction with chemical reactions is quiet difficult and closely related to fluid dynamics. Chemical reaction effects on heat and mass transfer has been analyzed by many researchers over various geometries with various boundary conditions in porous and nonporous media. Wahduzzaman *et al*. (2015) considered MHD Flow of Fluid over a Rotating Inclined Permeable plate Variable Reactive Index. They found out that increase in thermal conductivity and Eckert number increases heat transfer rate and primary velocity.

Razman *et al.* (2017) explored Buoyancy Effects on the Radiative Magneto Micropolar nanofluid flow with double stratification, Activation Energy and Binary Chemical Reaction. Among the findings, Heat transfer rate improves for increasing thermal stratification. Concentration profile ascents for increasing chemical reaction parameter and diminish for enhancing solutal stratification parameter.

One of the important flows in fluid mechanics is the flow over a shrinking sheet. The flow induced by an exponentially shrinking sheet is not studied much, though it is very important and realistic flow that frequently appears in many engineering process and is a new field of research at present and few literatures are available on this area of research now. The study of boundary layer flow over a shrinking surface is an ideal concept in several industrial processes, such situation take place in manufacturing of glass sheet, polymer dispensation, paper manufacturing, in textile industries and many others. The most common applications of shrinking sheet problem in engineering and industries are shrinking films. (Anuradha and Priyadharshini, 2016)

Wang (1990) was the first to apply a shrinking sheet problem. Recently, Miklavcic and Wang (2006), Anuradha and priyadharshini (2016), Anuradha *et al*. (2017) have worked on shrinking problems using different approaches

Anuradha and Priyadharshini (2016) studied MHD Convection Boundary Layer Flow of a Nano Fluid over a Permeable Shrinking Sheet in the Presence of Thermal Radiation and Chemical Reaction. They observed that increasing magnetic parameter values increases the velocity and diminish concentration and temperature profiles. The temperature of the fluid is decreased with increase in unsteadiness parameter and thus increase heat transfer rate.

They derived the governing equations to be

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.6}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial z^2}-\frac{\mu}{k_\rho}u-\frac{\sigma_e B^2 u}{\rho_f}+(1-C_\infty)\rho_\infty\beta g(T-T_\infty) \\ -(\rho_f-\rho f_\infty)g(C-C_\infty) \tag{2.7}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\partial}{\partial z}\left(\alpha\frac{\partial T}{\partial z}\right)+\tau\left\{D_B\left(\frac{\partial T}{\partial z}\frac{\partial C}{\partial z}\right)+\frac{D_\tau}{T_\infty}\left(\frac{\partial T}{\partial z}\right)^2\right\}+ \\ \frac{Q}{(\rho c)}(T-T_\infty)-\frac{1}{(\rho c)_f}\left(\frac{\partial q_r}{\partial z}\right) \tag{2.8}$$

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_B\frac{\partial^2 C}{\partial Z^2}+\frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial Z^2} \tag{2.9}$$

Anuradha and Yegammai (2017) worked MHD Free Convectional Boundary Layer Flow over an Exponentially Permeable Stretching Sheet with Chemical Reaction and Activation Energy. They realized that nanoparticle volume fraction escalates with increase in prandtl number thermophoresis parameter, and non dimensional energy (E). Nano particle volume fraction profile decreases with increase in Brownian motion parameter (Nb), increasing value of temperature difference parameter (δ) and increasing value of dimensionless reaction rate (σ) within the boundary layer region.

Zaib *et al.* (2019) revised the Impact of nonlinear radiative nanoparticles on an unsteady flow of a Williamson fluid toward a permeable convective heated shrinking sheet. Their results point out that multiple solutions are achieved for certain values of the suction parameter and for decelerating flow, while for accelerating flow, the solution is unique. Further, the non- Newtonian parameter reduces the fluid velocity and boost the temperature distribution Razman *et al.* (2017)

Maleque (2013) investigated Exothermic/Endothermic Chemical Reaction with Arrehenius Activation Energy on MHD Free Convection and Mass Transfer Flow in Presence of Thermal Radiation. He observed that increase in chemical reaction rate lead to increase in velocity and temperature profiles for exothermic reaction but opposite effects for endothermic reaction. Velocity profile increase the value of activation energy for endothermic chemical reaction but negligible effect is found for exothermic reaction. Temperature reduces for increasing value of activation energy for exothermic reaction but the reverse found for endothermic reaction.

2.2 Summary of Review and Gaps to fill

In reviewing the above literatures, several works have been carried out on magnetohydrodynamics boundary layer flow of nanofluid. Some authors worked on shrinking sheet problem without considering unsteadiness, others concentrated on one dimensional problem and ignoring magnetic, porosity, permeability and Arrhenius chemical reaction.

However, this research work seeks to consider the magnetohydrodynamics boundary layer flow of a nanofluid over permeable sheet by incorporating

(i) Unsteadiness

(ii) Magnetic field effect

(iii) Porosity

(iv) Arrhenius chemical reaction

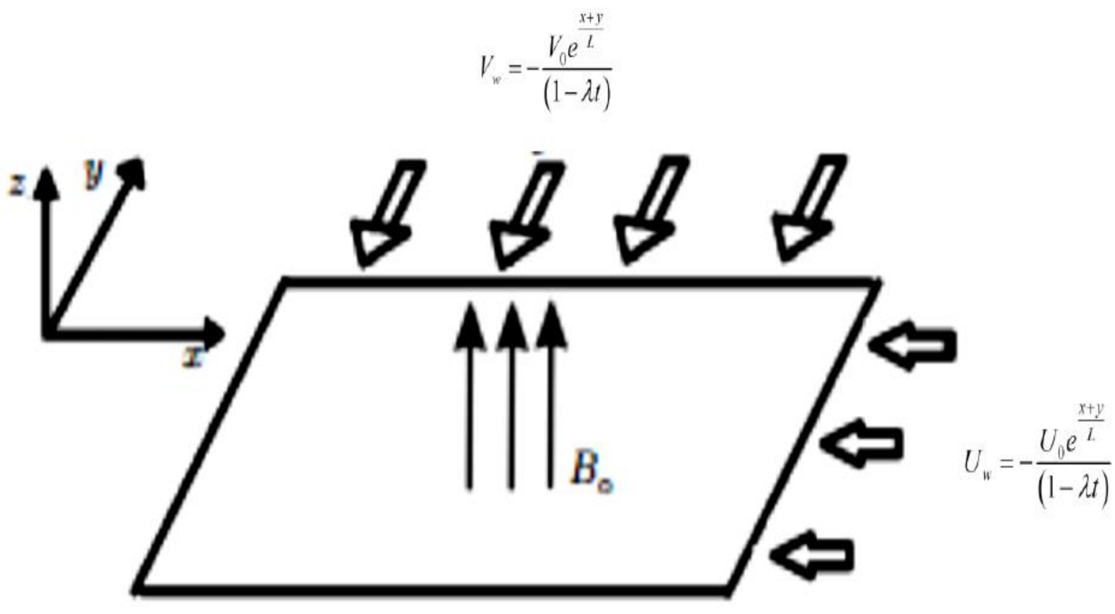
(v) 3 D problems

CHAPTER THREE

3.0 MATERIALS AND METHODS

3.1 Mathematical Formulation

We consider the transient three dimensional incompressible boundary layer flow of nanofluid over a permeable sheet shrank exponentially along $x\,y$ direction in the presence of magnetic field of strength B_0 which is applied perpendicular to the flow direction in the z-axis as shown Figure 3.1. Arrhenius chemical reaction with thermal radiation is considered in the flow region, we suppose that the sheet was shrinking with velocities $U_w = -\frac{U_0 e^{\frac{x+y}{L}}}{(1-\lambda t)}$ and $V_w = -\frac{V_0 e^{\frac{x+y}{L}}}{(1-\lambda t)}$ along the $x\,y$ plane where U_0 and V_0 are constants,



$\lambda \geq 0$ with $\lambda t < 1$.

Figure 3.1: The diagram of the physical system

Under the above assumptions, the boundary layer equations of the three dimensional incompressible fluid governing the prevailing flow, thermal, concentration fields, conservation of mass, momentum, energy and nanoparticles mass are as (Anuradha and Priyadharshini 2016; Anuradha *et al*., 2017 and Khan *et al*., 2015)

Continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{3.1}$$

Momentum equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\upsilon\frac{\partial^2 u}{\partial z^2}-\frac{\upsilon}{k_\rho}u-\Gamma u^2-\frac{\sigma_e B^2 u}{\rho_f}+(1-C_\infty)\rho_f\beta_T g_v(T-T_\infty)$$
$$-(\rho_P-\rho_f)g_v\beta_C(C-C_\infty) \tag{3.2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=\upsilon\frac{\partial^2 v}{\partial z^2}-\frac{\upsilon}{k_\rho}v-\Gamma v^2-\frac{\sigma_e B^2 v}{\rho_f} \tag{3.3}$$

Energy equation:

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\frac{\partial}{\partial z}\left(\alpha\frac{\partial T}{\partial z}\right)+\tau\left\{D_B\left(\frac{\partial T}{\partial z}\frac{\partial C}{\partial z}\right)+\frac{D_\tau}{T_\infty}\left(\frac{\partial T}{\partial z}\right)^2\right\}+$$
$$\frac{Q}{(\rho c)}(T-T_\infty)-\frac{1}{(\rho c)_f}\left(\frac{\partial q_r}{\partial z}\right)+\beta k_r^{\,2}(T-T_\infty)^\omega(C-C_\infty)e^{-\left(\frac{Ea}{K(T-T_\infty)}\right)}+\frac{\sigma_e B_0^{\,2}\left(u^2+v^2\right)}{\rho_f} \tag{3.4}$$

Species equation:

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=D_B\frac{\partial^2 C}{\partial z^2}+\frac{D_\tau}{T_\infty}\frac{\partial^2 T}{\partial z^2}-k_r^{\,2}(T-T_\infty)(C-C_\infty)^\omega e^{-\left(\frac{Ea}{K(T-T_\infty)}\right)} \tag{3.5}$$

Implementing Rosseland's approximation, the radiative heat flux

$$q_r = -\frac{4\sigma_1}{3k_1}\frac{\partial\left(T^4\right)}{\partial z} \tag{3.6}$$

where σ_1 and k_1 are the Stefan-Boltzmann constant and mean absorption coefficient respectively.

If we suppose that the temperature difference within the flow are small such that T^4 can be expressed as a function of temperature linearly, then the expansion of T^4 about T_∞ in taylor form is written as

$$T^4 = T_\infty{}^4 + 4T_\infty{}^3\left(T - T_\infty\right) + 6T_\infty{}^2\left(T - T_\infty\right)^2 + \dots. \tag{3.7}$$

Neglecting higher order terms after the first degree, we have

$$T^4 = 4T_\infty{}^3 T - 3T_\infty{}^4 \tag{3.8}$$

Putting (3.8) into (3.6), gives

$$q_r = -\frac{4\sigma_1}{3k_1}\frac{\partial\left(4T_\infty{}^3 T - 3T_\infty{}^4\right)}{\partial z} \tag{3.9}$$

$$\frac{\partial q_r}{\partial z} = \frac{\partial}{\partial z}\left(-\frac{4\sigma_1}{3k_1}\frac{\partial\left(4T_\infty{}^3 T - 3T_\infty{}^4\right)}{\partial T}\frac{\partial T}{\partial z}\right) = -\frac{16T_\infty{}^3\sigma_1}{3k_1}\frac{\partial^2 T}{\partial z^2} \tag{3.10}$$

$$\frac{\partial q_r}{\partial z} = -\frac{16T_\infty{}^3\sigma_1}{3k_1}\frac{\partial^2 T}{\partial z^2} \tag{3.11}$$

The initial and boundary conditions are stated as:

$$u(z,t)=0, v(z,t)=0, T(z,t)=T_\infty, C(z,t)=C_\infty \quad for\, t\le 0 \; for \; all \; z$$
$$u=-U_w, v=-V_w, w=0, T=T_w, C=C_w \; at \; z=0 \quad , \tag{3.12}$$
$$u\to 0, v\to 0, T\to T_\infty, C\to C_\infty \; at \; z\to\infty$$

3.2 Method of Solution

3.2.1 Similarity Transformation

Using the following similarity variables

$$u=\frac{U_0}{(1-\lambda t)}e^{\frac{x+y}{L}}f', v=\frac{U_0}{(1-\lambda t)}e^{\frac{x+y}{L}}g', \eta=\sqrt{\frac{U_0}{2\upsilon L}}\frac{e^{\frac{x+y}{2L}}}{(1-\lambda t)^{\frac{1}{2}}}z$$
$$T=T_\infty+\frac{(T_w-T_\infty)e^{\frac{x+y}{L}}}{(1-\lambda t)}(1+\varepsilon\theta), C=C_\infty+\frac{(C_w-C_\infty)e^{\frac{x+y}{L}}}{(1-\lambda t)}\phi, \; \varepsilon=\frac{K(T_w-T_\infty)e^{\frac{x+y}{L}}}{E_a(1-\lambda t)} \tag{3.13}$$

From equation (3.1)

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=-\frac{\partial w}{\partial z} \tag{3.14}$$

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(\frac{U_0}{(1-\lambda t)}e^{\frac{x+y}{L}}f'\right)$$

$$=\frac{U_0}{L(1-\lambda t)}e^{\frac{x+y}{L}}f'+\frac{U_0}{(1-\lambda t)}e^{\frac{x+y}{L}}\frac{\partial f'}{\partial\eta}\frac{\partial\eta}{\partial x}$$

$$=\frac{U_0}{L(1-\lambda t)}e^{\frac{x+y}{L}}f'+\frac{U_0}{(1-\lambda t)}e^{\frac{x+y}{L}}f''\frac{\partial}{\partial x}\left(\sqrt{\frac{U_0}{2\upsilon L}}\frac{e^{\frac{x+y}{2L}}}{(1-\lambda t)^{\frac{1}{2}}}z\right)$$

$$\frac{\partial u}{\partial x}=\frac{U_0 e^{\frac{x+y}{L}}}{L(1-\lambda t)}\left(f'+\frac{\eta}{2}f''\right) \tag{3.15}$$

Similarly,

$$\frac{\partial v}{\partial y}=\frac{U_0 e^{\frac{x+y}{L}}}{L(1-\lambda t)}\left(g'+\frac{\eta}{2}g''\right) \tag{3.16}$$

Then, from 3.14

$$-\frac{\partial w}{\partial z}=\frac{U_0}{L(1-\lambda t)}e^{\frac{x+y}{L}}\left(f'+\frac{\eta}{2}f''+g'+\frac{\eta}{2}g''\right)$$

$$-\frac{\partial w}{\partial \eta}\cdot\frac{\partial \eta}{\partial z}=\frac{U_0}{L(1-\lambda t)}e^{\frac{x+y}{L}}\left(f'+\frac{\eta}{2}f''+g'+\frac{\eta}{2}g''\right)$$

$$-\frac{\partial w}{\partial \eta}\cdot\left(\sqrt{\frac{U_0}{2\upsilon L}}\frac{e^{\frac{x+y}{2L}}}{(1-\lambda t)^{\frac{1}{2}}}\right)=\frac{U_0}{L(1-\lambda t)}e^{\frac{x+y}{L}}\left(f'+\frac{\eta}{2}f''+g'+\frac{\eta}{2}g''\right)$$

$$\frac{\partial w}{\partial \eta}=-\sqrt{\frac{2\upsilon U_0}{L(1-\lambda t)}}e^{\frac{x+y}{2L}}\left(f'+\frac{\eta}{2}f''+g'+\frac{\eta}{2}g''\right) \tag{3.17}$$

Taking the integral of both side of (3.17), gives

$$w=-\sqrt{\frac{2\upsilon U_0}{L(1-\lambda t)}}e^{\frac{x+y}{2L}}\left(\int f'd\eta+\frac{1}{2}\int \eta f''d\eta+\int g'd\eta+\frac{1}{2}\int \eta g''d\eta\right)$$

$$w=-\sqrt{\frac{2\upsilon U_0}{L(1-\lambda t)}}e^{\frac{x+y}{2L}}\left(f+\frac{1}{2}\eta f'-\frac{1}{2}f+g+\frac{1}{2}\eta g'-\frac{1}{2}g\right)$$

$$w = -\sqrt{\frac{2\upsilon U_0}{L(1-\lambda t)}} e^{\frac{x+y}{2L}} \left(\frac{1}{2} f + \frac{1}{2} \eta f' + \frac{1}{2} g + \frac{1}{2} \eta g' \right)$$

$$w = -\frac{1}{2}\sqrt{\frac{2\upsilon U_0}{L(1-\lambda t)}} e^{\frac{x+y}{2L}} \left(f + \eta f' + g + \eta g' \right)$$

$$w = -\sqrt{\frac{2\upsilon U_0}{4L(1-\lambda t)}} e^{\frac{x+y}{2L}} \left(f + \eta f' + g + \eta g' \right)$$

$$w = -\sqrt{\frac{\upsilon U_0}{2L(1-\lambda t)}} e^{\frac{x+y}{2L}} \left(f(\eta) + \eta f'(\eta) + g(\eta) + \eta g'(\eta) \right) \tag{3.18}$$

We set

$$\left.\begin{aligned}
&Q = \frac{Q_0 e^{\frac{x+y}{L}}}{(1-\lambda t)}, \quad K = \frac{K_0(1-\lambda t)}{e^{\frac{x+y}{L}}}, \quad \beta_T = \frac{\beta_{T0} e^{\frac{x+y}{L}}}{(1-\lambda t)}, \\
&\beta_C = \frac{\beta_{C0} e^{\frac{x+y}{L}}}{(1-\lambda t)}, \quad k_\rho = \frac{k_{\rho 0}(1-\lambda t)}{e^{\frac{x+y}{L}}}, \quad k_r = \frac{k_{r0} e^{\frac{\frac{x+y}{2}}{1-\omega}L}}{(1-\lambda t)^{\frac{1-\omega}{2}}}, \quad \tau = \frac{1}{\tau_0}\frac{(1-\lambda t)}{e^{\frac{x+y}{L}}}, \\
&\in = \frac{K(T_w - T_\infty)}{E_a(1-\lambda t)} e^{\frac{x+y}{L}}, \quad T_w = T_\infty + \frac{(T_w - T_\infty)}{(1-\lambda t)} e^{\frac{x+y}{L}} (1+\in), \\
&T_\infty = T_\infty + \frac{(T_w - T_\infty)}{(1-\lambda t)} e^{\frac{x+y}{L}}
\end{aligned}\right\} \tag{3.19}$$

and then solving for terms in equations (3.1) to (3.11) gives

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\frac{U_0}{(1-\lambda t)} e^{\frac{x+y}{L}} f' \right)$$

$$f' U_0 (-1)(-\lambda)(1-\lambda t)^{-2} e^{\frac{x+y}{L}} + \frac{U_0}{(1-\lambda t)} e^{\frac{x+y}{L}} \frac{\partial f'}{\partial \eta}\frac{\partial \eta}{\partial t}$$

$$f'(\eta)U_0(-1)(-\lambda)(1-\lambda t)^{-2}\, e^{\frac{x+y}{L}} + \frac{U_0}{(1-\lambda t)} e^{\frac{x+y}{L}} f'' \frac{\partial}{\partial t}\left(\sqrt{\frac{U_0}{2\upsilon L}}\, \frac{e^{\frac{x+y}{2L}}}{(1-\lambda t)^{\frac{1}{2}}}\, z \right)$$

$$\frac{\partial u}{\partial t} = \frac{\lambda U_0 e^{\frac{x+y}{L}}}{(1-\lambda t)^2}\left(f' + \frac{\eta}{2} f'' \right) \tag{3.20}$$

Similarly,

$$\frac{\partial v}{\partial t} = \frac{\lambda U_0 e^{\frac{x+y}{L}}}{(1-\lambda t)^2}\left(g' + \frac{\eta}{2} g'' \right) \tag{3.21}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{U_0}{(1-\lambda t)} e^{\frac{x+y}{L}} f' \right)$$

$$= \frac{U_0}{L(1-\lambda t)} e^{\frac{x+y}{L}} f' + \frac{U_0}{(1-\lambda t)} e^{\frac{x+y}{L}} \frac{\partial f'}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= \frac{U_0}{L(1-\lambda t)} e^{\frac{x+y}{L}} f' + \frac{U_0}{(1-\lambda t)} e^{\frac{x+y}{L}} f'' \frac{\partial}{\partial x}\left(\sqrt{\frac{U_0}{2\upsilon L}}\, \frac{e^{\frac{x+y}{2L}}}{(1-\lambda t)^{\frac{1}{2}}}\, z \right)$$

$$\frac{\partial u}{\partial x} = \frac{U_0 e^{\frac{x+y}{L}}}{L(1-\lambda t)}\left(f' + \frac{\eta}{2} f'' \right) \tag{3.22}$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{U_0 e^{\frac{x+y}{L}}}{L(1-\lambda t)}\left(f' + \frac{\eta}{2} f'' \right) \tag{3.23}$$

$$\frac{\partial v}{\partial x} = \frac{U_0 e^{\frac{x+y}{L}}}{L(1-\lambda t)}\left(g' + \frac{\eta}{2} g'' \right) \tag{3.24}$$

$$\frac{\partial v}{\partial y} = \frac{U_0 e^{\frac{x+y}{L}}}{L(1-\lambda t)}\left(g' + \frac{\eta}{2} g'' \right) \tag{3.25}$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(\frac{U_0}{(1-\lambda t)} e^{\frac{x+y}{L}} f' \right)$$

$$= \frac{U_0}{(1-\lambda t)} e^{\frac{x+y}{L}} \frac{\partial f'}{\partial \eta} \frac{\partial \eta}{\partial z}$$

$$= \frac{U_0}{(1-\lambda t)} e^{\frac{x+y}{L}} f'' \frac{\partial}{\partial z}\left(\sqrt{\frac{U_0}{2\upsilon L}} \frac{e^{\frac{x+y}{2L}}}{(1-\lambda t)^{\frac{1}{2}}} z \right)$$

$$\frac{\partial u}{\partial z} = \frac{U_0 e^{\frac{3(x+y)}{2L}}}{(1-\lambda t)^{\frac{1}{2}}(1-\lambda t)} \sqrt{\frac{U_0}{2\upsilon L}} f'' \tag{3.26}$$

Similarly,

$$\frac{\partial v}{\partial z} = \frac{U_0 e^{\frac{3(x+y)}{2L}}}{(1-\lambda t)^{\frac{1}{2}}(1-\lambda t)} \sqrt{\frac{U_0}{2\upsilon L}} g'' \tag{3.27}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} \right)$$

$$= \frac{\partial}{\partial z}\left(\frac{U_0}{(1-\lambda t)^{\frac{1}{2}}(1-\lambda t)} e^{\frac{3(x+y)}{2L}} f'' \right)$$

$$=\frac{U_0}{(1-\lambda t)^{\frac{1}{2}}(1-\lambda t)}e^{\frac{3(x+y)}{2L}}\sqrt{\frac{U_0}{2\upsilon L}}\frac{\partial f''}{\partial \eta}\frac{\partial \eta}{\partial z}$$

$$=\frac{U_0}{(1-\lambda t)^{\frac{1}{2}}(1-\lambda t)}e^{\frac{3(x+y)}{2L}}\sqrt{\frac{U_0}{2\upsilon L}}f'''\frac{\partial}{\partial z}\left(\sqrt{\frac{U_0}{2\upsilon L}}\frac{e^{\frac{x+y}{2L}}}{(1-\lambda t)^{\frac{1}{2}}}z\right)$$

$$\frac{\partial^2 u}{\partial z^2}=\frac{U_0{}^2 e^{\frac{2(x+y)}{L}}}{2\upsilon L(1-\lambda t)^2}f''' \qquad (3.28)$$

Similarly,

$$\frac{\partial^2 v}{\partial z^2}=\frac{U_0{}^2 e^{\frac{2(x+y)}{L}}}{2\upsilon L(1-\lambda t)^2}g''' \qquad (3.29)$$

$$\frac{\partial T}{\partial z}=\frac{\partial}{\partial z}\left(T_\infty+\frac{(T_w-T_\infty)e^{\frac{x+y}{L}}}{(1-\lambda t)}(1+\varepsilon\theta)\right)=\frac{\partial}{\partial z}\left(\frac{(T_w-T_\infty)e^{\frac{x+y}{L}}}{(1-\lambda t)}(1+\varepsilon\theta)\right)$$

$$=\frac{(T_w-T_\infty)}{(1-\lambda t)}e^{\frac{x+y}{L}}\frac{\partial(1+\varepsilon\theta)}{\partial \eta}\frac{\partial \eta}{\partial z}$$

$$=\frac{(T_w-T_\infty)}{(1-\lambda t)}e^{\frac{x+y}{L}}\varepsilon\theta'\frac{\partial}{\partial z}\left(\sqrt{\frac{U_0}{2\upsilon L}}\frac{e^{\frac{x+y}{2L}}}{(1-\lambda t)^{\frac{1}{2}}}z\right)$$

$$\frac{\partial T}{\partial z}=\frac{(T_w-T_\infty)e^{\frac{3(x+y)}{2L}}}{(1-\lambda t)^{\frac{1}{2}}(1-\lambda t)}\sqrt{\frac{U_0}{2\upsilon L}}\varepsilon\theta' \qquad (3.30)$$

$$\frac{\partial^2 T}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)$$

$$= \frac{\partial}{\partial z}\left(\frac{(T_w - T_\infty)}{(1-\lambda t)^{\frac{1}{2}}(1-\lambda t)} e^{\frac{3(x+y)}{2L}} \sqrt{\frac{U_0}{2\upsilon L}}\varepsilon\theta'\right)$$

$$= \frac{(T_w - T_\infty)}{(1-\lambda t)^{\frac{1}{2}}(1-\lambda t)} e^{\frac{3(x+y)}{2L}} \sqrt{\frac{U_0}{2\upsilon L}}\varepsilon \frac{\partial \theta'}{\partial \eta}\frac{\partial \eta}{\partial z}$$

$$= \frac{(T_w - T_\infty)}{(1-\lambda t)^{\frac{1}{2}}(1-\lambda t)} e^{\frac{3(x+y)}{2L}} \sqrt{\frac{U_0}{2\upsilon L}}\varepsilon\theta'' \frac{\partial}{\partial z}\left(\sqrt{\frac{U_0}{2\upsilon L}} \frac{e^{\frac{x+y}{2L}}}{(1-\lambda t)^{\frac{1}{2}}} z\right)$$

$$\frac{\partial^2 T}{\partial z^2} = \frac{(T_w - T_\infty)U_0 e^{\frac{2(x+y)}{L}}}{2\upsilon L(1-\lambda t)^2}\varepsilon\theta'' \tag{3.31}$$

Similarly,

$$\frac{\partial C}{\partial z} = \frac{\partial}{\partial z}\left(C_\infty + \frac{(C_w - C_\infty)e^{\frac{x+y}{L}}}{(1-\lambda t)}\phi\right) = \frac{\partial}{\partial z}\left(\frac{(C_w - C_\infty)e^{\frac{x+y}{L}}}{(1-\lambda t)}\phi\right)$$

$$= \frac{(C_w - C_\infty)}{(1-\lambda t)} e^{\frac{x+y}{L}} \frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial z}$$

$$= \frac{(C_w - C_\infty)}{(1-\lambda t)} e^{\frac{x+y}{L}} \phi' \frac{\partial}{\partial z}\left(\sqrt{\frac{U_0}{2\upsilon L}} \frac{e^{\frac{x+y}{2L}}}{(1-\lambda t)^{\frac{1}{2}}} z\right)$$

$$\frac{\partial C}{\partial z}=\frac{\left(C_w-C_\infty\right)e^{\frac{3(x+y)}{2L}}}{\left(1-\lambda t\right)^{\frac{1}{2}}\left(1-\lambda t\right)}\sqrt{\frac{U_0}{2\upsilon L}}\phi' \tag{3.32}$$

$$\frac{\partial^2 C}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial C}{\partial z}\right)$$

$$=\frac{\partial}{\partial z}\left(\frac{\left(C_w-C_\infty\right)}{\left(1-\lambda t\right)^{\frac{1}{2}}\left(1-\lambda t\right)}e^{\frac{3(x+y)}{2L}}\sqrt{\frac{U_0}{2\upsilon L}}\phi'\right)$$

$$=\frac{\left(C_w-C_\infty\right)}{\left(1-\lambda t\right)^{\frac{1}{2}}\left(1-\lambda t\right)}e^{\frac{3(x+y)}{2L}}\sqrt{\frac{U_0}{2\upsilon L}}\frac{\partial \phi'}{\partial \eta}\frac{\partial \eta}{\partial z}$$

$$\frac{\partial^2 C}{\partial z^2}=\frac{\left(C_w-C_\infty\right)U_0 e^{\frac{2(x+y)}{L}}}{2\upsilon L\left(1-\lambda t\right)^2}\phi'' \tag{3.33}$$

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial t}\left(T_\infty+\frac{\left(T_w-T_\infty\right)e^{\frac{x+y}{L}}}{\left(1-\lambda t\right)}\left(1+\varepsilon\theta\right)\right)=\frac{\partial}{\partial t}\left(\frac{\left(T_w-T_\infty\right)e^{\frac{x+y}{L}}}{\left(1-\lambda t\right)}\left(1+\varepsilon\theta\right)\right)$$

$$=\frac{\lambda\left(T_w-T_\infty\right)e^{\frac{x+y}{L}}}{\left(1-\lambda t\right)}\left(1+\varepsilon\theta\right)+\frac{\left(T_w-T_\infty\right)}{\left(1-\lambda t\right)}e^{\frac{x+y}{L}}\varepsilon\frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial t}$$

$$=\frac{\lambda\left(T_w-T_\infty\right)e^{\frac{x+y}{L}}}{\left(1-\lambda t\right)}\left(1+\varepsilon\theta\right)+\frac{\left(T_w-T_\infty\right)}{\left(1-\lambda t\right)}e^{\frac{x+y}{L}}\varepsilon\theta'\frac{\partial}{\partial t}\left(\sqrt{\frac{U_0}{2\upsilon L}}\frac{e^{\frac{x+y}{2L}}}{\left(1-\lambda t\right)^{\frac{1}{2}}}z\right)$$

$$\frac{\partial T}{\partial t}=\frac{\lambda\left(T_w-T_\infty\right)e^{\frac{x+y}{L}}}{\left(1-\lambda t\right)^2}\left(\left(1+\varepsilon\theta\right)+\frac{\eta}{2}\varepsilon\theta'\right) \tag{3.34}$$

$$\frac{\partial C}{\partial t}=\frac{\partial}{\partial t}\left(C_\infty+\frac{C_0 e^{\frac{x+y}{L}}}{(1-\lambda t)}\phi\right)=\frac{\partial}{\partial t}\left(\frac{C_0 e^{\frac{x+y}{L}}}{(1-\lambda t)}\phi\right)$$

$$=\frac{\lambda C_0 e^{\frac{x+y}{L}}}{(1-\lambda t)}\phi+\frac{(C_w-C_\infty)}{(1-\lambda t)}e^{\frac{x+y}{L}}\frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial t}$$

$$=\frac{\lambda(C_w-C_\infty)e^{\frac{x+y}{L}}}{(1-\lambda t)}\phi+\frac{(C_w-C_\infty)}{(1-\lambda t)}e^{\frac{x+y}{L}}\phi'\frac{\partial}{\partial t}\left(\sqrt{\frac{U_0}{2\upsilon L}}\frac{e^{\frac{x+y}{2L}}}{(1-\lambda t)^{\frac{1}{2}}}z\right)$$

$$\frac{\partial C}{\partial t}=\frac{\lambda(C_w-C_\infty)e^{\frac{x+y}{L}}}{(1-\lambda t)^2}\left(\phi+\frac{\eta}{2}\phi'\right) \tag{3.35}$$

$$\frac{\partial T}{\partial x}=\frac{\partial}{\partial x}\left(T_\infty+\frac{(T_w-T_\infty)e^{\frac{x+y}{L}}}{(1-\lambda t)}(1+\varepsilon\theta)\right)=\frac{\partial}{\partial x}\left(\frac{(T_w-T_\infty)e^{\frac{x+y}{L}}}{(1-\lambda t)}(1+\varepsilon\theta)\right)$$

$$=\frac{(T_w-T_\infty)e^{\frac{x+y}{L}}}{L(1-\lambda t)}(1+\varepsilon\theta)+\frac{(T_w-T_\infty)}{(1-\lambda t)}e^{\frac{x+y}{L}}\varepsilon\frac{\partial\theta}{\partial\eta}\frac{\partial\eta}{\partial x}$$

$$=\frac{(T_w-T_\infty)e^{\frac{x+y}{L}}}{L(1-\lambda t)}(1+\varepsilon\theta)+\frac{(T_w-T_\infty)}{(1-\lambda t)}e^{\frac{x+y}{L}}\varepsilon\theta'\frac{\partial}{\partial x}\left(\sqrt{\frac{U_0}{2\upsilon L}}\frac{e^{\frac{x+y}{2L}}}{(1-\lambda t)^{\frac{1}{2}}}z\right)$$

$$\frac{\partial T}{\partial x}=\frac{(T_w-T_\infty)e^{\frac{x+y}{L}}}{L(1-\lambda t)}\left((1+\varepsilon\theta)+\frac{\eta}{2}\varepsilon\theta'\right) \tag{3.36}$$

Similarly,

$$\frac{\partial T}{\partial y}=\frac{\left(T_w-T_\infty\right)e^{\frac{x+y}{L}}}{L\left(1-\lambda t\right)}\left(\left(1+\varepsilon\theta\right)+\frac{\eta}{2}\varepsilon\theta'\right) \tag{3.37}$$

$$\frac{\partial C}{\partial x}=\frac{\partial}{\partial x}\left(C_\infty+\frac{\left(C_w-C_\infty\right)e^{\frac{x+y}{L}}}{\left(1-\lambda t\right)}\phi\right)=\frac{\partial}{\partial x}\left(\frac{\left(C_w-C_\infty\right)e^{\frac{x+y}{L}}}{\left(1-\lambda t\right)}\phi\right)$$

$$=\frac{\left(C_w-C_\infty\right)e^{\frac{x+y}{L}}}{L\left(1-\lambda t\right)}\phi+\frac{\left(C_w-C_\infty\right)}{\left(1-\lambda t\right)}e^{\frac{x+y}{L}}\frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial x}$$

$$=\frac{\left(C_w-C_\infty\right)e^{\frac{x+y}{L}}}{L\left(1-\lambda t\right)}\phi+\frac{\left(C_w-C_\infty\right)}{\left(1-\lambda t\right)}e^{\frac{x+y}{L}}\phi'\frac{\partial}{\partial x}\left(\sqrt{\frac{U_0}{2\upsilon L}}\frac{e^{\frac{x+y}{2L}}}{\left(1-\lambda t\right)^{\frac{1}{2}}}z\right)$$

$$\frac{\partial C}{\partial x}=\frac{\left(C_w-C_\infty\right)e^{\frac{x+y}{L}}}{L\left(1-\lambda t\right)}\left(\phi+\frac{\eta}{2}\phi'\right) \tag{3.38}$$

Similarly,

$$\frac{\partial C}{\partial y}=\frac{\left(C_w-C_\infty\right)e^{\frac{x+y}{L}}}{L\left(1-\lambda t\right)}\left(\phi+\frac{\eta}{2}\phi'\right) \tag{3.39}$$

Also, note that

$$
\begin{aligned}
e^{-\frac{E_a}{K(T-T_\infty)}} &= e^{-\frac{E_a(1-\lambda t)}{K(T-T_\infty)e^{-\frac{x+y}{L}}}} \cdot e^{\frac{E_a(1-\lambda t)}{K(T-T_\infty)e^{-\frac{x+y}{L}}}} \cdot e^{-\frac{E_a}{K(T-T_\infty)}} \\
&= e^{-\frac{E_a(1-\lambda t)}{K(T-T_\infty)e^{-\frac{x+y}{L}}}} \cdot e^{-\frac{E_a}{K}\left[\frac{1}{\frac{(T-T_\infty)e^{-\frac{x+y}{L}}}{(1-\lambda t)}}-\frac{1}{(T-T_\infty)}\right]} \\
&= e^{-\frac{E_a(1-\lambda t)}{K(T-T_\infty)e^{-\frac{x+y}{L}}}} \cdot e^{-\frac{E_a}{K}\left[\frac{(T-T_\infty)-\frac{(T-T_\infty)e^{-\frac{x+y}{L}}}{(1-\lambda t)}}{(T-T_\infty)\frac{(T-T_\infty)e^{-\frac{x+y}{L}}}{(1-\lambda t)}}\right]} = e^{-\frac{1}{\varepsilon}} \cdot e^{-\frac{1}{\varepsilon}\left(\frac{1+\varepsilon\theta-1}{1+\varepsilon\theta}\right)} \\
&= e^{-\frac{1}{\varepsilon}} \cdot e^{\left(\frac{\theta}{1+\varepsilon\theta}\right)}
\end{aligned} \tag{3.40}
$$

Substituting equations (3.18), (3.19), (3.20), (3.22), (3.26), (3.28) with transformations in (3.13) into (3.2), we have

$$
\begin{aligned}
&\frac{\lambda U_0 e^{\frac{x+y}{L}}}{(1-\lambda t)^2}\left(f'+\frac{\eta}{2}f''\right)+\frac{U^2{}_0 e^{\frac{2(x+y)}{L}}}{L(1-\lambda t)^2}f'\left(f'+\frac{\eta}{2}f''\right)+\frac{U^2{}_0 e^{\frac{2(x+y)}{L}}}{L(1-\lambda t)^2}g'\left(f'+\frac{\eta}{2}f''\right) \\
&-\frac{U^2{}_0 e^{\frac{2(x+y)}{L}}}{2L(1-\lambda t)^2}f''\left(f+\eta f'+g+\eta g'\right)=\frac{U_0{}^2 e^{\frac{2(x+y)}{L}}}{2L(1-\lambda t)^2}f'''-\frac{\upsilon U_0 e^{\frac{2(x+y)}{L}}}{k_{\rho 0}(1-\lambda t)^2}f' \\
&-\Gamma\left(\frac{U_0}{(1-\lambda t)}e^{\frac{x+y}{L}}f'\right)^2-\frac{\sigma_e\left(\frac{B_0}{(1-\lambda t)^{\frac{1}{2}}}e^{\frac{x+y}{2L}}\right)^2\left(\frac{U_0}{(1-\lambda t)}e^{\frac{x+y}{L}}f'\right)}{\rho_f} \\
&+\frac{\rho_{f\infty}\beta_{T0}{}^2(T_w-T_\infty)e^{\frac{2(x+y)}{L}}g_v(1-C_\infty)}{(1-\lambda t)^2}(1+\varepsilon\theta)-\frac{(C_w-C_\infty)g_v(\rho_p-\rho_f)e^{\frac{2(x+y)}{L}}}{(1-\lambda t)^2}\phi
\end{aligned} \tag{3.41}
$$

Multiplying (3.41) by $\dfrac{2L(1-\lambda t)^2}{U^2{}_0 e^{\frac{2(x+y)}{L}}}$, gives

$$
\begin{aligned}
&\frac{2\lambda L}{U_0 e^{\frac{x+y}{L}}}\left(f' + \frac{\eta}{2} f'' \right) + 2f'\left(f' + \frac{\eta}{2} f'' \right) + 2g'\left(f' + \frac{\eta}{2} f'' \right) \\
&- f''\left(f + \eta f' + g + \eta g' \right) = f''' - \frac{2\upsilon L}{k_{\rho 0} U_0} f' - 2L\Gamma f'^2 - \frac{2\sigma_e B_0^{\ 2} L}{\rho_f U_0} f' \\
&+ \frac{2\beta_{T0} g_v \rho_{f_\infty} L\left(T_w - T_\infty\right)\left(1 - C_\infty\right)}{U_0^{\ 2}}\left(1 + \varepsilon\theta\right) - \frac{2\beta_{C0} g_v L\left(C_w - C_\infty\right)\left(\rho_f - \rho_{f_\infty}\right)}{U_0^{\ 2}}\phi
\end{aligned} \tag{3.42}
$$

$$
\begin{aligned}
&f''' + G_{r\theta}\left(1 + \varepsilon\theta\right) - G_{r\phi}\phi + f''\left(f + \eta f' + g + \eta g' \right) - \frac{a}{R_e}\left(f' + \frac{\eta}{2} f'' \right) \\
&-2f'\left(f' + \frac{\eta}{2} f'' \right) - 2g'\left(f' + \frac{\eta}{2} f'' \right) - \gamma f' - \Omega f'^2 - Mf' = 0
\end{aligned} \tag{3.43}
$$

where $\frac{a}{R_e} = \frac{2\lambda L}{U_0 e^{\frac{x+y}{L}}}$ $M = \frac{2\sigma_e B_0^{\ 2} L}{\rho_f U_0}$, $G_{r\theta} = \frac{2\beta_{T0} g_v \rho_f L\left(T_w - T_\infty\right)\left(1 - C_\infty\right)}{U_0^{\ 2}}$,

$G_{r\phi} = \frac{2\beta_{C0} g_v L\left(C_w - C_\infty\right)\left(\rho_p - \rho_f\right)}{U_0^{\ 2}}$, $\Omega = 2L\Gamma$, $\gamma = \frac{2\upsilon L}{k_{\rho 0} U_0}$

Substituting equations (3.18), (3.19), (3.21), (3.24), (3.25), (3.27), (3.29) with transformations in equation (3.13) into (3.3) results.

$$\frac{\lambda U_0 e^{\frac{x+y}{L}}}{(1-\lambda t)^2}\left(g'+\frac{\eta}{2}g''\right)+\frac{U_0}{(1-\lambda t)}e^{\frac{x+y}{L}}f'\left(\frac{U_0 e^{\frac{x+y}{L}}}{L(1-\lambda t)^2}\left(g'+\frac{\eta}{2}g''\right)\right)$$

$$+\frac{U_0}{(1-\lambda t)}e^{\frac{x+y}{L}}g'\left(\frac{U_0 e^{\frac{x+y}{L}}}{L(1-\lambda t)^2}\left(g'+\frac{\eta}{2}g''\right)\right)$$

$$-\frac{1}{(1-\lambda t)^{\frac{1}{2}}}\sqrt{\frac{\upsilon U_0}{2L}}e^{\frac{x+y}{2L}}\left(f+\eta f'+g+\eta g'\right)\left(\frac{U_0 e^{\frac{3(x+y)}{2L}}}{(1-\lambda t)^{\frac{1}{2}}(1-\lambda t)}\sqrt{\frac{U_0}{2\upsilon L}}g''\right)$$

$$=\upsilon\left(\frac{U_0{}^2 e^{\frac{2(x+y)}{L}}}{2\upsilon L(1-\lambda t)^2}g'''\right)-\frac{\upsilon}{\frac{k_{\rho 0}(1-\lambda t)}{e^{\frac{x+y}{L}}}}\left(\frac{U_0}{(1-\lambda t)}e^{\frac{x+y}{L}}g'\right)-\Gamma\left(\frac{U_0}{(1-\lambda t)}e^{\frac{x+y}{L}}g'\right)^2$$

$$-\frac{\sigma_e\left(\frac{B_0}{(1-\lambda t)^{\frac{1}{2}}}e^{\frac{x+y}{2L}}\right)^2\left(\frac{U_0}{(1-\lambda t)}e^{\frac{x+y}{L}}g'\right)}{\rho_f} \tag{3.44}$$

Simplifying (3.44) gives

$$\frac{\lambda U_0 e^{\frac{x+y}{L}}}{(1-\lambda t)^2}\left(g'+\frac{\eta}{2}g''\right)+\frac{U^2{}_0 e^{\frac{2(x+y)}{L}}}{L(1-\lambda t)^2}f'\left(g'+\frac{\eta}{2}g''\right)+\frac{U^2{}_0 e^{\frac{2(x+y)}{L}}}{L(1-\lambda t)^2}g'\left(g'+\frac{\eta}{2}g+''\right)$$

$$-\frac{U^2{}_0 e^{\frac{2(x+y)}{L}}}{2L(1-\lambda t)^2}g''\left(f+\eta f'+g+\eta g'\right)=\frac{U_0{}^2 e^{\frac{2(x+y)}{L}}}{2L(1-\lambda t)^2}g'''-\frac{\upsilon U_0 e^{\frac{2(x+y)}{L}}}{k_{\rho 0}(1-\lambda t)^2}g' \tag{3.45}$$

$$-2L\Gamma g'^2-\frac{\sigma_e B_0{}^2 U_0 e^{\frac{2(x+y)}{L}}}{\rho_f(1-\lambda t)^2}g'$$

Multiplying (3.45) by $\dfrac{2L(1-\lambda t)^2}{U^2{}_0 e^{\frac{2(x+y)}{L}}}$ gives

$$g''' + g''\left(f + \eta f' + g + \eta g' \right) - \frac{a}{R_e}\left(g' + \frac{\eta}{2} g'' \right) - 2f'\left(g' + \frac{\eta}{2} g'' \right) - 2g'\left(g' + \frac{\eta}{2} g'' \right) \tag{3.46}$$
$$-\gamma g' - \Omega g'^2 - Mg' = 0$$

where

$$\frac{1}{R_e} = \frac{\upsilon}{LU_0}, \quad \frac{a}{R_e} = \frac{2\lambda L}{U_0 e^{\frac{x+y}{L}}}, \quad \gamma = \frac{2\upsilon L}{k_{\rho 0} U_0}, \quad M = \frac{2\sigma_e B_0{}^2 L}{\rho_f U_0}, \quad \Omega = 2L\Gamma$$

Substituting equations (3.18), (3.19), (3.30), (3.32), (3.34), (3.36), (3.37),(3.40) with transformations in equation (3.13) into (3.4), results

$$\frac{\lambda\left(T_w - T_\infty\right)e^{\frac{x+y}{L}}}{\left(1-\lambda t\right)^2}\left(\left(1+\varepsilon\theta\right) + \frac{\eta}{2}\varepsilon\theta'\right) + \frac{U_0}{\left(1-\lambda t\right)} e^{\frac{x+y}{L}} f'\left(\frac{\left(T_w - T_\infty\right)e^{\frac{x+y}{L}}}{L\left(1-\lambda t\right)}\left(1+\varepsilon\theta\right) + \frac{\eta}{2}\varepsilon\theta'\right) +$$
$$\frac{U_0}{\left(1-\lambda t\right)} e^{\frac{x+y}{L}} g'\left(\frac{\left(T_w - T_\infty\right)e^{\frac{x+y}{L}}}{L\left(1-\lambda t\right)}\left(\left(1+\varepsilon\theta\right) + \frac{\eta}{2}\varepsilon\theta'\right)\right)$$

$$
-\frac{1}{(1-\lambda t)^{\frac{1}{2}}}\sqrt{\frac{\upsilon U_0}{2L}}e^{\frac{x+y}{2L}}\left(f+\eta f'+g+\eta g'\right)\left(\frac{(T_w-T_\infty)e^{\frac{3(x+y)}{2L}}}{(1-\lambda t)^{\frac{1}{2}}(1-\lambda t)}\sqrt{\frac{U_0}{2\upsilon L}}\varepsilon\theta'\right)=
$$

$$
\alpha\left(\frac{(T_w-T_\infty)U_0 e^{\frac{2(x+y)}{L}}}{2\upsilon L(1-\lambda t)^2}\varepsilon\theta''\right)+\frac{\frac{Qe^{\frac{x+y}{L}}}{(1-\lambda t)}\left(T_\infty+\frac{(T_w-T_\infty)e^{\frac{x+y}{L}}}{(1-\lambda t)}(1+\varepsilon\theta)-T_\infty\right)}{(\rho C)_f}
$$

$$
+\left(\frac{1}{\tau_0}\frac{(1-\lambda t)}{e^{\frac{x+y}{L}}}\right)\left(\begin{array}{l} D_B\left(\frac{(T_w-T_\infty)e^{\frac{3(x+y)}{2L}}}{(1-\lambda t)^{\frac{1}{2}}(1-\lambda t)}\sqrt{\frac{U_0}{2\upsilon L}}\varepsilon\theta'\right)\left(\frac{(C_w-C_\infty)e^{\frac{3(x+y)}{2L}}}{(1-\lambda t)^{\frac{1}{2}}(1-\lambda t)}\sqrt{\frac{U_0}{2\upsilon L}}\phi'\right)\\ +\frac{D_\tau}{T_\infty}\left(\frac{(T_w-T_\infty)e^{\frac{3(x+y)}{2L}}}{(1-\lambda t)^{\frac{1}{2}}(1-\lambda t)}\sqrt{\frac{U_0}{2\upsilon L}}\varepsilon\theta'\right)^2 \end{array}\right)
$$

$$
+\frac{\sigma_e B^2}{\rho_f}\left(\left(\frac{U_0}{(1-\lambda t)}e^{\frac{x+y}{L}}f'\right)^2+\left(\frac{U_0}{(1-\lambda t)}e^{\frac{x+y}{L}}g'\right)^2\right)-
$$

$$
\frac{1}{(\rho C)_f}\left(\frac{-16T_\infty{}^3}{3k_1}\right)\left(\frac{(T_w-T_\infty)U_0 e^{\frac{2(x+y)}{L}}}{2\upsilon L(1-\lambda t)^2}\varepsilon\theta''\right)
$$

$$
+\beta\left(\frac{k_{r0}e^{\frac{\frac{x+y}{2}L}{1-\omega}}}{(1-\lambda t)^{\frac{1-\omega}{2}}}\right)^2*\left(\frac{(T_w-T_\infty)e^{\frac{x+y}{L}}}{(1-\lambda t)}(1+\varepsilon\theta)\right)^{\omega}*\left(\frac{(C_w-C_\infty)e^{\frac{x+y}{L}}}{(1-\lambda t)}\phi(\eta)\right)e^{-\frac{1}{\varepsilon}}e^{\frac{\theta}{(1+\varepsilon\theta)}} \tag{3.47}
$$

Simplifying (3.47) gives

$$\frac{\lambda\left(T_w-T_\infty\right)e^{\frac{x+y}{L}}}{\left(1-\lambda t\right)^2}\left(\left(1+\varepsilon\theta\right)+\frac{\eta}{2}\varepsilon\theta'\right)+\frac{U_0\left(T_w-T_\infty\right)e^{\frac{2(x+y)}{L}}}{L\left(1-\lambda t\right)^2}\begin{pmatrix}\left(1+\varepsilon\theta\right)+\\ \frac{\eta}{2}\varepsilon\theta'\end{pmatrix}\left(f'+g'\right)-$$

$$\frac{U_0\left(T_w-T_\infty\right)e^{\frac{2(x+y)}{L}}}{2L\left(1-\lambda t\right)^2}\left(f+\eta f'+g+\eta g'\right)\varepsilon\theta'=$$

$$\frac{\left(T_w-T_\infty\right)e^{\frac{2(x+y)}{L}}}{2\upsilon\tau_0 L\left(1-\lambda t\right)^2}\varepsilon\theta'\left(D_B\left(C_w-C_\infty\right)U_0\phi'+\left(T_w-T_\infty\right)\varepsilon\theta'\right)$$

$$+\frac{Q_0\left(T_w-T_\infty\right)e^{\frac{2(x+y)}{L}}}{\left(\rho C\right)_f\left(1-\lambda t\right)^2}\left(1+\varepsilon\theta\right)+\frac{U_0{}^2\sigma_e B^2 e^{\frac{2(x+y)}{L}}}{\rho_f\left(1-\lambda t\right)^2}\left(f'^2+g'^2\right)+$$

$$\frac{\left(T_w-T_\infty\right)U_0 e^{\frac{2(x+y)}{L}}}{2\upsilon L\left(1-\lambda t\right)^2}\varepsilon\theta''\left(\alpha+\frac{16T_\infty{}^3}{3k_1\left(\rho C\right)_f}\right)$$

$$+\frac{\beta k_{r0}{}^2\left(C_w-C_\infty\right)\left(T_w-T_\infty\right)^\omega e^{-\frac{1}{\varepsilon}}e^{\frac{2(x+y)}{L}}}{\left(1-\lambda t\right)^2}\phi\left(1+\varepsilon\theta\right)^\omega e^{\frac{\theta}{(1+\varepsilon\theta)}} \qquad (3.48)$$

Multiplying (3.48) by $\dfrac{2\upsilon\left(1-\lambda t\right)^2}{\left(T_w-T_\infty\right)U_0^2 e^{\frac{2(x+y)}{L}}}$ gives

$$\begin{aligned}&\frac{2\lambda\upsilon}{U_0^2 e^{\frac{x+y}{L}}}\left(\left(1+\varepsilon\theta\right)+\frac{\eta}{2}\varepsilon\theta'\right)+\frac{2\upsilon}{LU_0}\left(\left(1+\varepsilon\theta\right)+\frac{\eta}{2}\varepsilon\theta'\right)\left(f'+g'\right)-\\&\frac{2\upsilon}{2LU_0}\left(f+\eta f'+g'+\eta g'\right)\varepsilon\theta'=\frac{\varepsilon}{\tau_0 LU_0^2}\theta'\left(D_B\left(C_w-C_\infty\right)\phi'+\frac{D_\tau\left(T_w-T_\infty\right)\varepsilon\theta'}{T_\infty U_0}\right)+\\&\frac{2\upsilon Q_0}{\left(\rho C\right)_f U_0^2}\left(1+\varepsilon\theta\right)+\frac{2\upsilon\sigma_e B^2}{\rho_f\left(T_w-T_\infty\right)}\left(f'^2+g'^2\right)+\frac{1}{LU_0}\varepsilon\theta''\left(\alpha+\frac{16T_\infty{}^3}{3k_1\left(\rho C\right)_f}\right)+\\&\frac{2\beta k_{r0}{}^2\left(C_w-C_\infty\right)\left(T_w-T_\infty\right)^{\omega-1}e^{-\frac{1}{\varepsilon}}}{U_0^2}\phi\left(1+\varepsilon\theta\right)^\omega e^{\frac{\theta}{(1+\varepsilon\theta)}}\end{aligned} \qquad (3.49)$$

Simplifying (3.49), results

$$
\begin{aligned}
&\frac{2\lambda\upsilon}{U_0^2 e^{\frac{x+y}{L}}}\left(\left(1+\varepsilon\theta\right)+\frac{\eta}{2}\varepsilon\theta'\right)+\frac{2\upsilon}{LU_0}\left(\left(1+\varepsilon\theta\right)+\frac{\eta}{2}\varepsilon\theta'\right)\left(f'+g'\right)- \\
&\frac{2\upsilon}{2LU_0}\left(f+\eta f'+g'+\eta g'\right)\varepsilon\theta'=\varepsilon\theta'\left(\frac{D_B\left(C_w-C_\infty\right)}{LU_0\tau_0U_0}\phi'+\frac{D_\tau\varepsilon\left(T_w-T_\infty\right)}{LU_0\tau_0U_0^2T_\infty}\theta'\right)+ \\
&\frac{2\upsilon Q_0}{\left(\rho C\right)_f U_0^2}\left(1+\varepsilon\theta\right)+\frac{2\upsilon\sigma_e B^2}{\rho_f\left(T_w-T_\infty\right)}\left(f'^2+g'^2\right)+\left(\frac{\alpha}{LU_0}+\frac{16T_\infty{}^3}{3k_1\left(\rho C\right)_f LU_0}\right)\varepsilon\theta''+ \\
&\frac{2\beta k_{r0}{}^2\left(C_w-C_\infty\right)\left(T_w-T_\infty\right)^{\omega-1}e^{-\frac{1}{\varepsilon}}}{U_0^2}\phi\left(1+\varepsilon\theta\right)^{\omega}e^{\frac{\theta}{(1+\varepsilon\theta)}}
\end{aligned}
\qquad (3.50)
$$

Simplifying (3.50), gives

$$
\begin{aligned}
&\frac{a}{R_e}\left(\left(\frac{1}{\varepsilon}+\theta\right)+\frac{\eta}{2}\theta'\right)+2\left(\left(\frac{1}{\varepsilon}+\theta\right)+\frac{\eta}{2}\theta'\right)\left(f'+g'\right)-\left(f+\eta f'+g'+\eta g'\right)\theta'= \\
&N_b\theta'\phi'+N_t\theta'^2+Q_h\left(1+\varepsilon\theta\right)+E_cM\left(f'^2+g'^2\right)+R_1\theta''+ \\
&\delta\phi\left(1+\varepsilon\theta\right)^{\omega}e^{\frac{\theta}{(1+\varepsilon\theta)}}
\end{aligned}
\qquad (3.51)
$$

where

$$
\begin{aligned}
&\frac{1}{R_e}=\frac{\upsilon}{LU_0},\quad a=\frac{2\lambda L}{U_0e^{\frac{x+y}{L}}},\quad \frac{1}{P_{em1}}=\frac{D_B}{LU_0},\quad d=\frac{\left(C_w-C_\infty\right)}{\tau_0U_0},\quad \frac{1}{P_{em2}}=\frac{D_\tau}{LU_0},\quad c=\frac{\varepsilon\left(T_w-T_\infty\right)}{\tau_0U_0^2T_\infty}, \\
&Q_h=\frac{Q_0L}{\left(\rho C\right)_f\varepsilon U_0},\quad M=\frac{\sigma_eB^2L}{\rho_fU_0},\quad E_c=\frac{U_0^2}{\varepsilon\left(T_w-T_\infty\right)},\quad P_e=\frac{\alpha}{LU_0},\quad R=\frac{16T_\infty{}^3}{3k_1\left(\rho C\right)_f LU_0}, \\
&\delta=\frac{2\beta k_{r0}{}^2\left(C_w-C_\infty\right)\left(T_w-T_\infty\right)^{\omega-1}e^{-\frac{1}{\varepsilon}}}{\varepsilon U_0^2},\quad R_1=\frac{1}{\Pr}\left(1+R\right),\quad P_r=\frac{R_e}{P_e}=\frac{\upsilon}{\alpha},\ N_t=\frac{cR_e}{P_{em2}},\quad N_b=\frac{dR_e}{P_{em1}}
\end{aligned}
$$

Substituting equations (3.18), (3.19) (3.31), (3.32), (3.33), (3.35), (3.38), (3.39) with transformations in equation (3.13) into (3.5) gives

$$\frac{\lambda\left(C_w-C_\infty\right)e^{\frac{x+y}{L}}}{\left(1-\lambda t\right)^2}\left(\phi+\frac{\eta}{2}\phi'\right)+\frac{U_0}{\left(1-\lambda t\right)}e^{\frac{x+y}{L}}f'\left(\frac{\left(C_w-C_\infty\right)e^{\frac{x+y}{L}}}{L\left(1-\lambda t\right)}\left(\phi+\frac{\eta}{2}\phi'\right)\right)$$

$$+\frac{U_0}{\left(1-\lambda t\right)}e^{\frac{x+y}{L}}g'\left(\frac{\left(C_w-C_\infty\right)e^{\frac{x+y}{L}}}{L\left(1-\lambda t\right)}\left(\phi+\frac{\eta}{2}\phi'\right)\right)$$

$$-\frac{1}{\left(1-\lambda t\right)^{\frac{1}{2}}}\sqrt{\frac{\upsilon U_0}{2L}}e^{\frac{x+y}{2L}}\left(f+\eta f'+g+\eta g'\right)\left(\frac{\left(C_w-C_\infty\right)e^{\frac{3(x+y)}{2L}}}{\left(1-\lambda t\right)^{\frac{1}{2}}\left(1-\lambda t\right)}\sqrt{\frac{U_0}{2\nu L}}\phi'\right)$$

$$=D_B\left(\frac{\left(C_w-C_\infty\right)U_0e^{\frac{2(x+y)}{L}}}{2\upsilon L\left(1-\lambda t\right)^2}\phi''\right)+\frac{D_\tau}{T_\infty}\left(\frac{\left(T_w-T_\infty\right)U_0e^{\frac{3(x+y)}{2L}}}{2\upsilon L\left(1-\lambda t\right)^2}\varepsilon\theta''\right)$$

$$-\left(\frac{k_{r0}e^{\frac{\frac{x+y}{2}}{1-\omega}L}}{\left(1-\lambda t\right)^{\frac{1-\omega}{2}}}\right)^2*\left(\frac{\left(T_w-T_\infty\right)e^{\frac{x+y}{L}}}{\left(1-\lambda t\right)}\left(1+\varepsilon\theta\right)\right)^\omega*\left(\frac{\left(C_w-C_\infty\right)e^{\frac{x+y}{L}}}{\left(1-\lambda t\right)}\phi(\eta)\right)e^{-\frac{1}{\varepsilon}}e^{\frac{\theta}{(1+\varepsilon\theta)}} \qquad (3.52)$$

Simplifying (3.52) results

$$\frac{\lambda\left(C_w-C_\infty\right)e^{\frac{x+y}{L}}}{\left(1-\lambda t\right)^2}\left(\phi+\frac{\eta}{2}\phi'\right)+\frac{\left(C_w-C_\infty\right)U_0e^{\frac{2(x+y)}{L}}}{L\left(1-\lambda t\right)^2}\left(\phi+\frac{\eta}{2}\phi'\right)\left(f'+g'\right)-$$

$$\frac{\left(C_w-C_\infty\right)U_0e^{\frac{2(x+y)}{L}}}{2L\left(1-\lambda t\right)^2}\left(f+\eta f'+g+\eta g'\right)\phi'=\frac{D_B\left(C_w-C_\infty\right)U_0e^{\frac{2(x+y)}{L}}}{2\upsilon L\left(1-\lambda t\right)^2}\phi''+ \qquad (3.53)$$

$$\frac{D_\tau\left(T_w-T_\infty\right)U_0e^{\frac{2(x+y)}{L}}}{2\upsilon LT_\infty\left(1-\lambda t\right)^2}\varepsilon\theta''-\frac{k_{r0}{}^2\left(C_w-C_\infty\right)\left(T_w-T_\infty\right)^\omega e^{-\frac{1}{\varepsilon}}e^{\frac{2(x+y)}{L}}}{\left(1-\lambda t\right)^2}\phi\left(1+\varepsilon\theta\right)^\omega e^{\frac{\theta}{(1+\varepsilon\theta)}}$$

Multiplying (3.53) by $\dfrac{2\nu\left(1-\lambda t\right)^2}{\left(C_w-C_\infty\right)U_0^2e^{\frac{2(x+y)}{L}}}$, gives

$$\frac{2\lambda\upsilon L}{LU_0^2 e^{\frac{x+y}{L}}}\left(\phi+\frac{\eta}{2}\phi'\right)+2\frac{\upsilon}{LU_0}\left(\phi+\frac{\eta}{2}\phi'\right)(f'+g')-\frac{\upsilon}{LU_0}(f+\eta f'+g+\eta g')\phi'=$$
$$\frac{D_B}{LU_0}\phi''+\frac{D_\tau}{LU_0}\frac{\varepsilon(T_w-T_\infty)}{T_\infty(C_w-C_\infty)}\theta''-\frac{2\upsilon k_{r0}{}^2(T_w-T_\infty)^\omega e^{-\frac{1}{\varepsilon}}}{U_0^2}\phi(1+\varepsilon\theta)^\omega e^{\frac{\theta}{(1+\varepsilon\theta)}}=0 \tag{3.54}$$

Simplifying (3.54), yeilds

$$\frac{a}{R_e}\left(\phi+\frac{\eta}{2}\phi'\right)+2\left(\phi+\frac{\eta}{2}\phi'\right)(f'+g')-(f+\eta f'+g+\eta g')\phi'=$$
$$\frac{1}{Sc}\phi''+\frac{N_{t1}}{S_c}\theta''-\sigma\phi(1+\varepsilon\theta)^\omega e^{\frac{\theta}{(1+\varepsilon\theta)}} \tag{3.55}$$

where

$$\sigma=\frac{2\upsilon k_{r0}{}^2(T_w-T_\infty)^\omega e^{-\frac{1}{\varepsilon}}}{U_0^2},\quad h=\frac{\varepsilon(T_w-T_\infty)}{T_\infty(C_w-C_\infty)},\ \frac{N_{t1}}{S_c}=\frac{hR_e}{P_{em2}},\quad \frac{1}{Sc}=\frac{R_e}{P_{em1}}$$

Considering the initial and boundary conditions (3.12)

$$\begin{aligned}&u(z,t)=0, v(z,t)=0, T(z,t)=T_\infty, C(z,t)=C_\infty \ \ for\, t\le 0\ for\ all\ z\\&u=-U_w, v=-V_w, w=0, T=T_w, C=C_w\ at\ z=0\\&u\to 0, v\to 0, T\to T_\infty, C\to C_\infty\ at\ z\to\infty\end{aligned} \tag{3.56}$$

$$u(z,t)=\frac{U_0}{(1-\lambda t)}e^{\frac{x+y}{L}}f' \text{ and } U_w=\frac{U_0 e^{\frac{x+y}{L}}}{(1-\lambda t)} \tag{3.57}$$

Substituting (3.57) into (3.56), gives

$$u(0,t)=\frac{U_0}{(1-\lambda t)}e^{\frac{x+y}{L}}f'(0)=-\frac{U_0 e^{\frac{x+y}{L}}}{(1-\lambda t)} \tag{3.58}$$

Multiplying (3.58) through by $\frac{(1-\lambda t)}{U_0 e^{\frac{x+y}{L}}}$, gives

$$f'(0) = -1 \tag{3.59}$$

Similarly,

$$v(0,t) = \frac{U_0}{(1-\lambda t)} e^{\frac{x+y}{L}} g'(0) = -\frac{V_0 e^{\frac{x+y}{L}}}{(1-\lambda t)} \tag{3.60}$$

Multiplying (3.60) by $\frac{(1-\lambda t)}{U_0 e^{\frac{x+y}{L}}}$ gives

$$v(0,t) = g'(0) = \frac{V_0}{U_0} = -\psi$$

$$g'(0) = -\psi \tag{3.61}$$

Similarly,

$$T(z,t) = T_\infty + \frac{(T_w - T_\infty)}{(1-\lambda t)} e^{\frac{x+y}{L}} \theta(\eta) = T_\infty + \frac{(T_w - T_\infty)}{(1-\lambda t)} e^{\frac{x+y}{L}}$$

$$T(0,t) = T_\infty + \frac{(T_w - T_\infty)}{(1-\lambda t)} e^{\frac{x+y}{L}} \theta(0) = T_\infty + \frac{(T_w - T_\infty)}{(1-\lambda t)} e^{\frac{x+y}{L}}$$

$$\frac{(T_w - T_\infty)}{(1-\lambda t)} e^{\frac{x+y}{L}} \theta(0) = \frac{(T_w - T_\infty)}{(1-\lambda t)} e^{\frac{x+y}{L}} \tag{3.62}$$

Multiplying (3.62) through by $\frac{(1-\lambda t)}{(T_w - T_\infty)e^{\frac{x+y}{L}}}$, gives

$$\theta(0)=1 \tag{3.63}$$

Similarly

$$C(z,t)=C_\infty+\frac{(C_w-C_\infty)}{(1-\lambda t)}e^{\frac{x+y}{L}}\phi(\eta)=C_\infty+\frac{(C_w-C_\infty)}{(1-\lambda t)}e^{\frac{x+y}{L}}$$

$$C(0,t)=C_\infty+\frac{(C_w-C_\infty)}{(1-\lambda t)}e^{\frac{x+y}{L}}\phi(0)=C_\infty+\frac{(C_w-C_\infty)}{(1-\lambda t)}e^{\frac{x+y}{L}}$$

$$\frac{(C_w-C_\infty)}{(1-\lambda t)}e^{\frac{x+y}{L}}\theta(0)=\frac{(C_w-C_\infty)}{(1-\lambda t)}e^{\frac{x+y}{L}} \tag{3.64}$$

Multiplying (3.64) through by $\frac{(1-\lambda t)}{(C_w - C_\infty)e^{\frac{x+y}{L}}}$, gives

$$\phi(0)=1 \tag{3.65}$$

Similarly, when

$z \to \infty$ *and* $u \to 0$, then

$$u=\frac{U_0}{(1-\lambda t)}e^{\frac{x+y}{L}}f'(\eta\to 0) \tag{3.66}$$

This implies that

$$f' \to 0 \text{ as } \eta \to \infty \tag{3.67}$$

Similarly when

$z \to \infty \ and \ v \to 0$, then

$$v = \frac{U_0}{(1-\lambda t)} e^{\frac{x+y}{L}} g'(\eta \to 0) \text{ as } \eta \to \infty \tag{3.68}$$

This implies

$$g' \to 0 \ as \ \eta \to \infty \tag{3.69}$$

When

$T \to T_\infty \ at \ z \to \infty$, we have

$$T = T_\infty + \frac{(T_w - T_\infty)}{(1-\lambda t)} e^{\frac{x+y}{L}} \theta(\eta \to \infty) \to T_\infty \ as \, \eta \to \infty$$

Then

$$\theta(\eta \to \infty) \to \frac{T - T_\infty}{\frac{(T_w - T_\infty)}{(1-\lambda t)} e^{\frac{x+y}{L}}} \ as \, \eta \to \infty$$

(3.74)

This gives

$$\theta \to 0 \ as \, \eta \to \infty \tag{3.70}$$

Similarly

$C \to C_\infty \ at \ z \to \infty$, we have

$$C = C_\infty + \frac{(C_w - C_\infty)}{(1-\lambda t)} e^{\frac{x+y}{L}} \phi(\eta \to \infty) \to C_\infty \, as \, \eta \to \infty$$

This implies

$$C_\infty + \frac{(C_w - C_\infty)}{(1-\lambda t)} e^{\frac{x+y}{L}} \phi(\eta \to \infty) \to C_\infty \, as \, \eta \to \infty$$

$$\phi(\eta \to \infty) \to \frac{C - C_\infty}{\frac{(C_w - C_\infty)}{(1-\lambda t)} e^{\frac{x+y}{L}}} \; as \, \eta \to \infty$$

This gives

$$\phi \to 0 \; as \, \eta \to \infty \tag{3.71}$$

$$w = -\sqrt{\frac{\upsilon U_0}{2L(1-\lambda t)}} e^{\frac{x+y}{2L}} \left(f + \eta f' + g + \eta g' \right)$$

$$w(\eta, t) = w_w = 0$$

$$w(0, t) = w_w = 0$$

$$w(0,t) = -\sqrt{\frac{\upsilon U_0}{2L(1-\lambda t)}} e^{\frac{x+y}{2L}} \left(f(0) + 0 + g(0) + 0 \right) = 0$$

$$-\sqrt{\frac{\upsilon U_0}{2L(1-\lambda t)}} e^{\frac{x+y}{2L}} \left(f(0) + 0 + g(0) + 0 \right) = 0$$

$$f(0) + g(0) = 0 \Rightarrow f(0) = 0 \text{ and } g(0) = 0 \tag{3.72}$$

Therefore, the similarity equations together with the corresponding boundary conditions are:

$$f''' + f''\left(f + \eta f' + g + \eta g'\right) - \frac{a}{R_e}\left(f' + \frac{\eta}{2}f''\right) - 2f'\left(f' + \frac{\eta}{2}f''\right)$$
$$-2g'\left(f' + \frac{\eta}{2}f''\right) - \Omega f'^2 - \left(M + \gamma\right)f' + G_{r\theta}\left(1 + \varepsilon\theta\right) + G_{r\phi}\phi = 0 \tag{3.73}$$

$$g''' + g''\left(f + \eta f' + g + \eta g'\right) - \frac{a}{\mathrm{R}_e}\left(g' + \frac{\eta}{2}g''\right) - 2f'\left(g' + \frac{\eta}{2}g''\right)$$
$$-2g'\left(g' + \frac{\eta}{2}g''\right) - \Omega g'^2 - \left(M + \gamma\right)g' = 0 \tag{3.74}$$

$$\frac{a}{R_e}\left(\left(\frac{1}{\varepsilon} + \theta\right) + \frac{\eta}{2}\theta'\right) + 2\left(\left(\frac{1}{\varepsilon} + \theta\right) + \frac{\eta}{2}\theta'\right)\left(f' + g'\right) - \left(f + \eta f' + g + \eta g'\right)\theta' =$$
$$N_b\theta'\phi' + N_t\theta'^2 + Q_h\left(1 + \varepsilon\theta\right) + E_c M_1\left(f'^2 + g'^2\right) + R_1\theta'' + \tag{3.75}$$
$$\delta\phi\left(1 + \varepsilon\theta\right)^{\omega} e^{\frac{\theta}{(1+\varepsilon\theta)}}$$

$$\frac{a}{R_e}\left(\phi + \frac{\eta}{2}\phi'\right) + 2\left(\phi + \frac{\eta}{2}\phi'\right)\left(f' + g'\right) - \left(f + \eta f' + g + \eta g'\right)\phi' =$$
$$\frac{1}{Sc}\phi'' + \frac{N_{t1}}{S_c}\theta'' - \sigma\phi\left(1 + \varepsilon\theta\right)^{\omega} e^{\frac{\theta}{(1+\varepsilon\theta)}} \tag{3.76}$$

$$f(0) = 0, g(0) = 0, f'(0) = -1, g'(0) = -\psi, \theta(0) = 1, \phi(0) = 1$$
$$f' \to 0 \text{ as } \eta \to \infty, g' \to 0 \text{ as } \eta \to \infty \tag{3.77}$$
$$\theta \to 0 \text{ as } \eta \to \infty, \phi \to 0 \text{ as } \eta \to \infty$$

Next, we shall establish the conditions for the existence of unique solution of the model equations.

3.2.3 Existence and uniqueness of solution

First, we consider the dimensional equations (3.1) – (3.11) satisfying (3.12) when α, u, v and q_r are constants, $\alpha = D_B$ and $\sigma_e \to 0$, $Q \to 0$, $\tau \to 0$, $D_\tau \to 0$. Then equations (3.1) – (3.11) reduces to

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \alpha\frac{\partial^2 T}{\partial z^2} + \beta k_r^2\left(T - T_\infty\right)^{\omega}.\left(C - C_\infty\right)e^{-\frac{E_a}{k(T - T_\infty)}} \tag{3.78}$$

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=\alpha\frac{\partial^2 C}{\partial z^2}-k_r^2\left(T-T_\infty\right)^{\varpi}.\left(C-C_\infty\right)e^{-\frac{E_a}{k(T-T_\infty)}} \tag{3.79}$$

Multiplying (3.79) by β gives

$$\frac{\partial}{\partial t}(\beta C)+u\frac{\partial}{\partial x}(\beta C)+v\frac{\partial}{\partial y}(\beta C)+w\frac{\partial}{\partial z}(\beta C)=\alpha\frac{\partial^2}{\partial z^2}(\beta C)- \beta k_r^2\left(T-T_\infty\right)^{\varpi}.\left(C-C_\infty\right)e^{-\frac{E_a}{k(T-T_\infty)}} \tag{3.80}$$

Adding (3.78) and (3.80), result to

$$\frac{\partial}{\partial t}(T+\beta C)+u\frac{\partial}{\partial x}(T+\beta C)+v\frac{\partial}{\partial y}(T+\beta C)+w\frac{\partial}{\partial z}(T+\beta C)=\alpha\frac{\partial^2}{\partial z^2}(T+\beta C) \tag{3.81}$$

Introducing a new space variable (Olayiwola *et al*., 2014) as:

$$\vartheta = x+y+z\sqrt{\alpha} \tag{3.82}$$

Then equation (3.81) becomes

$$\frac{\partial}{\partial t}(T+\beta C)+U\frac{\partial}{\partial \vartheta}(T+\beta C)=\alpha\frac{\partial^2}{\partial \vartheta^2}(T+\beta C) \tag{3.83}$$

where

$$U = u+v+w\sqrt{\alpha}$$

Let

$$\varphi = T+\beta C \tag{3.84}$$

Then (3.83) becomes

$$\frac{\partial \varphi}{\partial t}+U\frac{\partial \varphi}{\partial \vartheta}=\alpha\frac{\partial^2 \varphi}{\partial \vartheta^2} \tag{3.85}$$

We make the variable dimensionless by introducing (Olayiwola, 2011)

$$\eta = \left(\rho^2\alpha\right)^{-\frac{1}{2}}\int_0^{\vartheta}\rho\, ds \tag{3.86}$$

Then the coordinate transformation becomes:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} - U\frac{\partial}{\partial \eta} \tag{3.87}$$

$$\frac{\partial}{\partial \vartheta} = \frac{\partial}{\partial \eta} \tag{3.88}$$

Using (3.87) and (3.89), then equation (3.85) in terms of η and t becomes

$$\frac{\partial \varphi}{\partial t} - \frac{\partial^2 \varphi}{\partial \eta^2} = 0 \tag{3.89}$$

With the initial and boundary conditions:

$$\varphi(\eta,0) = T_\infty + \beta C_\infty,\ \ \varphi(0,t) = T_w + \beta C_w,\ \ \varphi(\infty,t) \to T_\infty + \beta C_\infty \tag{3.90}$$

Theorem 3.1: There exists a unique solution of equations (3.78) and (3.79) satisfying (3.12).

Proof: We multiply (3.79) by β and obtain (3.80) satisfying (3.12) as earlier done.

Using the Fourier sine transform (see Myint-U and Debnath (1987), p. 333 - 335), we obtain the solution of the problem (3.89) in compact form as:

$$\varphi(\eta,t) = \frac{2}{\pi}(T_w + \beta C_w)\int_0^\infty s \sin s\eta \int_0^t e^{-s^2(t-\tau)} d\tau\, ds \tag{3.91}$$

That is

$$\varphi(\vartheta,t) = \frac{2}{\pi}(T_w + \beta C_w)\int_0^\infty s \sin \frac{\vartheta}{\sqrt{\alpha}} s \int_0^t e^{-s^2(t-\tau)} d\tau\, ds \tag{3.92}$$

Then, we obtain

$$T(\vartheta,t) = \frac{2}{\pi}(T_w + \beta C_w)\int_0^\infty s \sin \frac{\vartheta}{\sqrt{\alpha}} s \int_0^t e^{-s^2(t-\tau)} d\tau\, ds - \beta C(\vartheta,t) \tag{3.93}$$

$$C(\vartheta,t)=\frac{1}{\beta}\left(\frac{2}{\pi}\left(T_w+\beta C_w\right)\int_0^\infty s\sin\frac{\vartheta}{\sqrt{k_b}}s\int_0^t e^{-s^2(t-\tau)}d\tau\,ds-T(\vartheta,t)\right) \tag{3.94}$$

Hence, there exists a unique solution of problem (3.78) and (3.79). This completes the proof.

We shall now consider an alternative method for the existence of unique solution of the model equations.

Next, we consider the similarity transformed equations (3.73) – (3.76) satisfying (3.77) and establish the conditions for the existence of unique solution.

Theorem 3.2 Let $|f''(\eta)|\le b_1, |f'(\eta)|\le b_2, |g''(\eta)|\le b_3, |g'(\eta)|\le b_4, |\theta'(\eta)|\le b_5, |\phi'(\eta)|\le b_6,$ $|\theta'(0)|\le c_1, |\phi'(0)|\le c_2, |f''(0)|\le c_3, |g''(0)|\le c_4,$ where $M, G_{r\theta}, \delta, a, \mathrm{P_r}, \gamma, \gamma_1, N_b, N_t,$ $N_{t_1}, S_c, b_i,\ i=1,2,...,6,\ c_j, j=1,2....,4$ are real constants and $0<\eta<\infty$ and $0\le\varepsilon\le1$. Then the problem (3.73) – (3.76) satisfies (3.77) have a unique solution.

In the proof we shall need the following theorem 3.3.

Theorem 3.3 (Derrick and Grossman (1976)):

Let

$$\begin{aligned}
y_1' &= f_1(y_1, y_2, ...y_n, t) \qquad y_1(t_0)=y_{10}\\
y_2' &= f_2(y_1, y_2, ...y_n, t) \qquad y_2(t_0)=y_{20}\\
y_3' &= f_3(y_1, y_2, ...y_n, t) \qquad y_3(t_0)=y_{30}\\
&\vdots\\
y_n' &= f_n(y_1, y_2, ...y_n, t) \qquad y_n(t_0)=y_{n0}
\end{aligned}$$

If the partial derivatives $\dfrac{\partial f_j}{\partial y_i}, i = 1, 2, \ldots n$ are continuous in the region D of the definition then the problem has a unique solution.

All we are left to do is to show that our problems satisfy the hypothesis of Theorem 3.3.

Proof of Theorem 3.2: We now return to our problem (3.73) to (3.77) and transform it to meet the requirement of Theorem 3.3 as follows.

Let

$$\left.\begin{array}{l} Y_1 = \eta, \quad Y_2 = f, \quad Y_3 = g, \quad Y_4 = \theta, \quad Y_5 = \phi, \quad Y_6 = f', \quad Y_7 = g', \\ Y_8 = \theta', \quad Y_9 = \phi', Y_{10} = f'', \quad Y_{11} = g'' \end{array}\right\} \tag{3.95}$$

Then equations (3.73)-(3.76) becomes

$$\begin{aligned} f''' = d_1 = {} & 2Y_6\left(Y_6 + \frac{Y_1Y_{10}}{2}\right) + 2Y_7\left(Y_6 + \frac{Y_1Y_{10}}{2}\right) + (M+\gamma)Y_6 + \frac{a}{\mathrm{R}_e}\left(Y_6 + \frac{Y_1Y_{10}}{2}\right) \\ & + \Omega Y_6^2 - Y_{10}\left(Y_2 + Y_1Y_6 + Y_3 + Y_1Y_7\right) - \left(G_{r\theta}\left(1+\varepsilon Y_4\right) + G_{r\phi}Y_5\right) \end{aligned} \tag{3.96}$$

$$\begin{aligned} g''' = d_2 = {} & 2Y_7\left(Y_7 + \frac{Y_1Y_{11}}{2}\right) + 2Y_6\left(Y_7 + \frac{Y_1Y_{11}}{2}\right) + \frac{a}{\mathrm{R_e}}\left(Y_7 + \frac{Y_1Y_{11}}{2}\right) + \Omega Y_7^2 \\ & + (M+\gamma)Y_7 - Y_{11}\left(Y_2 + Y_1Y_6 + Y_3 + Y_1Y_7\right) \end{aligned} \tag{3.97}$$

$$\theta'' = d_3 = \frac{1}{R_1}\left(\begin{array}{l} 2\left(\left(\dfrac{1}{\varepsilon} + Y_4\right) + \dfrac{Y_1Y_8}{2}\right)\left(Y_6 + Y_7\right) + \dfrac{a}{\mathrm{R_e}}\left(\left(\dfrac{1}{\varepsilon} + Y_4\right) + \dfrac{Y_1Y_8}{2}\right) \\ -\left(Y_2 + Y_1Y_6 + Y_3 + Y_1Y_7\right)Y_8 - N_tY_8^2 - N_bY_8Y_9 - \\ \delta Y_5\left(1+\varepsilon Y_4\right)^{\omega} e^{\frac{Y_4}{(1+\varepsilon Y_4)}} - E_cM\left(Y_6^2 + Y_7^2\right) - Q_h\left(1+\varepsilon Y_4\right) \end{array}\right) \tag{3.98}$$

$$\phi'' = d_4 = S_c \begin{pmatrix} a_1 Y_5 \left(1+\varepsilon Y_4\right)^{\omega} e^{\frac{Y_4}{(1+\varepsilon Y_4)}} + 2\left(Y_5 + \frac{Y_1 Y_9}{2}\right)\left(Y_6 + Y_7\right) + \frac{a}{\mathrm{R_e}}\left(Y_5 + \frac{Y_1 Y_9}{2}\right) \\ -a_2\left(\left(\frac{1}{\varepsilon} + Y_4\right) + \frac{Y_1 Y_8}{2}\right)\left(Y_6 + Y_7\right) - a_3\left(\left(\frac{1}{\varepsilon} + Y_4\right) + \frac{Y_1 Y_8}{2}\right) \\ +\left(a_4 Y_8 - Y_9\right)\left(Y_2 + Y_1 Y_6 + Y_3 + Y_1 Y_7\right) + a_5 Y_8^{\,2} + a_6 Y_8 Y_9 + \\ a_7\left(Y_6^{\,2} + Y_7^{\,2}\right) + a_8\left(1+\varepsilon Y_4\right) \end{pmatrix} \tag{3.99}$$

where

$$a_1 = \left(\sigma + \frac{\delta N_{t_1}}{R_1 S_c}\right),\ a_2 = \frac{2N_{t_1}}{R_1 S_c},\ a_3 = \frac{a N_{t_1}}{\mathrm{R_e}\, R_1 S_c},\ a_4 = \frac{N_{t_1}}{R_1 S_c},\ a_5 = \frac{N_t N_{t_1}}{R_1 S_c},\ a_6 = \frac{N_b N_{t_1}}{R_1 S_c},$$

$$a_7 = \frac{E_{\mathrm{c}} M N_{t_1}}{R_1 S_c},\quad a_8 = \frac{Q_h N_{t_1}}{R_1 S_c}$$

The system (3.95) can be written in vector form using

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \\ Y_7 \\ Y_8 \\ Y_9 \\ Y_{10} \\ Y_{11} \end{pmatrix} = \begin{pmatrix} \eta \\ f \\ g \\ \theta \\ \phi \\ f' \\ g' \\ \theta' \\ \phi' \\ f'' \\ g'' \end{pmatrix} \tag{3.100}$$

and the derivative of (3.100) results

$$\begin{pmatrix} Y_1' \\ Y_2' \\ Y_3' \\ Y_4' \\ Y_5' \\ Y_6' \\ Y_7' \\ Y_8' \\ Y_9' \\ Y_{10}' \\ Y_{11}' \end{pmatrix} = \begin{pmatrix} 1 \\ Y_6 \\ Y_7 \\ Y_8 \\ Y_9 \\ Y_{10} \\ Y_{11} \\ d_3 \\ d_4 \\ d_1 \\ d_2 \end{pmatrix} \tag{3.101}$$

where

$$d_1 = 2Y_6\left(Y_6 + \frac{Y_1Y_{10}}{2}\right) + 2Y_7\left(Y_6 + \frac{Y_1Y_{10}}{2}\right) + (M+\gamma)Y_6 + \frac{a}{\mathrm{R}_e}\left(Y_6 + \frac{Y_1Y_{10}}{2}\right)$$
$$+\Omega Y_6^{\,2} - Y_{10}\left(Y_2 + Y_1Y_6 + Y_3 + Y_1Y_7\right) - \left(G_{r\theta}\left(1+\varepsilon Y_4\right) + G_{r\phi}Y_5\right)$$

$$d_2 = 2Y_7\left(Y_7 + \frac{Y_1Y_{11}}{2}\right) + 2Y_6\left(Y_7 + \frac{Y_1Y_{11}}{2}\right) + \frac{a}{\mathrm{R_e}}\left(Y_7 + \frac{Y_1Y_{11}}{2}\right) + \Omega Y_7^{\,2}$$
$$+(M+\gamma)Y_7 - Y_{11}\left(Y_2 + Y_1Y_6 + Y_3 + Y_1Y_7\right)$$

$$d_3 = \frac{1}{R_1}\begin{pmatrix} 2\left(\left(\frac{1}{\varepsilon} + Y_4\right) + \frac{Y_1Y_8}{2}\right)\left(Y_6 + Y_7\right) + \frac{a}{\mathrm{R_e}}\left(\left(\frac{1}{\varepsilon} + Y_4\right) + \frac{Y_1Y_8}{2}\right) \\ -\left(Y_2 + Y_1Y_6 + Y_3 + Y_1Y_7\right)Y_8 - N_tY_8^{\,2} - N_bY_8Y_9 - \\ \delta Y_5\left(1+\varepsilon Y_4\right)^{\varpi} e^{\frac{Y_4}{(1+\varepsilon Y_4)}} - E_cM\left(Y_6^{\,2} + Y_7^{\,2}\right) - Q_h\left(1+\varepsilon Y_4\right) \end{pmatrix}$$

$$d_4 = S_c \begin{pmatrix} a_1 Y_5 \left(1+\varepsilon Y_4\right)^{\varpi} e^{\frac{Y_4}{(1+\varepsilon Y_4)}} + 2\left(Y_5 + \frac{Y_1 Y_9}{2}\right)\left(Y_6 + Y_7\right) + \frac{a}{\mathrm{R_e}}\left(Y_5 + \frac{Y_1 Y_9}{2}\right) \\ -a_2\left(\left(\frac{1}{\varepsilon} + Y_4\right) + \frac{Y_1 Y_8}{2}\right)\left(Y_6 + Y_7\right) - a_3\left(\left(\frac{1}{\varepsilon} + Y_4\right) + \frac{Y_1 Y_8}{2}\right) \\ +\left(a_4 Y_8 - Y_9\right)\left(Y_2 + Y_1 Y_6 + Y_3 + Y_1 Y_7\right) + a_5 Y_8^{\ 2} + a_6 Y_8 Y_9 + \\ a_7\left(Y_6^{\ 2} + Y_7^{\ 2}\right) + a_8\left(1+\varepsilon Y_4\right) \end{pmatrix}$$

Satisfying

$$\begin{pmatrix} Y_1(0) \\ Y_2(0) \\ Y_3(0) \\ Y_4(0) \\ Y_5(0) \\ Y_6(0) \\ Y_7(0) \\ Y_8(0) \\ Y_9(0) \\ Y_{10}(0) \\ Y_{11}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -\psi \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} \qquad (3.102)$$

$0 < Y_i < \infty$, where s_1, s_2, s_3 and s_4 are guessed values such that $Y_i(\infty) \to 0, \quad i = 4,5,6,7$.

We define

$$
\begin{aligned}
&f_1(Y_1, Y_2, ..., Y_{11}) = 1 \\
&f_2(Y_1, Y_2, ..., Y_{11}) = Y_6 \\
&f_3(Y_1, Y_2, ..., Y_{11}) = Y_7 \\
&f_4(Y_1, Y_2, ..., Y_{11}) = Y_8 \\
&f_5(Y_1, Y_2, ..., Y_{11}) = Y_9 \\
&f_6(Y_1, Y_2, ..., Y_{11}) = Y_{10} \\
&f_7(Y_1, Y_2, ..., Y_{11}) = Y_{11} \\
&f_8(Y_1, Y_2, ..., Y_{11}) = d_3 \\
&f_9(Y_1, Y_2, ..., Y_{11}) = d_4 \\
&f_{10}(Y_1, Y_2, ..., Y_{11}) = d_1 \\
&f_{11}(Y_1, Y_2, ..., Y_{11}) = d_2
\end{aligned}
\qquad (3.103)
$$

Then

$$\left|\frac{\partial f_1}{\partial Y_i}\right| = 0, \quad i = 1, 2, ...11 \qquad (3.104)$$

$$\left|\frac{\partial f_2}{\partial Y_i}\right| = 0, \quad i = 1, 2, ...5, 7...11 \qquad (3.105)$$

$$\left|\frac{\partial f_2}{\partial Y_6}\right| = 1 \qquad (3.106)$$

$$\left|\frac{\partial f_3}{\partial Y_i}\right| = 0, \quad i = 1, 2, ...6, 8, ...11 \qquad (3.107)$$

$$\left|\frac{\partial f_3}{\partial Y_7}\right| = 1 \qquad (3.108)$$

$$\left|\frac{\partial f_4}{\partial Y_i}\right| = 0, \quad i = 1, 2, ...7, 9, ...11 \qquad (3.109)$$

$$\left|\frac{\partial f_4}{\partial Y_8}\right| = 1 \tag{3.110}$$

$$\left|\frac{\partial f_5}{\partial Y_i}\right| = 0, \quad i = 1, 2, ...8, 10, 11 \tag{3.111}$$

$$\left|\frac{\partial f_5}{\partial Y_9}\right| = 1 \tag{3.112}$$

$$\left|\frac{\partial f_6}{\partial Y_i}\right| = 0, \quad i = 1, 2, ...9, 11 \tag{3.113}$$

$$\left|\frac{\partial f_6}{\partial Y_{10}}\right| = 1 \tag{3.114}$$

$$\left|\frac{\partial f_7}{\partial Y_i}\right| = 0, \quad i = 1, 2, ...10 \tag{3.115}$$

$$\left|\frac{\partial f_7}{\partial Y_{11}}\right| = 1 \tag{3.116}$$

$$\left|\frac{\partial f_8}{\partial Y_1}\right| = \left|\frac{1}{R_1}\left(\frac{a}{\mathrm{R_e}}\frac{Y_8}{2}\right)\right| \leq \frac{1}{R_1}\left(\frac{ab_5}{2R_e}\right) = \xi_1 < \infty \tag{3.117}$$

$$\left|\frac{\partial f_8}{\partial Y_2}\right| = \left|-\frac{Y_8}{R_1}\right| \leq \frac{b_5}{R_1} = \xi_2 < \infty \tag{3.118}$$

$$\left|\frac{\partial f_8}{\partial Y_3}\right| = \left|-\frac{Y_8}{R_1}\right| \leq \frac{b_5}{R_1} = \xi_2 < \infty \tag{3.119}$$

$$\left|\frac{\partial f_8}{\partial Y_4}\right| = \left|\frac{1}{R_1}\begin{pmatrix} 2\left(Y_6+Y_7\right)+\dfrac{a}{\mathrm{R_e}}-\delta\omega\varepsilon Y_5\left(1+\varepsilon Y_4\right)^{\omega} e^{\frac{Y_4}{\left(1+\varepsilon Y_4\right)}} \\ -\delta Y_5\left(1+\varepsilon Y_4\right)^{\omega}\dfrac{e^{\frac{Y_4}{\left(1+\varepsilon Y_4\right)}}}{\left(1+\varepsilon Y_4\right)^2}-Q_h\varepsilon \end{pmatrix}\right|$$

$$\le \frac{1}{R_1}\begin{pmatrix} 2\left(b_2+b_4\right)+\dfrac{a}{\mathrm{R_e}}-\delta\omega\varepsilon Y_5\left(1+\varepsilon Y_4\right)^{\omega} e^{\frac{1}{\varepsilon}} \\ -\delta Y_5\left(1+\varepsilon Y_4\right)^{\omega} e^{\frac{1}{\varepsilon}}-Q_h\varepsilon \end{pmatrix} = \xi_3 < \infty, -1 \le \omega \le 1 \tag{3.120}$$

$$\left|\frac{\partial f_8}{\partial Y_5}\right| = \left|\frac{1}{R_1}\left(-\delta\left(1+\varepsilon Y_4\right)^{\omega} e^{\frac{Y_4}{\left(1+\varepsilon Y_4\right)}}\right)\right| \le \frac{1}{R_1}\left(\delta\left(1+\varepsilon Y_4\right)^{\omega} e^{\frac{1}{\varepsilon}}\right) = \xi_4 < \infty \tag{3.121}$$

$$\left|\frac{\partial f_8}{\partial Y_6}\right| = \left|\frac{1}{R_1}\left(2\left(\frac{1}{\varepsilon}+Y_4\right)-2E_c M Y_6\right)\right| \le \frac{1}{R_1}\left(2\left(\frac{1}{\varepsilon}+Y_4\right)-2E_c M b_2\right) = \xi_5 < \infty \tag{3.122}$$

$$\left|\frac{\partial f_8}{\partial Y_7}\right| = \left|\frac{1}{R_1}\left(2\left(\frac{1}{\varepsilon}+Y_4\right)-2E_c M Y_7\right)\right| \le \frac{1}{R_1}\left(2\left(\frac{1}{\varepsilon}+Y_4\right)-2E_c M b_4\right) = \xi_6 < \infty \tag{3.123}$$

$$\left|\frac{\partial f_8}{\partial Y_8}\right| = \left|\frac{1}{R_1}\left(Y_1\left(Y_6+Y_7\right)+\frac{aY_1}{2\mathrm{R_e}}-\left(Y_2+Y_1Y_6+Y_3+Y_1Y_7\right)-2N_tY_8-N_bY_9\right)\right|$$

$$\le \frac{1}{R_1}\left(Y_1\left(b_2+b_4\right)+\frac{aY_1}{2\mathrm{R_e}}-\left(Y_2+Y_1b_2+Y_3+Y_1b_4\right)-2N_tb_5-N_bb_6\right) = \xi_7 < \infty \tag{3.124}$$

$$\left|\frac{\partial f_8}{\partial Y_9}\right| = \left|-\frac{N_bY_8}{R_1}\right| \le \frac{N_bb_5}{R_1} = \xi_8 < \infty \tag{3.125}$$

$$\left|\frac{\partial f_8}{\partial Y_{10}}\right| = \left|\frac{\partial f_8}{\partial Y_{11}}\right| = 0 \tag{3.126}$$

$$\left|\frac{\partial f_9}{\partial Y_1}\right| = \left| S_c\left(\frac{a}{\mathrm{R_e}}\frac{Y_9}{2} - a_3\frac{Y_8}{2} + \left(a_4 - \frac{a_2}{2}\right)Y_8\left(Y_6+Y_7\right)\right)\right|$$
$$\leq S_c\left(\frac{a}{\mathrm{R_e}}\frac{b_6}{2} - a_3\frac{b_5}{2} + \left(a_4 - \frac{a_2}{2}\right)b_5\left(b_2+b_4\right)\right) = \xi_9 < \infty \tag{3.127}$$

$$\left|\frac{\partial f_9}{\partial Y_2}\right| = \left| S_c\left(a_4Y_8 - Y_9\right)\right| \leq S_c\left(a_4b_5 - b_6\right) = \xi_{10} < \infty \tag{3.128}$$

$$\left|\frac{\partial f_9}{\partial Y_3}\right| = \left| S_c\left(a_4Y_8 - Y_9\right)\right| \leq S_c\left(a_4b_5 - b_6\right) = \xi_{10} < \infty \tag{3.129}$$

$$\left|\frac{\partial f_9}{\partial Y_4}\right| = \left| S_c\begin{pmatrix} a_1\omega\varepsilon Y_5\left(1+\varepsilon Y_4\right)^{\omega-1} e^{\frac{Y_4}{\left(1+\varepsilon Y_4\right)}} + a_1Y_5\left(1+\varepsilon Y_4\right)^{\omega}\dfrac{e^{\frac{Y_4}{\left(1+\varepsilon Y_4\right)}}}{\left(1+\varepsilon Y_4\right)^2} \\ -a_2\left(Y_6+Y_7\right) - a_3 + a_8 \in \end{pmatrix}\right|$$
$$\leq S_c\begin{pmatrix} a_1\omega\varepsilon Y_5\left(1+\varepsilon Y_4\right)^{\omega-1} e^{\frac{1}{\varepsilon}} + a_1Y_5\left(1+\varepsilon Y_4\right)^{\omega} e^{\frac{1}{\varepsilon}} \\ -a_2\left(b_2+b_4\right) - a_3 + a_8 \in \end{pmatrix} = \xi_{11} < \infty \tag{3.130}$$

$$\left|\frac{\partial f_9}{\partial Y_5}\right| = \left| S_c\left(a_1\left(1+\varepsilon Y_4\right)^{\omega} e^{\frac{Y_4}{\left(1+\varepsilon Y_4\right)}} + 2\left(Y_6+Y_7\right) + \frac{a}{\mathrm{R_e}}\right)\right|$$
$$\leq S_c\left(a_1\left(1+\varepsilon Y_4\right)^{\omega} e^{\frac{1}{\varepsilon}} + 2\left(b_2+b_4\right) + \frac{a}{\mathrm{R_e}}\right) = \xi_{12} < \infty \tag{3.131}$$

$$\left|\frac{\partial f_9}{\partial Y_6}\right| = \left| S_c\left(2\left(Y_5 + \frac{Y_1Y_9}{2}\right) - a_2\left(\left(\frac{1}{\varepsilon}+Y_4\right) + \frac{Y_1Y_8}{2}\right) + \left(a_4Y_8 - Y_9\right)Y_1 + 2a_7Y_6\right)\right|$$
$$\leq S_c\left(2\left(Y_5 + \frac{Y_1b_6}{2}\right) - a_2\left(\left(\frac{1}{\varepsilon}+Y_4\right) + \frac{Y_1b_5}{2}\right) + \left(a_4b_5 - b_6\right)Y_1 + 2a_7b_2\right) = \xi_{13} < \infty \tag{3.132}$$

$$\left|\frac{\partial f_9}{\partial Y_7}\right| = \left|S_c\left(2\left(Y_5+\frac{Y_1Y_9}{2}\right)-a_2\left(\left(\frac{1}{\varepsilon}+Y_4\right)+\frac{Y_1Y_8}{2}\right)+\left(a_4Y_8-Y_9\right)Y_1+2a_7Y_7\right)\right|$$
$$\le S_c\left(2\left(Y_5+\frac{Y_1b_6}{2}\right)-a_2\left(\left(\frac{1}{\varepsilon}+Y_4\right)+\frac{Y_1b_5}{2}\right)+\left(a_4b_5-b_6\right)Y_1+2a_7b_4\right)=\xi_{14}<\infty \tag{3.133}$$

$$\left|\frac{\partial f_9}{\partial Y_8}\right| = \left|S_c\left(-a_2\frac{Y_1}{2}\left(Y_6+Y_7\right)-a_3\frac{Y_1}{2}+a_4\left(Y_2+Y_1Y_6+Y_3+Y_1Y_7\right)+2a_5Y_8+a_6Y_9\right)\right|$$
$$\le S_c\left(-a_2\frac{Y_1}{2}\left(b_2+b_4\right)-a_3\frac{Y_1}{2}+a_4\left(Y_2+Y_1b_2+Y_3+Y_1b_4\right)+2a_5b_5+a_6b_6\right)=\xi_{15}<\infty \tag{3.134}$$

$$\left|\frac{\partial f_9}{\partial Y_9}\right| = \left|S_c\left(Y_1\left(Y_6+Y_7\right)+\frac{a}{\mathrm{R_e}}\frac{Y_1}{2}-\left(Y_2+Y_1Y_6+Y_3+Y_1Y_7\right)+a_6Y_8\right)\right|$$
$$\le S_c\left(Y_1\left(b_2+b_4\right)+\frac{a}{\mathrm{R_e}}\frac{Y_1}{2}-\left(Y_2+Y_1b_2+Y_3+Y_1b_4\right)+a_6b_5\right)=\xi_{16}<\infty \tag{3.135}$$

$$\left|\frac{\partial f_9}{\partial Y_{10}}\right| = \left|\frac{\partial f_9}{\partial Y_{11}}\right| = 0 \tag{3.136}$$

$$\left|\frac{\partial f_{10}}{\partial Y_1}\right| = \left|\frac{a}{\mathrm{R}_e}\frac{Y_{10}}{2}\right| \le \frac{ab_1}{2\,\mathrm{R_e}} = \xi_{17} < \infty \tag{3.137}$$

$$\left|\frac{\partial f_{10}}{\partial Y_2}\right| = \left|-Y_{10}\right| \le b_1 < \infty \tag{3.138}$$

$$\left|\frac{\partial f_{10}}{\partial Y_3}\right| = \left|-Y_{10}\right| \le b_1 < \infty \tag{3.139}$$

$$\left|\frac{\partial f_{10}}{\partial Y_4}\right| = \left|-\left(G_{r\theta}\varepsilon\right)\right| \le G_{r\theta}\varepsilon < \infty \tag{3.140}$$

$$\left|\frac{\partial f_{10}}{\partial Y_5}\right| = \left|-G_{r\phi}\right| \le G_{r\phi} < \infty \tag{3.141}$$

$$\left|\frac{\partial f_{10}}{\partial Y_6}\right| = \left|2Y_6 + 2Y_7 + \left(M + \gamma\right) + \frac{a}{\mathrm{R}_e} + 2\Omega Y_6\right|$$

$$\le 2b_2 + 2b_4 + \left(M + \gamma\right) + \frac{a}{\mathrm{R}_e} + 2\Omega b_2 = \xi_{18} < \infty \tag{3.142}$$

$$\left|\frac{\partial f_{10}}{\partial Y_7}\right| = \left|2Y_6\right| \le 2b_2 < \infty \tag{3.143}$$

$$\left|\frac{\partial f_{10}}{\partial Y_8}\right| = \left|\frac{\partial f_{10}}{\partial Y_9}\right| = \left|\frac{\partial f_{10}}{\partial Y_{11}}\right| = 0 \tag{3.144}$$

$$\left|\frac{\partial f_{10}}{\partial Y_{10}}\right| = \left|\frac{a}{\mathrm{R}_e}\frac{Y_1}{2} - \left(Y_2 + Y_3\right)\right| = \left|\frac{aY_1}{2\,\mathrm{R_e}} - \left(Y_2 + Y_3\right)\right| = \xi_{19} < \infty \tag{3.145}$$

$$\left|\frac{\partial f_{11}}{\partial Y_1}\right| = \left|\frac{a}{\mathrm{R_e}}\frac{Y_{11}}{2}\right| \le \frac{ab_3}{2\,\mathrm{R_e}} = \xi_{20} < \infty \tag{3.146}$$

$$\left|\frac{\partial f_{11}}{\partial Y_2}\right| = \left|-Y_{11}\right| \le b_3 < \infty \tag{3.147}$$

$$\left|\frac{\partial f_{11}}{\partial Y_3}\right| = \left|-Y_{11}\right| \le b_3 < \infty \tag{3.148}$$

$$\left|\frac{\partial f_{11}}{\partial Y_4}\right| = \left|\frac{\partial f_{11}}{\partial Y_5}\right| = \left|\frac{\partial f_{11}}{\partial Y_8}\right| = \left|\frac{\partial f_{11}}{\partial Y_9}\right| = \left|\frac{\partial f_{11}}{\partial Y_{10}}\right| = 0 \tag{3.149}$$

$$\frac{\partial f_{11}}{\partial Y_6} = \left|2Y_7\right| \le 2b_4 < \infty \tag{3.150}$$

$$\left|\frac{\partial f_{11}}{\partial Y_7}\right| = \left|2Y_7 + 2Y_6 + \frac{a}{\mathrm{R_e}} + 2\Omega Y_7 + (M+\gamma)\right|$$

$$\leq 2b_4 + 2b_2 + \frac{a}{\mathrm{R_e}} + 2\Omega b_4 + (M+\gamma) = \xi_{21} < \infty \tag{3.151}$$

$$\left|\frac{\partial f_{11}}{\partial Y_{11}}\right| = \left|\frac{a}{\mathrm{R_e}}\frac{Y_1}{2} - (Y_2 + Y_3)\right| = \left|\frac{a}{2\,\mathrm{R_e}}Y_1 - (Y_2+Y_3)\right| = \xi_{22} < \infty \tag{3.152}$$

Let $K = \max\left\{0, 1, \mathrm{b}_1, 2b_2, b_3, 2b_4, G_{r\theta}\varepsilon, \mathrm{G}_{r\phi}\,\varepsilon, \xi_i\right\} < \infty, \quad i = 1, 2, ..., 22$

Therefore the partial derivatives $\dfrac{\partial f_j}{\partial Y_i}, \;\; i, j = 1, 2, ..., 11$ are continuous and bounded.

Hence by Theorem (3.3), equations (3.73) – (3.76) satisfying (3.77) has a unique solution.

Next, we shall examine the properties of solution of the dimensionless equations (3.29) - (3.33).

3.2.4 Properties of solution

Here, we show that $f(\eta), g(\eta), \theta(\eta)$ and $\phi(\eta)$ are bounded. If we consider equations (3.73) – (3.77) when $a \to 0$, then equations (3.73) – (3.77) reduces to :

$$\left.\begin{aligned} &(f+g)f'' - (M+\gamma)f' = 2(f'+g')f' + \Omega f'^2 - f''' - G_{r\theta}(1+\varepsilon\theta) - G_{r\phi}\phi \\ &f(0) = 0,\; f'(0) = -1,\; f'(\infty) \to 0 \end{aligned}\right\} \tag{3.153}$$

$$\left.\begin{aligned} &(f+g)g'' - (M+\gamma)g' = 2(f'+g')g' + \Omega g'^2 - g''' \\ &g(0) = 0,\; g'(0) = -\psi,\; g'(\infty) \to 0 \end{aligned}\right\} \tag{3.154}$$

$$\left.\begin{aligned} &R_1\theta'' + (f+g)\theta' + Q_h\varepsilon\theta = 2(f'+g')\left(\frac{1}{\varepsilon} + \theta\right) - Q_h \\ &-N_b\theta'\phi' - N_t\theta'^2 - E_cM\left(f'^2 + g'^2\right) - \delta\phi(1+\varepsilon\theta)^{\omega} e^{\frac{\theta}{1+\varepsilon\theta}} \\ &\theta(0) = 1\;\theta(\infty) \to 0 \end{aligned}\right\} \tag{3.155}$$

$$\left.\begin{aligned}&\frac{1}{Sc}\phi''+(f+g)\phi'=2(f+g)\phi-\frac{N_{t1}}{S_c}\theta''+\sigma\phi(1+\varepsilon\theta)^{\omega}e^{\frac{\theta}{(1+\varepsilon\theta)}}\\&\phi(0)=1,\ \phi(\infty)\to 0\end{aligned}\right\}\tag{3.156}$$

Theorem 3.6: Let $\varepsilon > 0, \psi > 0$. Then the equations (3.153) – (3.156) have solution.

Proof:

Equations (153) – (156) can be written respectively as:

$$Lf = F(\eta, f), Lg = F(\eta, g), L\theta = F(\eta,\theta) \text{ and } L\phi = F(\eta,\phi)\tag{3.157}$$

where

$$Lf = (f+g)f'' - (M+\gamma)f'$$

$$F(\eta, f) = 2(f'+g')f' + \Omega f'^2 - f''' - G_{r\theta}(1+\varepsilon\theta) - G_{r\phi}\phi$$

$$Lg = (f+g)g'' - (M+\gamma)g'$$

$$F(\eta, g) = 2(f'+g')g' + \Omega g'^2 - g'''$$

$$L\theta = R_1\theta'' + (f+g)\theta' + Q_h\varepsilon\theta$$

$$F(\eta,\theta) = 2(f'+g')\left(\frac{1}{\varepsilon}+\theta\right) - Q_h - N_b\theta'\phi' - N_t\theta'^2 - E_c M\left(f'^2 + g'^2\right)$$
$$-\delta\phi(1+\varepsilon\theta)^{\omega}e^{\frac{\theta}{1+\varepsilon\theta}}$$

$$L\phi = \frac{1}{Sc}\phi'' + (f+g)\phi'$$

$$F(\eta,\phi)=2(f+g)\phi-\frac{N_{t1}}{S_c}\theta''+\sigma\phi(1+\varepsilon\theta)^{\omega}e^{\frac{\theta}{(1+\varepsilon\theta)}}$$

Consider

$$\left.\begin{aligned}&\underline{f}(\eta)=\left(e^{-\eta}-1\right)\\&\underline{g}(\eta)=\psi\left(e^{-\eta}-1\right)\\&\underline{\theta}(\eta)=-\frac{1}{\varepsilon}\\&\underline{\phi}(\eta)=0\end{aligned}\right\}\tag{3.158}$$

We shall show that (3.158) are the lower solutions

Clearly,

$$\left.\begin{aligned}&\underline{f}(0)=0,\ \underline{f'}(0)=-1,\ \underline{f'}(\eta\to\infty)\to 0\\&\underline{g}(0)=0,\ \underline{g'}(0)=-\psi,\ \underline{g'}(\eta\to\infty)\to 0\\&\underline{\theta}(0)=-\frac{1}{\in},\ \underline{\theta}(\eta\to\infty)=-\frac{1}{\varepsilon}\\&\underline{\phi}(0)=0,\ \underline{\phi}(\eta\to\infty)=0\end{aligned}\right\}\tag{3.159}$$

Now

$$\left.\begin{aligned}&\underline{f}=\left(e^{-\eta}-1\right),\ \underline{f'}=-e^{-\eta},\ \underline{f''}=e^{-\eta}\\&\underline{g}=\psi\left(e^{-\eta}-1\right),\ \underline{g'}=-\psi e^{-\eta},\ \underline{g''}=\psi e^{-\eta}\\&\underline{\theta}(0)=-\frac{1}{\in},\ \underline{\theta'}=0,\ \underline{\theta''}=0\\&\underline{\phi}(0)=0,\ \underline{\phi'}=0,\ \underline{\phi''}=0\end{aligned}\right\}\tag{3.160}$$

This gives

$$\left.\begin{aligned}&L\underline{f}=(M+\gamma)e^{-\eta}-(\psi+1)\left(1-e^{-\eta}\right)e^{-\eta}\\&L\underline{g}=(M+\gamma)e^{-\eta}-(\psi+1)\left(1-e^{-\eta}\right)e^{-\eta}\\&L\underline{\theta}=-Q_h\\&L\underline{\phi}=0\end{aligned}\right\}\tag{3.161}$$

and

$$\left.\begin{aligned}
F\left(\eta,\underline{f}\right)&=\left(2\left(\psi+1\right)+\Omega\right)e^{-2\eta}+e^{-\eta}\\
F\left(\eta,\underline{g}\right)&=\left(2\left(\psi+1\right)+\Omega\psi\right)e^{-2\eta}+e^{-\eta}\\
F\left(\eta,\underline{\theta}\right)&=-\left(Q_h+E_cM\left(\psi^2+1\right)+e^{-2\eta}\right)\\
F\left(\eta,\underline{\phi}\right)&=0
\end{aligned}\right\}\qquad(3.162)$$

Hence,

$$\left.\begin{aligned}
&L\underline{f}\geq F\left(\eta,\underline{f}\right), &&\text{provided} &&\left(M+\gamma\right)\geq\left(2\psi+\Omega+3\right)\\
&L\underline{g}\geq F\left(\eta,\underline{g}\right), &&\text{provided} &&\left(M+\gamma\right)\geq\left(\left(2+\Omega\right)\psi+3\right)\\
&L\underline{\theta}\geq F\left(\eta,\underline{\theta}\right)\\
&L\underline{\phi}\geq F\left(\eta,\underline{\phi}\right)
\end{aligned}\right\}\qquad(3.163)$$

By definition 3, equations (3.158) are the lower solutions.

Also consider

$$\left.\begin{aligned}
\overline{f}\left(\eta\right)&=\left(1-e^{-\eta}\right)\\
\overline{g}\left(\eta\right)&=\psi\left(1-e^{-\eta}\right)\\
\overline{\theta}\left(\eta\right)&=\frac{1}{\varepsilon}\left(2-e^{-\eta}\right)\\
\overline{\phi}\left(\eta\right)&=\left(2-e^{-\eta}\right)
\end{aligned}\right\}\qquad(3.164)$$

We shall show that (3.164) are the upper solutions.

Clearly,

$$\left.\begin{aligned}
&\overline{f}\left(0\right)=0,\ \overline{f'}\left(0\right)=1,\overline{f'}\left(\eta\to\infty\right)=0\\
&\overline{g}\left(0\right)=0,\ \overline{g'}\left(0\right)=1,\overline{g'}\left(\eta\to\infty\right)=0\\
&\overline{\theta}\left(0\right)=\frac{1}{\varepsilon}>1,\ \overline{\theta}\left(\eta\to\infty\right)=\frac{2}{\varepsilon}>0\\
&\overline{\phi}\left(0\right)=1,\overline{\phi}\left(\eta\to\infty\right)=2>0
\end{aligned}\right\}\qquad(3.165)$$

Now

$$\left.\begin{aligned}
&\overline{f'''}(\eta)=e^{-\eta},\ \overline{f''}(\eta)=-e^{-\eta},\ \overline{f'}(\eta)=e^{-\eta},\ \overline{f}=\left(1-e^{-\eta}\right)\\
&\overline{g'''}(\eta)=\psi e^{-\eta},\ \overline{g''}(\eta)=-\psi e^{-\eta},\ \overline{g'}(\eta)=\psi e^{-\eta},\ \overline{g}=\psi\left(1-e^{-\eta}\right)\\
&\overline{\theta''}(\eta)=-\frac{1}{\varepsilon}e^{-\eta},\ \overline{\theta'}(\eta)=\frac{1}{\varepsilon}e^{-\eta},\ \overline{\theta}(\eta)=\frac{1}{\varepsilon}\left(2-e^{-\eta}\right)\\
&\overline{\phi''}(\eta)=-e^{-\eta},\ \overline{\phi'}(\eta)=e^{-\eta},\ \overline{\phi}(\eta)=\left(2-e^{-\eta}\right)
\end{aligned}\right\}\tag{3.166}$$

This gives

$$\left.\begin{aligned}
&L\overline{f}=\left((\psi+1)\left(1-e^{-\eta}\right)+M+\gamma\right)e^{-\eta}\\
&L\overline{g}=-\left((\psi+1)\left(1-e^{-\eta}\right)+M+\gamma\right)e^{-\eta}\\
&L\overline{\theta}=\frac{1}{\varepsilon}\left(R_1-2(\psi+1)\left(1-e^{-\eta}\right)\right)e^{-\eta}-Q_h\left(2-e^{-\eta}\right)\\
&L\overline{\phi}=\left((\psi+1)\left(1-e^{-\eta}\right)-\frac{1}{S_c}\right)e^{-\eta}
\end{aligned}\right\}\tag{3.167}$$

and

$$\left.\begin{aligned}
&F\left(\eta,\overline{f}\right)=G_{r\theta}\left(1+\varepsilon\overline{\theta}\right)+G_{r\phi}\overline{\phi}+e^{-\eta}+\left(2(\psi+1)+\Omega\right)e^{-2\eta}\\
&F\left(\eta,\overline{g}\right)=\left((2+\Omega)\psi+2\right)e^{-2\eta}-e^{-\eta}\\
&F\left(\eta,\overline{\theta}\right)=Q_h+\delta\overline{\phi}\left(1+\varepsilon\overline{\theta}\right)^{\omega}e^{\frac{\overline{\theta}}{1+\varepsilon\overline{\theta}}}+\left(\frac{N_b}{\varepsilon}+\frac{N_t}{\varepsilon^2}+E_cM\left(\psi^2+1\right)+\frac{2}{\varepsilon}(\psi+1)\right)e^{-2\eta}\\
&-\frac{6}{\varepsilon}(\psi+1)e^{-\eta}\\
&F\left(\eta,\overline{\phi}\right)=\sigma\overline{\phi}\left(1+\varepsilon\overline{\theta}\right)^{\omega}e^{\frac{\overline{\theta}}{1+\varepsilon\overline{\theta}}}+\left(4(\psi+1)+\frac{N_{t1}}{\varepsilon S_c}\right)e^{-\eta}-2(\psi+1)e^{-2\eta}
\end{aligned}\right\}\tag{3.168}$$

Hence,

$$\left.\begin{array}{l} L\overline{f} \le F\left(\eta, \overline{f}\right), \text{provided } \left(2G_{r\theta} + G_{r\phi} + 1\right) \ge \left(M + \gamma\right) \\ L\overline{g} \le F\left(\eta, \overline{g}\right), \text{provided } \left(M + \gamma\right) \le 1 \\ L\overline{\theta} \le F\left(\eta, \overline{\theta}\right), \text{provided } \dfrac{4}{\varepsilon}\left(\psi + 1\right) \ge Q_h \ \textit{and} \\ \left(Q_h + \left(\dfrac{N_b}{\varepsilon} + \dfrac{N_t}{\varepsilon^2}\right) + E_c M\left(\psi^2 + 1\right) + \delta 2^{\omega} e^{\frac{\overline{\theta}}{1+\varepsilon\overline{\theta}}}\right) \ge \dfrac{R_1}{\varepsilon} \\ L\overline{\phi} \le F\left(\eta, \overline{\phi}\right) \end{array}\right\} \tag{3.169}$$

By definition 4, equations (3.164) are the upper solutions.

Thus, there exist a solution of problem (3.153) – (3.156). This completes the proof.

Next, we shall consider the fitted rate constant to be zero, that is, we set $\omega = 0$ and the above equations (3.73) - (3.76) satisfying (3.77) will be considered in four forms:

Case 1: When the reaction is unsteady with Arrhenius chemical reaction.

Case 2: When the reaction is steady with Arrhenius chemical reaction.

Case 3: When the reaction is unsteady with chemical reaction of constant reaction rate.

Case 4: When the reaction is steady with chemical reaction of constant reaction rate.

3.2.4 Case 1: When the reaction is unsteady with Arrhenius chemical reaction: $a \neq 0$

3.2.4.1 Solution of Case 1

Here, as in Mohammed *et al*. (2015) and Olayiwola (2016), equations (3.73) – (3.76) was solved satisfying (3.77) using Iteration Perturbation technique.

Note that

$$
\begin{aligned}
e^{\frac{\theta}{1+\varepsilon\theta}} &= 1+\frac{\theta}{1+\varepsilon\theta}+\frac{1}{2}\left(\frac{\theta}{1+\varepsilon\theta}\right)^2+\dots \\
&\approx 1+\frac{\theta}{1+\varepsilon\theta}+\frac{1}{2}\left(\frac{\theta}{1+\varepsilon\theta}\right)^2 \\
&= 1+\theta(1+\varepsilon\theta)^{-1}+\frac{1}{2}\theta^2\left((1+\varepsilon\theta)^{-1}\right)^2 \\
&= 1+\theta\left(1-\varepsilon\theta+\varepsilon^2\theta^2+\dots\right)+\frac{1}{2}\theta^2\left(1-\varepsilon\theta+\varepsilon^2\theta^2+\dots\right)^2
\end{aligned}
$$

In the limit of $\varepsilon \to 0$,

$$e^{\frac{\theta}{1+\varepsilon\theta}} \to e^{\theta} \approx 1+\theta+\frac{1}{2}\theta^2 \tag{3.170}$$

Ayeni (1978) has shown that $\exp(\theta)$ can be approximated as

$$1+(e-2)\theta+\theta^2 \tag{3.171}$$

We start with the initial approximate solutions:

$$f_0(\eta)=\frac{1}{b}\left(e^{-b\eta}-1\right) \tag{3.172}$$

$$g_0(\eta)=\frac{\psi}{b}\left(e^{-b\eta}-1\right) \tag{3.173}$$

where b is an unknown constant

Substituting (3.171), (3.172) and (3.173) into (3.94) – (3.97) and hence the following approximated equations was obtained:

$$f''' + \frac{1}{b}\left(e^{-b\eta} - 1\right) f'' + \frac{\psi}{b}\left(e^{-b\eta} - 1\right) f'' + \eta\left(f' + g'\right) f'' - \frac{a}{R_e}\left(f' + \frac{\eta}{2} f''\right) -$$
$$2 f'\left(f' + \frac{\eta}{2} f''\right) - 2g'\left(f' + \frac{\eta}{2} f''\right) - \Omega f'^2 - \left(M + \gamma\right) f' + G_{r\theta}\left(1 + \varepsilon\theta\right) + G_{r\phi}\phi = 0 \tag{3.174}$$

$$g''' + \frac{1}{b}\left(e^{-b\eta} - 1\right) g'' + \frac{\psi}{b}\left(e^{-b\eta} - 1\right) g'' + \eta\left(f' + g'\right) g'' - \frac{a}{R_e}\left(g' + \frac{\eta}{2} g''\right) -$$
$$2 f'\left(g' + \frac{\eta}{2} g''\right) - 2g'\left(g' + \frac{\eta}{2} g''\right) - \Omega g'^2 - \left(M + \gamma\right) g' = 0 \tag{3.175}$$

$$R_1\theta'' + \frac{1}{b}\left(e^{-b\eta} - 1\right)\theta' + \frac{\psi}{b}\left(e^{-b\eta} - 1\right)\theta' + \eta\left(f' + g'\right)\theta' - \frac{a}{R_e}\left(\left(\frac{1}{\varepsilon} + \theta\right) + \frac{\eta}{2}\theta'\right) -$$
$$2\left(\left(\frac{1}{\varepsilon} + \theta\right) + \frac{\eta}{2}\theta'\right)\left(f' + g'\right) + E_c M\left(f'^2 + g'^2\right) + N_b\theta'\phi' + N_t\theta'^2 + Q_h\left(1 + \varepsilon\theta\right) +$$
$$\delta\phi\left(1 + \left(e - 2\right)\theta + \theta^2\right) = 0 \tag{3.176}$$

$$\phi'' + \frac{S_c}{b}\left(e^{-b\eta} - 1\right)\phi' + \frac{S_c\psi}{b}\left(e^{-b\eta} - 1\right)\phi' + S_c\eta\left(f' + g'\right)\phi' - \frac{a}{R_e}S_c\left(\phi + \frac{\eta}{2}\phi'\right) -$$
$$2S_c\left(\phi + \frac{\eta}{2}\phi'\right)\left(f' + g'\right) + N_{t1}\theta'' - S_c\sigma\phi\left(1 + \left(e - 2\right)\theta + \theta^2\right) = 0 \tag{3.177}$$

We rewrite equations (3.174) – (3.177) in the form:

$$f''' + bf'' + \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta} - 1\right) - b\right) f'' + \eta\left(f' + g'\right) f'' - \frac{a}{R_e}\left(f' + \frac{\eta}{2} f''\right) -$$
$$2 f'\left(f' + \frac{\eta}{2} f''\right) - 2g'\left(f' + \frac{\eta}{2} f''\right) - \Omega f'^2 - \left(M + \gamma\right) f' + G_{r\theta}\left(1 + \varepsilon\theta\right) + G_{r\phi}\phi = 0 \tag{3.178}$$

$$g''' + bg'' + \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta} - 1\right) - b\right) g'' + \eta\left(f' + g'\right) g'' - \frac{a}{R_e}\left(g' + \frac{\eta}{2} g''\right) -$$
$$2 f'\left(g' + \frac{\eta}{2} g''\right) - 2g'\left(g' + \frac{\eta}{2} g''\right) - \Omega g'^2 - \left(M + \gamma\right) g' = 0 \tag{3.179}$$

$$R_1\theta'' + \theta' + \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\theta' + \eta\left(f'+g'\right)\theta' - \frac{a}{R_e}\left(\left(\frac{1}{\varepsilon}+\theta\right)+\frac{\eta}{2}\theta'\right) - 2\left(\left(\frac{1}{\varepsilon}+\theta\right)+\frac{\eta}{2}\theta'\right)\left(f'+g'\right) + E_c M\left(f'^2+g'^2\right) + N_b\theta'\phi' + N_t\theta'^2 + Q_h\left(1+\varepsilon\theta\right) + \delta\phi\left(1+\left(e-2\right)\theta+\theta^2\right) = 0 \quad (3.180)$$

$$\phi'' + S_c\phi' + S_c\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\phi' + S_c\eta\left(f'+g'\right)\phi' - \frac{a}{R_e}S_c\left(\phi+\frac{\eta}{2}\phi'\right) - 2S_c\left(\phi+\frac{\eta}{2}\phi'\right)\left(f'+g'\right) + N_{t1}\theta'' - S_c\sigma\phi\left(1+\left(e-2\right)\theta+\theta^2\right) = 0 \quad (3.181)$$

Introducing an artificial parameter $\in$ into equations (3.178) – (3.181) gives

$$f''' + bf'' + \in\left(\begin{array}{l}\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-b\right)f'' + \eta\left(f'+g'\right)f'' - \frac{a}{R_e}\left(f'+\frac{\eta}{2}f''\right) - \\ 2f'\left(f'+\frac{\eta}{2}f''\right) - 2g'\left(f'+\frac{\eta}{2}f''\right) - \Omega f'^2 - \left(M+\gamma\right)f' + \\ G_{r\theta}\left(1+\varepsilon\theta\right) + G_{r\phi}\phi\end{array}\right) = 0 \quad (3.182)$$

$$g''' + bg'' + \in\left(\begin{array}{l}\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-b\right)g'' + \eta\left(f'+g'\right)g'' - \frac{a}{R_e}\left(g'+\frac{\eta}{2}g''\right) - \\ 2f'\left(g'+\frac{\eta}{2}g''\right) - 2g'\left(g'+\frac{\eta}{2}g''\right) - \Omega g'^2 - \left(M+\gamma\right)g'\end{array}\right) = 0 \quad (3.183)$$

$$R_1\theta'' + \theta' + \in\left(\begin{array}{l}\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\theta' + \eta\left(f'+g'\right)\theta' - \frac{a}{R_e}\left(\left(\frac{1}{\varepsilon}+\theta\right)+\frac{\eta}{2}\theta'\right) - \\ 2\left(\left(\frac{1}{\varepsilon}+\theta\right)+\frac{\eta}{2}\theta'\right)\left(f'+g'\right) + E_c M\left(f'^2+g'^2\right) + N_b\theta'\phi' + N_t\theta'^2 + \\ Q_h\left(1+\varepsilon\theta\right) + \delta\phi\left(1+\left(e-2\right)\theta+\theta^2\right)\end{array}\right) = 0 \quad (3.184)$$

$$\phi'' + S_c\phi' + \in \begin{pmatrix} S_c\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\phi' + S_c\eta\left(f'+g'\right)\phi' - \frac{a}{R_e}S_c\left(\phi+\frac{\eta}{2}\phi'\right) - \\ 2S_c\left(\phi+\frac{\eta}{2}\phi'\right)\left(f'+g'\right) + N_{t1}\theta'' - S_c\sigma\phi\left(1+\left(e-2\right)\theta+\theta^2\right) \end{pmatrix} = 0 \quad (3.185)$$

We suppose the solution of equations (3.182) – (3.185) can be expressed as:

$$\left.\begin{aligned} f(\eta) &= f_0(\eta) + \in f_1(\eta) + \ldots \\ g(\eta) &= g_0(\eta) + \in g_1(\eta) + \ldots \\ \theta(\eta) &= \theta_0(\eta) + \in \theta_1(\eta) + \ldots \\ \phi(\eta) &= \phi_0(\eta) + \in \phi_1(\eta) + \ldots \end{aligned}\right\} \quad (3.186)$$

Substituting (3.186) into (3.182) – (3.185) and processing and collecting the like powers of $\in$, we have for

$\in^0$:

$$f_0''' + bf_0'' = 0 \quad (3.187)$$

$$g_0''' + bg_0'' = 0 \quad (3.188)$$

$$\theta_0'' + \frac{1}{R_1}\theta_0' = 0 \quad (3.189)$$

$$\phi_0'' + S_c\phi_0' = 0 \quad (3.190)$$

$\in^1$ **:**

$$f_1''' + bf_1'' + \left(\left(\frac{1+\psi}{b} \right) \left(e^{-b\eta} - 1 \right) - b \right) f_0'' + \eta \left(f_0' + g_0' \right) f_0'' - \frac{a}{R_e} \left(f_0' + \frac{\eta}{2} f_0'' \right) -$$
$$2 f_0' \left(f_0' + \frac{\eta}{2} f_0'' \right) - 2 g_0' \left(f_0' + \frac{\eta}{2} f_0'' \right) - \Omega f_0'^2 - \left(M + \gamma \right) f_0' + \qquad (2.191)$$
$$G_{r\theta} \left(1 + \varepsilon \theta_0 \right) + G_{r\phi} \phi_0 = 0$$

$$g_1''' + bg_1'' + \left(\left(\frac{1+\psi}{b} \right) \left(e^{-b\eta} - 1 \right) - b \right) g_0'' + \eta \left(f_0' + g_0' \right) g_0'' - \frac{a}{R_e} \left(g_0' + \frac{\eta}{2} g_0'' \right) -$$
$$2 f_0' \left(g_0' + \frac{\eta}{2} g_0'' \right) - 2 g_0' \left(g_0' + \frac{\eta}{2} g_0'' \right) - \Omega g_0'^2 - \left(M + \gamma \right) g_0' = 0 \qquad (3.192)$$

$$\theta_1'' + \frac{1}{R_1} \theta_1' + \frac{1}{R_1} \left(\begin{array}{l} \left(\left(\left(\frac{1+\psi}{b} \right) \left(e^{-b\eta} - 1 \right) - 1 \right) \theta_0' + \eta \left(f_0' + g_0' \right) \theta_0' - \right. \\ \frac{a}{R_e} \left(\left(\frac{1}{\varepsilon} + \theta_0 \right) + \frac{\eta}{2} \theta_0' \right) - 2 \left(\left(\frac{1}{\varepsilon} + \theta_0 \right) + \frac{\eta}{2} \theta_0' \right) \left(f_0' + g_0' \right) + \\ E_c M \left(f_0'^2 + g_0'^2 \right) + N_b \theta_0' \phi_0' + N_t \theta_0'^2 + Q_h \left(1 + \varepsilon \theta_0 \right) + \\ \delta \phi_0 \left(1 + \left(e - 2 \right) \theta_0 + \theta_0^2 \right) \end{array} \right) = 0 \qquad (3.193)$$

$$\phi_1'' + S_c \phi_1' + S_c \left(\left(\frac{1+\psi}{b} \right) \left(e^{-b\eta} - 1 \right) - 1 \right) \phi_0' + S_c \eta \left(f_0' + g_0' \right) \phi_0' - \frac{a}{R_e} S_c \left(\phi_0 + \frac{\eta}{2} \phi_0' \right) -$$
$$2 S_c \left(\phi_0 + \frac{\eta}{2} \phi_0' \right) \left(f_0' + g_0' \right) + N_{t1} \theta_0'' - S_c \sigma \phi_0 \left(1 + \left(e - 2 \right) \theta_0 + \theta_0^2 \right) = 0 \qquad (3.194)$$

Recall from (3.186) that

$$f(\eta) = f_0(\eta) + \in f_1(\eta) + \cdots$$

Then

$$f(0) = f_0(0) + \in f_1(0) = 0 + \in \cdot 0 + \cdots \qquad (3.195)$$

This implies that

$$f_0(0) = 0 \quad \text{and} \quad f_1(0) = 0 \tag{3.196}$$

Similarly,

$$f'(\eta) = f_0'(\eta) + \in f_1'(\eta) + \cdots \tag{3.197}$$

Then

$$f'(0) = f_0'(0) + \in f_1'(0) = -1 + \in \cdot 0 + \cdots \tag{3.198}$$

This implies that

$$f_0'(0) = -1 \quad \text{and} \quad f_1'(0) = 0 \tag{3.199}$$

Similarly,

$$f'(\infty) = f_0'(\infty) + \in f_1'(\infty) = 0 + \in \cdot 0 + \cdots \tag{3.200}$$

This implies that

$$f_0'(\infty) = 0 \quad \text{and} \quad f_1'(\infty) = 0 \tag{3.201}$$

Likewise, from (3.186)

$$g(\eta) = g_0(\eta) + \in g_1(\eta) + \cdots$$

Then

$$g(0) = g_0(0) + \in g_1(0) = 0 + \in \cdot 0 + \cdots \tag{3.202}$$

This implies that

$$g_0(0) = 0 \quad \text{and} \quad g_1(0) = 0 \tag{3.203}$$

Similarly,

$$g'(0) = g_0'(0) + \in g_1'(0) = -\psi + \in \cdot 0 + \cdots \tag{3.204}$$

This implies that

$$g_0(0) = -\psi \text{ and } g_1'(0) = 0 \tag{3.205}$$

Likewise,

$$g'(\infty) = g_0'(\infty) + \in g_1'(\infty) = 0 + \in \cdot 0 + \cdots \tag{3.206}$$

This implies that

$$g_0'(\infty) = 0 \text{ and } g_1'(\infty) = 0 \tag{3.207}$$

Similarly,

$$\theta(\eta) = \theta_0(\eta) + \in \theta_1(\eta) + \cdots \tag{3.208}$$

Then

$$\theta(0) = \theta_0(0) + \in \theta_1(0) = 1 + \in \cdot 0 + \cdots \tag{3.209}$$

This implies that

$$\theta_0(0) = 1 \text{ and } \theta_1(0) = 0 \tag{3.210}$$

Likewise,

$$\theta(\infty) = \theta_0(\infty) + \in \theta_0(\infty) = 0 + \in \cdot 0 + \cdots \tag{3.211}$$

This implies that

$\theta_0(\infty) = 0$ and $\theta_1(\infty) = 0$ (3.212)

Similarly,

$$\phi(\eta) = \phi_0(\eta) + \in \phi_1(\eta) + \cdots \tag{3.213}$$

Then

$$\phi(0) = \phi_0(0) + \in \phi_1(0) = 1 + \in \cdot 0 + \cdots \tag{3.214}$$

This implies that

$\phi_0(0) = 1$ and $\phi_1(0) = 0$ (3.215)

Likewise,

$$\phi(\infty) = \phi_0(\infty) + \in \phi_0(\infty) = 0 + \in \cdot 0 + \cdots \tag{3.216}$$

This implies that

$\phi_0(\infty) = 0$ and $\phi_1(\infty) = 0$ (3.217)

The order zero and one equations with their respective boundary conditions are given below

$\in^0$:

$$\begin{gathered} f_0''' + b f_0'' = 0 \\ f_0(0) = 0, \quad f_0'(0) = -1, \quad f_0'(\eta \to \infty) \to 0 \end{gathered} \tag{3.218}$$

$$\begin{gathered} g_0''' + b g_0'' = 0 \\ g_0(0) = 0, \quad g_0'(0) = -\psi, \quad g_0'(\eta \to \infty) \to 0 \end{gathered} \tag{3.219}$$

$$\theta_0'' + \frac{1}{R_1}\theta_0' = 0$$
$$\theta_0(0) = 1, \quad \theta_0(\eta \to \infty) \to 0 \tag{3.220}$$

$$\phi_0'' + S_c\phi_0' = 0$$
$$\phi_0(0) = 1, \quad \phi_0(\eta \to \infty) \to 0 \tag{3.221}$$

$\in^1$ **:**

$$f_1''' + bf_1'' + \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta} - 1\right) - b\right) f_0'' + \eta\left(f_0' + g_0'\right) f_0'' - \frac{a}{R_e}\left(f_0' + \frac{\eta}{2} f_0''\right) -$$
$$2 f_0'\left(f_0' + \frac{\eta}{2} f_0''\right) - 2g_0'\left(f_0' + \frac{\eta}{2} f_0''\right) - \Omega f_0'^2 - \left(M + \gamma\right) f_0' + \tag{3.222}$$
$$G_{r\theta}\left(1 + \varepsilon\theta_0\right) + G_{r\phi}\phi_0 = 0$$
$$f_1(0) = 0, \quad f_1'(0) = 0, \quad f_1'(\eta \to \infty) \to 0$$

$$g_1''' + bg_1'' + \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta} - 1\right) - b\right) g_0'' + \eta\left(f_0' + g_0'\right) g_0'' - \frac{a}{R_e}\left(g_0' + \frac{\eta}{2} g_0''\right) -$$
$$2 f_0'\left(g_0' + \frac{\eta}{2} g_0''\right) - 2g_0'\left(g_0' + \frac{\eta}{2} g_0''\right) - \Omega g_0'^2 - \left(M + \gamma\right) g_0' = 0 \tag{3.223}$$
$$g_1(0) = 0, \quad g_1'(0) = 0, \quad g_1'(\eta \to \infty) \to 0$$

$$\theta_1'' + \frac{1}{R_1}\theta_1' + \frac{1}{R_1}\begin{pmatrix} \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\theta_0' + \eta\left(f_0'+g_0'\right)\theta_0' - \\ \frac{a}{R_e}\left(\left(\frac{1}{\varepsilon}+\theta_0\right)+\frac{\eta}{2}\theta_0'\right) - 2\left(\left(\frac{1}{\varepsilon}+\theta_0\right)+\frac{\eta}{2}\theta_0'\right)\left(f_0'+g_0'\right) + \\ E_c M\left(f_0'^2+g_0'^2\right) + N_b\theta_0'\phi_0' + N_t\theta_0'^2 + Q_h\left(1+\varepsilon\theta_0\right) + \\ \delta\phi_0\left(1+\left(e-2\right)\theta_0+\theta_0^{\ 2}\right) \end{pmatrix} = 0$$

$$\theta_1\left(0\right)=0, \quad \theta_1\left(\eta\to\infty\right)\to 0$$

$$\phi_1'' + S_c\phi_1' + S_c\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\phi_0' + S_c\eta\left(f_0'+g_0'\right)\phi_0' - \frac{a}{R_e}S_c\left(\phi_0+\frac{\eta}{2}\phi_0'\right) -$$

(3.224) $$2S_c\left(\phi_0+\frac{\eta}{2}\phi_0'\right)\left(f_0'+g_0'\right) + N_{t1}\theta_0'' - S_c\sigma\phi_0\left(1+\left(e-2\right)\theta_0+\theta_0^{\ 2}\right) = 0$$

$$\phi_1\left(0\right)=0, \quad \phi_1\left(\eta\to\infty\right)\to 0$$

(3.225)

Consider (3.218)

Let

$$p\left(\eta\right) = f_0''(\eta) \tag{3.226}$$

Then (3.218) becomes

$$p'\left(\eta\right) + bp\left(\eta\right) = 0 \tag{3.227}$$

$$\frac{p'\left(\eta\right)}{p\left(\eta\right)} = -b \tag{3.228}$$

Integrating both sides of (3.228) gives

$$\ln P\left(\eta\right) = -b\eta + c$$

$$p(\eta) = c_1 e^{-b\eta} \tag{3.229}$$

But $p(\eta) = f_0''(\eta)$

then

$$f_0''(\eta) = c_1 e^{-b\eta} \tag{3.230}$$

Integrating (3.230) with respect to η gives

$$f_0'(\eta) = -\frac{c_1}{b} e^{-b\eta} + c_2 \tag{3.231}$$

Integrating (3.231) we have

$$f_0(\eta) = \frac{c_1}{b^2} e^{-b\eta} + c_2\eta + c_3 \tag{3.232}$$

Applying the boundary conditions

$$f_0'(\infty) = -0 + c_2 = 0 \tag{3.233}$$

$$c_2 = 0 \tag{3.234}$$

$$f_0'(0) = -\frac{c_1}{b} + 0 = -1 \tag{3.235}$$

$$c_1 = b \tag{3.236}$$

$$f_0(0) = \frac{b}{b^2} + 0 + c_3 = 0 \tag{3.237}$$

$$c_3 = -\frac{1}{b} \tag{3.238}$$

Putting back equations (3.234), (3.236) and (3.238) into (3.232) gives

$$f_0(\eta) = \frac{1}{b}\left(e^{-b\eta} - 1\right) \tag{3.239}$$

Consider (3.219)

Let

$$p_1(\eta) = g_0''(\eta) \tag{3.240}$$

Then (3.219) becomes

$$p_1'(\eta) + bp_1(\eta) = 0 \tag{3.241}$$

$$\frac{p_1'(\eta)}{p_1(\eta)} = -b \tag{3.242}$$

Integrating both sides of (3.242) gives

$$\ln P_1(\eta) = -b\eta + c_4$$

$$p_1(\eta) = c_5 e^{-b\eta} \tag{3.243}$$

But $p_1(\eta) = g_0''(\eta)$

Then (3.243) becomes

$$g_0''(\eta) = c_5 e^{-b\eta} \tag{3.244}$$

Integrating (3.244) gives

$$g_0{}'(\eta) = -\frac{c_5}{b} e^{-b\eta} + c_6 \tag{3.245}$$

Integrating (3.245) we have

$$g_0(\eta) = \frac{c_5}{b^2} e^{-b\eta} + c_6\eta + c_7 \tag{3.246}$$

Applying the boundary conditions

$$g_0'(\infty) = -0 + c_6 = 0 \tag{3.247}$$

$$c_6 = 0 \tag{3.248}$$

$$g_0{}'(0) = -\frac{c_5}{b} + 0 = -\psi \tag{3.249}$$

$$c_5 = b\psi \tag{3.250}$$

$$g_0(0) = \frac{\psi}{b} + c_7 = 0 \tag{3.251}$$

$$c_7 = -\frac{\psi}{b} \tag{3.252}$$

Putting back equations (3.248), (3.250) and (3.252) into (3.246) gives

$$g_0(\eta) = \frac{\psi}{b}\left(e^{-b\eta} - 1\right) \tag{3.253}$$

Consider (3.220)

Let

$$q(\eta) = \theta_0'(\eta) \tag{3.254}$$

Then (3.220) becomes

$$q'(\eta) + \frac{1}{R_1} q(\eta) = 0 \tag{3.255}$$

$$\frac{q'(\eta)}{q(\eta)} = -\frac{1}{R_1} \tag{3.256}$$

Integrating both sides of (3.256) gives

$$\ln q(\eta) = -\frac{1}{R_1}\eta + c_8$$

$$q(\eta) = c_9 e^{-\frac{1}{R_1}\eta} \tag{3.257}$$

$$q(\eta) = c_9 e^{-\alpha\eta} \tag{3.258}$$

Where $\alpha = \frac{1}{R_1}$

But $q(\eta) = \theta_0'(\eta)$

then

$$\theta_0'(\eta) = c_9 e^{-\alpha\eta} \tag{3.259}$$

Integrating (3.259) gives

$$\theta_0(\eta) = -\frac{c_9}{\alpha} e^{-\alpha\eta} + c_{10} \tag{3.260}$$

Applying the boundary conditions

$$\theta_0(\infty) = -0 + c_{10} = 0 \tag{3.261}$$

$$c_{10} = 0 \tag{3.262}$$

$$\theta_0(0) = -\frac{c_9}{\alpha} + 0 = 1 \tag{3.263}$$

$$c_9 = -\alpha \tag{3.264}$$

Substituting back equations (3.262) and (3.264) into (3.260) gives

$$\theta_0(\eta) = e^{-\alpha\eta} \tag{3.265}$$

Similarly considering (3.221)

Let

$$q_1(\eta) = \phi_0'(\eta) \tag{3.266}$$

Then (3.221) becomes

$$q_1'(\eta) + S_c q_1(\eta) = 0 \tag{3.267}$$

$$\frac{q_1'(\eta)}{q_1(\eta)} = -S_c \tag{3.268}$$

Integrating both sides of (3.268) gives

$$\ln q_1(\eta) = -S_c\eta + c_{11}$$

$$q_1(\eta) = c_{12}e^{-S_c\eta} \qquad (3.269)$$

But $q_1(\eta) = \phi_0'(\eta)$

Then (3.269) becomes

$$\phi_0'(\eta) = c_{12}e^{-S_c\eta} \qquad (3.270)$$

Integrating (3.270) gives

$$\phi_0(\eta) = -\frac{c_{12}}{S_c}e^{-S_c\eta} + c_{13} \qquad (3.271)$$

Applying the boundary conditions

$$\phi_0(\infty) = -0 + c_{13} = 0 \qquad (3.272)$$

$$c_{13} = 0 \qquad (3.273)$$

$$\phi_0(0) = -\frac{c_{12}}{S_c} + 0 = 1 \qquad (3.274)$$

$$c_{12} = -S_c \qquad (3.275)$$

Substituting back equations (3.273) and (3.275) into (3.271) gives

$$\phi_0(\eta) = e^{-S_c\eta} \qquad (3.276)$$

$$\phi_0'(\eta) = -S_c e^{-S_c \eta} \tag{3.277}$$

$$f_0'(\eta) = -e^{-b\eta} \tag{3.278}$$

$$f_0''(\eta) = be^{-b\eta} \tag{3.279}$$

$$g_0'(\eta) = -\psi e^{-b\eta} \tag{3.280}$$

$$g_0''(\eta) = b\psi e^{-b\eta} \tag{3.281}$$

$$\theta_0'(\eta) = -\alpha e^{-\alpha\eta} \tag{3.282}$$

$$\theta_0''(\eta) = \alpha^2 e^{-\alpha\eta} \tag{3.283}$$

Consider (3.222)

Let $r = f_1''$ (3.284)

Then (3.222) reduces to

$$r' + br = -\begin{pmatrix} \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta} - 1\right) - b\right) f_0'' + \eta\left(f_0' + g_0'\right) f_0'' - \frac{a}{R_e}\left(f_0' + \frac{\eta}{2} f_0''\right) - \\ 2 f_0'\left(f_0' + \frac{\eta}{2} f_0''\right) - 2g_0'\left(f_0' + \frac{\eta}{2} f_0''\right) - \Omega f_0'^2 - \left(M + \gamma\right) f_0' + \\ G_{r\theta}\left(1 + \in \theta_0\right) + G_{r\phi}\phi_0 \end{pmatrix} \tag{3.285}$$

Substituting (3.265), (3.276), (3.278), (3.279), and (3.253) into (3.285) results

$$r'+br=-\begin{pmatrix} \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-b\right)\left(be^{-b\eta}\right)+\eta\left(-e^{-b\eta}+-\psi e^{-b\eta}\right)be^{-b\eta} \\ -\frac{a}{\mathrm{Re}}\left(-e^{-b\eta}+\frac{\eta}{2}\left(be^{-b\eta}\right)\right)-2\left(-e^{-b\eta}+\frac{\eta}{2}\left(be^{-b\eta}\right)\right)\left(-e^{-b\eta}\right) \\ -2\left(-e^{-b\eta}+\frac{\eta}{2}\left(be^{-b\eta}\right)\right)\left(be^{-b\eta}\right)-\Omega\left(e^{-2b\eta}\right)-\left(M+\gamma\right)\left(-e^{-b\eta}\right) \\ +G_{r\theta}\left(e^{-\alpha\eta}\right)+G_{r\phi}\left(e^{-S_c\eta}\right) \end{pmatrix} \tag{3.286}$$

Simplifying (3.286) gives

$$r'+br=-\begin{pmatrix} \left(1+\psi-2-2\psi-\Omega\right)e^{-2b\eta}+\left(M+\gamma-\left(1+\psi\right)-b^2+\frac{a}{R_e}\right)e^{-b\eta} \\ \left(b\psi+b-\left(1+\psi\right)b\right)\eta e^{-2b\eta}-\frac{ab}{2R_e}\eta e^{-b\eta}+G_{r\theta}\left(1+\in e^{-\alpha\eta}\right)+G_{r\phi}e^{-S_c\eta} \end{pmatrix} \tag{3.287}$$

Further simplification of (3.287) gives

$$r'+br=-\left(Z_1e^{-b\eta}+Z_2e^{-2b\eta}-Z_3\eta e^{-b\eta}+G_{r\theta}\left(1+\in e^{-\alpha\eta}\right)+G_{r\phi}e^{-S_c\eta}\right)$$

$$r'+br=Z_3\eta e^{-b\eta}-Z_1e^{-b\eta}-Z_2e^{-2b\eta}-G_{r\theta}\left(1+\in e^{-\alpha\eta}\right)-G_{r\phi}e^{-S_c\eta} \tag{3.288}$$

where

$$Z_1=\frac{a}{R_e}-b\left(\frac{1+\psi}{b}\right)-b^2+\left(M+\gamma\right)$$

$$Z_2=b\left(\frac{1+\psi}{b}\right)-2\left(1+\psi\right)$$

$$Z_3 = \frac{ab}{2R_e}$$

Consider $\frac{dy}{dt} + p(t)y = q(t)$ (3.289)

By the integrating factor method the solution for (3.289) is given by

$$y(t) = e^{-\int p(t)dt} \int_0^t e^{-\int p(x)dx} \cdot q(x) + ce^{-\int p(t)dt} \tag{3.290}$$

Similarly, the solution for (3.288) is given by

$$r(\eta) = e^{-\int bd\eta} \int_0^\eta e^{\int bdx} \cdot \begin{pmatrix} Z_3 x e^{-bx} - Z_1 e^{-bx} - Z_2 e^{-2bx} \\ -G_{r\theta}\left(1 + \in e^{-\alpha x}\right) - G_{r\phi} e^{-S_c x} \end{pmatrix} dx + c_{13} e^{-\int bd\eta} \tag{3.291}$$

$$r(\eta) = e^{-b\eta} \int_0^\eta \begin{pmatrix} Z_3 x - Z_1 - Z_2 e^{-bx} - G_{r\theta} e^{bx} - \\ G_{r\theta} \in e^{(b-\alpha)x} - G_{r\phi} e^{(b-S_c)x} \end{pmatrix} dx + c_{13} e^{-b\eta} \tag{3.292}$$

Simplifying (3.292) gives

$$r(\eta) = e^{-b\eta} \left(\frac{Z_3 x^2}{2} - Z_1 x + \frac{Z_2}{b} e^{-bx} - \frac{G_{r\theta}}{b} e^{bx} - \frac{G_{r\theta} \in}{b-\alpha} e^{(b-\alpha)x} - \frac{G_{r\theta}}{b-S_c} e^{(b-S_c)x} \right)\Bigg|_0^\eta + c_{13} e^{-b\eta}$$

$$r(\eta) = e^{-b\eta} \begin{pmatrix} \frac{Z_3 \eta^2}{2} - Z_1 \eta + \frac{Z_2}{b}\left(e^{-b\eta} - 1\right) - \frac{G_{r\theta}}{b}\left(e^{b\eta} - 1\right) - \\ \frac{G_{r\theta} \in}{b-\alpha}\left(e^{(b-\alpha)\eta} - 1\right) - \frac{G_{r\theta}}{b-S_c}\left(e^{(b-S_c)\eta} - 1\right) \end{pmatrix} + c_{13} e^{-b\eta}$$

$$r(\eta)=\frac{Z_3\eta^2}{2}e^{-b\eta}-Z_1\eta e^{-b\eta}-\frac{Z_2}{b}\left(e^{-2b\eta}-e^{-b\eta}\right)-\frac{G_{r\theta}}{b}\left(1-e^{-b\eta}\right)-\frac{G_{r\theta}\in}{b-\alpha}\left(e^{-\alpha\eta}-e^{-b\eta}\right)$$
$$-\frac{G_{r\phi}}{b-S_c}\left(e^{-S_c\eta}-e^{-b\eta}\right)+c_{13}e^{-b\eta} \tag{3.293}$$

Rearranging (3.293) in terms of similar power provides

$$r(\eta)=\frac{Z_3\eta^2}{2}e^{-b\eta}-Z_1\eta e^{-b\eta}+\left(\frac{G_{r\theta}\in}{b-\alpha}+\frac{G_{r\theta}}{b}+\frac{Z_2}{b}+\frac{G_{r\phi}}{b-S_c}\right)e^{-b\eta}-\frac{Z_2}{b}e^{-2b\eta}$$
$$-\frac{G_{r\theta}\in}{b-\alpha}e^{-\alpha\eta}-\frac{G_{r\phi}}{b-S_c}e^{-S_c\eta}-\frac{G_{r\theta}}{b}+c_{13}e^{-b\eta}$$

$$r(\eta)=\frac{Z_3\eta^2}{2}e^{-b\eta}-Z_1\eta e^{-b\eta}+Z_4e^{-b\eta}-\frac{Z_2}{b}e^{-2b\eta}-\frac{G_{r\theta}\in}{b-\alpha}e^{-\alpha\eta}-$$
$$\frac{G_{r\phi}}{b-S_c}e^{-S_c\eta}-\frac{G_{r\theta}}{b}+c_{13}e^{-b\eta} \tag{3.294}$$

Where $Z_4=\left(\frac{G_{r\theta}\in}{b-\alpha}+\frac{G_{r\theta}}{b}+\frac{Z_2}{b}+\frac{G_{r\phi}}{b-S_c}\right)$

Recalling that $r=f_1''$

Then equation (3.294) becomes

$$f_1''(\eta)=\frac{Z_3\eta^2}{2}e^{-b\eta}-Z_1\eta e^{-b\eta}+Z_4e^{-b\eta}-\frac{Z_2}{b}e^{-2b\eta}-\frac{G_{r\theta}\in}{b-\alpha}e^{-\alpha\eta}-$$
$$\frac{G_{r\phi}}{b-S_c}e^{-S_c\eta}-\frac{G_{r\theta}}{b}+c_{13}e^{-b\eta} \tag{3.295}$$

Integrating both sides of (3.295) gives

$$f_1'(\eta)=\frac{Z_3}{2}\left(-\frac{\eta^2 e^{-b\eta}}{b}+\frac{2}{b}\left(-\frac{\eta}{b}e^{-b\eta}-\frac{e^{-b\eta}}{b^2}\right)\right)-Z_1\left(-\frac{\eta}{b}e^{-b\eta}-\frac{e^{-b\eta}}{b^2}\right)$$
$$-\frac{Z_4}{b}e^{-b\eta}+\frac{Z_2}{2b^2}e^{-2b\eta}+\frac{G_{r\theta}\in}{\alpha(b-\alpha)}e^{-\alpha\eta}+\frac{G_{r\phi}}{S_c(b-S_c)}e^{-S_c\eta}-\frac{G_{r\theta}}{b}\eta-\frac{c_{13}}{b}e^{-b\eta}+c_{14} \tag{3.296}$$

Simplifying (3.296) in terms of similar power gives

$$f_1'(\eta)=Z_5e^{-b\eta}+Z_6e^{-2b\eta}+Z_7\eta e^{-b\eta}-Z_8\eta^2e^{-b\eta}+Z_9e^{-\alpha\eta}+Z_{10}e^{-S_c\eta}-$$
$$\frac{G_{r\theta}}{b}\eta-\frac{c_{13}}{b}e^{-b\eta}+c_{14} \tag{3.297}$$

So a solution is possible if $G_{r\theta}\to 0$.

Where

$$Z_5=\frac{Z_1}{b^2}-\frac{Z_3}{b^3}-\frac{Z_4}{b}$$

$$Z_6=\frac{Z_2}{2b^2}$$

$$Z_7=\frac{Z_1}{b}-\frac{Z_3}{b^2}$$

$$Z_8=\frac{Z_3}{2b}$$

$$Z_9=\frac{G_{r\theta}\in}{\alpha(b-\alpha)}$$

$$Z_{10}=\frac{G_{r\phi}}{S_c(b-S_c)}$$

Integrating both sides of (3.297) provides

$$f_1(\eta) = -\frac{Z_5}{b}e^{-b\eta} - \frac{Z_6}{2b}e^{-2b\eta} + Z_7\left(-\frac{\eta}{b}e^{-b\eta} - \frac{e^{-b\eta}}{b^2}\right)$$
$$-Z_8\left(\frac{\eta^2 e^{-b\eta}}{b} + \frac{2}{b}\left(-\frac{\eta}{b}e^{-b\eta} - \frac{e^{-b\eta}}{b^2}\right)\right) - \frac{Z_9}{\alpha}e^{-\alpha\eta} - \tag{3.298}$$
$$\frac{Z_{10}}{S_c}e^{-S_c\eta} + \frac{c_{13}}{b^2}e^{-b\eta} + c_{14}\eta + c_{15}$$

Simplifying (3.298) gives

$$f_1(\eta) = \left(\frac{2Z_8}{b^3} - \frac{Z_5}{b} - \frac{Z_7}{b^2}\right)e^{-b\eta} - \frac{Z_6}{2b}e^{-2b\eta} + \left(\frac{2Z_8}{b^2} - \frac{Z_7}{b}\right)\eta e^{-b\eta} - \frac{Z_8}{b}\eta^2 e^{-b\eta} - \frac{Z_9}{\alpha}e^{-\alpha\eta}$$
$$-\frac{Z_{10}}{S_c}e^{-S_c\eta} + \frac{c_{13}}{b^2}e^{-b\eta} + c_{14}\eta + c_{15}$$

$$f_1(\eta) = Z_{11}e^{-b\eta} - Z_{12}e^{-2b\eta} + Z_{13}\eta e^{-b\eta} - Z_{14}\eta^2 e^{-b\eta} - Z_{15}e^{-\alpha\eta} - Z_{16}e^{-S_c\eta} -$$
$$+\frac{c_{13}}{b^2}e^{-b\eta} + c_{14}\eta + c_{15} \tag{3.299}$$

Where

$$\mathrm{Z}_{11} = \frac{2Z_8}{b^3} - \frac{Z_5}{b} - \frac{Z_7}{b^2}$$

$$\mathrm{Z}_{12} = \frac{Z_6}{2b}$$

$$\mathrm{Z}_{13} = \frac{2Z_8}{b^2} - \frac{Z_7}{b}$$

$$\mathrm{Z}_{14} = \frac{Z_8}{b}$$

$$\mathrm{Z}_{15} = \frac{Z_9}{\alpha}$$

$$Z_{16} = \frac{Z_{10}}{S_c}$$

Introducing the boundary conditions gives

$$f_1'(\infty) = 0 + 0 + c_{14} = 0$$

$$c_{14} = 0 \tag{3.300}$$

$$f_1'(0) = Z_5 + Z_6 + Z_9 + Z_{10} - \frac{c_{13}}{b} = -1$$

$$c_{13} = b\left(1 + Z_5 + Z_6 + Z_9 + Z_{10}\right)$$

$$c_{13} = bZ_{17} \tag{3.301}$$

Where

$$Z_{17} = 1 + Z_5 + Z_6 + Z_9 + Z_{10}$$

$$f_1(0) = Z_{11} - Z_{12} - Z_{15} - Z_{16} + \frac{Z_{17}}{b} + c_{15} = 0$$

$$c_{15} = Z_{18} \tag{3.302}$$

Where

$$Z_{18} = Z_{12} + Z_{15} - Z_{11} + Z_{16} - \frac{Z_{17}}{b}$$

$$f_1(\eta) = Z_{11}e^{-b\eta} - Z_{12}e^{-2b\eta} + Z_{13}\eta e^{-b\eta} - Z_{14}\eta^2 e^{-b\eta} - Z_{15}e^{-\alpha\eta}$$
$$-Z_{16}e^{-S_c\eta} + \frac{Z_{17}}{b}e^{-b\eta} + Z_{18} \quad (3.303)$$

Similarly, consider (3.223)

Let $r_1 = g_1''$ (3.304)

Then (3.223) reduces to

$$r_1' + br_1 = -\left(\begin{array}{l} \left(\left(\frac{1+\psi}{b} \right)\left(e^{-b\eta} - 1\right) - b \right) g_0'' + \eta\left(f_0' + g_0' \right) g_0'' - \frac{a}{\text{Re}}\left(g_0' + \frac{\eta}{2} g_0'' \right) \\ -2\left(g_0' + \frac{\eta}{2} g_0'' \right) f_0' - 2\left(g_0' + \frac{\eta}{2} g_0'' \right) g_0' - \Omega g_0'^2 - \left(M + \gamma \right) g_0' \end{array} \right) \quad (3.305)$$

Substituting (3.278), (3.280) and (3.281), into (3.305) results

$$r_1' + br_1 = -\left(\begin{array}{l} \left(\left(\frac{1+\psi}{b} \right)\left(e^{-b\eta} - 1\right) - b \right)\left(b\psi e^{-b\eta} \right) + \eta\left(-e^{-b\eta} - \psi e^{-b\eta} \right) b\psi e^{-b\eta} \\ -\frac{a}{\text{Re}}\left(-\psi e^{-b\eta} + \frac{\eta}{2}\left(b\psi e^{-b\eta} \right) \right) - 2\left(-\psi e^{-b\eta} + \frac{\eta}{2}\left(b\psi e^{-b\eta} \right) \right)\left(-e^{-b\eta} \right) \\ -2\left(-\psi e^{-b\eta} + \frac{\eta}{2}\left(b\psi e^{-b\eta} \right) \right)\left(-\psi e^{-b\eta} \right) - \Omega\left(e^{-2b\eta} \right) - \left(M + \gamma \right)\left(-\psi e^{-b\eta} \right) \end{array} \right) \quad (3.306)$$

Simplifying (3.306) gives

$$r_1' + br_1 = -\left(\begin{array}{l} \psi\left(1+\psi\right)\left(e^{-2b\eta} - e^{-b\eta} \right) - b^2\psi e^{-b\eta} - \left(b\psi + b\psi^2 \right)\eta e^{-2b\eta} + \frac{a\psi}{R_e} e^{-b\eta} \\ -\frac{ab\psi}{2R_e}\eta e^{-b\eta} - 2\psi e^{-2b\eta} + b\psi\eta e^{-2b\eta} - 2\psi^2 e^{-2b\eta} + b\psi^2\eta e^{-2b\eta} - \Omega\psi^2 e^{-2b\eta} \\ +\left(M + \gamma \right)\psi e^{-b\eta} \end{array} \right) \quad (3.307)$$

Further simplification of (3.307) gives

$$r_1'+br_1=-\begin{pmatrix}\left(\dfrac{a\psi}{R_e}-\left(\psi+\psi^2\right)-b^2\psi+\left(M+\gamma\right)\psi\right)e^{-b\eta}+\left(\left(\psi+\psi^2\right)-2\left(\psi+\psi^2\right)-\Omega\psi^2\right)e^{-2b\eta}\\ \left(b\left(\psi+\psi^2\right)-b\left(\psi+\psi^2\right)\right)\eta e^{-2b\eta}-\dfrac{ab\psi}{2R_e}\eta e^{-b\eta}\end{pmatrix}$$

$$r_1'+br_1=-\left(Z_{19}e^{-b\eta}-Z_{20}e^{-2b\eta}-Z_{21}\eta e^{-b\eta}\right)$$

$$r_1'+br_1=Z_{20}e^{-2b\eta}+Z_{21}\eta e^{-b\eta}-Z_{19}e^{-b\eta} \tag{3.308}$$

where

$$Z_{19}=\frac{a\psi}{R_e}-\left(\psi+\psi^2\right)-b^2\psi+\left(M+\gamma\right)\psi$$

$$Z_{20}=\left(\psi+\psi^2\right)-\Omega\psi^2$$

$$Z_{21}=\frac{ab\psi}{2R_e}$$

Similarly, the solution for (3.308) is given by

$$r_1\left(\eta\right)=e^{-\int bd\eta}\int_0^{\eta}e^{\int bdx}\cdot\left(Z_{20}e^{-2bx}+Z_{21}\eta e^{-bx}-Z_{19}e^{-bx}\right)dx+c_{16}e^{-\int bd\eta} \tag{3.309}$$

$$r_1\left(\eta\right)=e^{-b\eta}\int_0^{\eta}e^{bx}\cdot\left(Z_{20}e^{-2bx}+Z_{21}xe^{-bx}-Z_{19}e^{-bx}\right)dx+c_{16}e^{-b\eta} \tag{3.310}$$

Simplifying (3.310) gives

$$r_1(\eta) = e^{-b\eta}\left(-\frac{Z_{20}}{b}e^{-bx} + Z_{21}\frac{x^2}{2} - Z_{19}x\right)\Bigg|_0^{\eta} + c_{16}e^{-b\eta}$$

$$r_1(\eta) = e^{-b\eta}\left(\frac{Z_{21}\eta^2}{2} - Z_{19}\eta - \frac{Z_{20}}{b}\left(e^{-b\eta} - 1\right)\right) + c_{16}e^{-b\eta} \tag{3.311}$$

$$r_1(\eta) = \frac{Z_{21}}{2}\eta^2 e^{-b\eta} - Z_{19}\eta e^{-b\eta} - \frac{Z_{20}}{b}\left(e^{-2b\eta} - e^{-b\eta}\right) + c_{16}e^{-b\eta}$$

$$r_1(\eta) = \frac{Z_{21}\eta^2}{2}e^{-b\eta} - Z_{19}\eta e^{-b\eta} - \frac{Z_{20}}{b}e^{-2b\eta} + \frac{Z_{20}}{b}e^{-b\eta} + c_{16}e^{-b\eta} \tag{3.312}$$

Recalling that $r_1 = g_1''$

Then equation (3.312) becomes

$$g_1''(\eta) = \frac{Z_{21}\eta^2}{2}e^{-b\eta} - Z_{19}\eta e^{-b\eta} - \frac{Z_{20}}{b}e^{-2b\eta} + \frac{Z_{20}}{b}e^{-b\eta} + c_{16}e^{-b\eta} \tag{3.313}$$

Integrating both sides of (3.313) gives

$$\begin{aligned} g_1'(\eta) = {} & \frac{Z_{21}}{2}\left(-\frac{\eta^2 e^{-b\eta}}{b} + \frac{2}{b}\left(-\frac{\eta}{b}e^{-b\eta} - \frac{e^{-b\eta}}{b^2}\right)\right) - Z_{19}\left(-\frac{\eta}{b}e^{-b\eta} - \frac{e^{-b\eta}}{b^2}\right) \\ & + \frac{Z_{20}}{2b^2}e^{-2b\eta} - \frac{Z_{20}}{b^2}e^{-b\eta} - \frac{c_{16}}{b}e^{-b\eta} + c_{17} \end{aligned} \tag{3.314}$$

Simplifying (3.314) in terms of similar power provides

$$g_1'(\eta) = Z_{22}e^{-b\eta} + Z_{23}\eta e^{-b\eta} - Z_{24}\eta^2 e^{-b\eta} + Z_{25}e^{-2b\eta} - \frac{c_{16}}{b}e^{-b\eta} + c_{17} \tag{3.315}$$

Where

$$Z_{22} = \frac{Z_{19}}{b^2} - \frac{2Z_{21}}{2b^3} - \frac{Z_{20}}{b^2}$$

$$Z_{23} = \frac{Z_{19}}{b} - \frac{2Z_{21}}{2b^2}$$

$$Z_{24} = \frac{Z_{21}}{2b}$$

$$Z_{25} = \frac{Z_{20}}{2b^2}$$

Integrating both sides of (3.315) gives

$$g_1(\eta) = -\frac{Z_{22}}{b}e^{-b\eta} + Z_{23}\left(-\frac{\eta}{b}e^{-b\eta} - \frac{e^{-b\eta}}{b^2}\right) - Z_{24}\left(-\frac{\eta^2 e^{-b\eta}}{b} + \frac{2}{b}\left(-\frac{\eta}{b}e^{-b\eta} - \frac{e^{-b\eta}}{b^2}\right)\right) - \frac{Z_{25}}{2b}e^{-2b\eta} + \frac{c_{16}}{b^2}e^{-b\eta} + c_{17}\eta + c_{18} \tag{3.316}$$

Simplifying (3.316) in terms of similar power gives

$$g_1(\eta) = Z_{26}e^{-b\eta} + Z_{27}\eta e^{-b\eta} + Z_{28}\eta^2 e^{-b\eta} - Z_{29}e^{-2b\eta} + \frac{c_{16}}{b^2}e^{-b\eta} + c_{18} \tag{3.317}$$

Where

$$Z_{26} = \frac{2Z_{24}}{b^3} - \frac{Z_{22}}{b} - \frac{Z_{23}}{b^2}$$

$$Z_{27} = \frac{2Z_{24}}{b^2} - \frac{Z_{23}}{b}$$

$$Z_{28} = \frac{Z_{24}}{b}$$

$$Z_{29} = \frac{Z_{25}}{2b}$$

Inserting the boundary conditions gives

$$g_1'(\infty) = 0 + 0 + c_{17} = 0$$

$$c_{17} = 0 \tag{3.318}$$

$$g_1'(0) = Z_{22} + Z_{25} - \frac{c_{16}}{b} = 0$$

$$c_{16} = b\left(Z_{22} + Z_{25}\right)$$

$$c_{16} = bk_1 \tag{3.319}$$

Where

$$k_1 = Z_{22} + Z_{25}$$

$$g_1(0) = Z_{26} - Z_{29} + \frac{k_1}{b} + c_{18} = 0$$

$$c_{18} = k_2 \tag{3.320}$$

Where

$$k_2 = Z_{29} - Z_{26} - \frac{k_1}{b}$$

Substituting back (3.318) (3.319) and (3.320) into (3.317) gives

$$g_1(\eta) = Z_{26}e^{-b\eta} + Z_{27}\eta e^{-b\eta} + Z_{28}\eta^2 e^{-b\eta} - Z_{29}e^{-2b\eta} + \frac{k_1}{b}e^{-b\eta} + k_2 \tag{3.321}$$

Similarly Consider (3.224)

Let

$$r_2 = \theta_1' \tag{3.322}$$

Then (3.224) reduces to

$$r_2' + \alpha r_2 = -\frac{1}{R_1}\begin{pmatrix} \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\theta_0' + \eta\left(f_0' + g_0'\right)\theta_0' - \\ \frac{a}{R_e}\left(\left(\frac{1}{\in}+\theta_0\right)+\frac{\eta}{2}\theta_0'\right) - 2\left(\left(\frac{1}{\in}+\theta_0\right)+\frac{\eta}{2}\theta_0'\right)\left(f_0' + g_0'\right) + \\ E_cM\left(f_0'^2 + g_0'^2\right) + N_b\theta_0'\phi_0' + N_t\theta_0'^2 + Q_h\left(1+\in\theta_0\right) + \\ \delta\phi_0\left(1+\left(e-2\right)\theta_0 + \theta_0^2\right) \end{pmatrix} \tag{3.323}$$

Substituting (3.265), (3.276), (3.278), (3.280),(3.277) and (3.282) into (3.323) provides

$$r_2' + \alpha r_2 = -\frac{1}{R_1}\begin{pmatrix} \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\left(-\alpha e^{-\alpha\eta}\right) + \eta\left(-e^{-b\eta} - \psi e^{-b\eta}\right)\left(-\alpha e^{-\alpha\eta}\right) \\ -\frac{a}{R_e}\left(\left(\frac{1}{\in}+e^{-\alpha\eta}\right)+\frac{\eta}{2}\left(-\alpha e^{-\alpha\eta}\right)\right) \\ -2\left(\left(\frac{1}{\in}+e^{-\alpha\eta}\right)+\frac{\eta}{2}\left(-\alpha e^{-\alpha\eta}\right)\right)\left(-e^{-b\eta} - \psi e^{-b\eta}\right) \\ +E_cM\left(\left(-e^{-b\eta}\right)^2 + \left(-\psi e^{-b\eta}\right)^2\right) \\ +Q_h\left(1+\in e^{-\alpha\eta}\right) + N_b\left(-S_c e^{-S_c\eta}\right)\left(-\alpha e^{-\alpha\eta}\right) \\ +N_t\left(-\alpha e^{-\alpha\eta}\right)^2 + \delta\left(e^{-S_c\eta}\right)\left(1+\left(e-2\right)e^{-\alpha\eta} + e^{-2\alpha\eta}\right) \end{pmatrix} \tag{3.324}$$

Simplifying (3.324) gives

$$r_2' + \alpha r_2 = -\frac{1}{R_1}\left(\begin{array}{l} -\frac{1+\psi}{b}\alpha e^{-(\alpha+b)\eta} + \frac{1+\psi}{b}\alpha e^{-\alpha\eta} + \alpha e^{-\alpha\eta} + (1+\psi)\alpha\eta e^{-(\alpha+b)\eta} - \frac{a}{R_e}\frac{1}{\in} - \\ \frac{a}{R_e}e^{-\alpha\eta} + \frac{a}{2R_e}\alpha\eta e^{-\alpha\eta} - 2\left(\begin{array}{l} -(1+\psi)\frac{1}{\in}e^{-b\eta} - (1+\psi)e^{-(\alpha+b)\eta} + \\ (1+\psi)\alpha\frac{\eta}{2}e^{-(\alpha+b)\eta} \end{array} \right) \\ +E_c M\left(1+\psi^2\right)e^{-2b\eta} + Q_h + Q_h \in e^{-\alpha\eta} + N_b S_c \alpha e^{-\left(\alpha+S_c\right)\eta} \\ +N_t \alpha^2 e^{-2\alpha\eta} + \delta e^{-S_c\eta} + \delta(e-2)e^{-\left(\alpha+S_c\right)\eta} + \delta e^{-\left(2\alpha+S_c\right)\eta} \end{array} \right) \quad (3.325)$$

Further simplification of (3.325) gives

$$r_2' + \alpha r_2 = \left(\begin{array}{l} Z_{30}e^{-\alpha\eta} + Z_{31}e^{-(\alpha+b)\eta} + Z_{32}e^{-2b\eta} + Z_{33}\eta e^{-\alpha\eta} + Z_{34}e^{-\left(\alpha+S_c\right)\eta} + Z_{35} + \\ Z_{36}e^{-2\alpha\eta} + Z_{37}e^{-b\eta} + \delta e^{-S_c\eta} + \delta e^{-\left(2\alpha+S_c\right)\eta} \end{array} \right) \quad (3.326)$$

where

$$Z_{30} = -\frac{1}{R_1}\left(Q_h \in + \frac{1+\psi}{b}\alpha + \alpha - \frac{a}{R_e} \right)$$

$$Z_{31} = -\frac{1}{R_1}\left(2(1+\psi) - \frac{1+\psi}{b}\alpha \right)$$

$$Z_{32} = -\frac{1}{R_1}E_c M\left(1+\psi^2\right)$$

$$Z_{33} = -\frac{1}{R_1}\frac{a}{2R_e}\alpha$$

$$Z_{34} = -\frac{1}{R_1}\left(\delta(e-2) + N_b S_c \alpha\right)$$

$$Z_{35} = -\frac{1}{R_1}\left(Q_h - \frac{a}{R_e}\frac{1}{\in} \right)$$

$$Z_{36} = -\frac{1}{R_1} N_t \alpha^2$$

$$Z_{37} = -2\frac{1}{R_1}\left(1+\psi\right)\frac{1}{\in}$$

Similarly, the solution for (3.326) is given by

$$r_2\left(\eta\right) = e^{-\int \alpha d\eta} \int_0^\eta e^{\int \alpha dx} \cdot \begin{pmatrix} Z_{30}e^{-\alpha x} + Z_{31}e^{-\left(\alpha+b\right)x} + Z_{32}e^{-2bx} + Z_{33}xe^{-\alpha x} + \\ Z_{34}e^{-\left(\alpha+S_c\right)x} + Z_{35} + Z_{36}e^{-2\alpha x} + Z_{37}e^{-bx} + \delta e^{-S_c x} \\ +\delta e^{-\left(2\alpha+S_c\right)x} \end{pmatrix} dx + c_{19}e^{-\int \alpha d\eta} \tag{3.327}$$

Simplifying (3.327) gives

$$r_2\left(\eta\right) = e^{-\alpha\eta} \begin{pmatrix} Z_{30}x - \frac{Z_{31}}{b}e^{-bx} + \frac{Z_{32}}{\left(\alpha-2b\right)}e^{\left(\alpha-2b\right)x} + \frac{Z_{33}x^2}{2} - \frac{Z_{34}}{S_c}e^{-S_c x} \\ +\frac{Z_{35}}{\alpha}e^{\alpha x} - \frac{Z_{36}}{\alpha}e^{-\alpha x} + \frac{Z_{37}}{\left(\alpha-b\right)}e^{\left(\alpha-b\right)x} + \frac{\delta}{\left(\alpha-S_c\right)}e^{\left(\alpha-S_c\right)x} \\ -\frac{\delta}{\left(\alpha+S_c\right)}e^{-\left(\alpha+S_c\right)x} \end{pmatrix}_0^\eta + c_{19}e^{-\alpha\eta} \tag{3.328}$$

i.e.

$$r_2\left(\eta\right) = e^{-\alpha\eta} \begin{pmatrix} Z_{30}\eta - \frac{Z_{31}}{b}\left(e^{-b\eta}-1\right) + \frac{Z_{32}}{\left(\alpha-2b\right)}\left(e^{\left(\alpha-2b\right)\eta}-1\right) + \frac{Z_{33}\eta^2}{2} - \\ \frac{Z_{34}}{S_c}\left(e^{-S_c\eta}-1\right) + \frac{Z_{35}}{\alpha}\left(e^{\alpha\eta}-1\right) - \frac{Z_{36}}{\alpha}\left(e^{-\alpha\eta}-1\right) + \\ \frac{Z_{37}}{\left(\alpha-b\right)}\left(e^{\left(\alpha-b\right)\eta}-1\right) + \frac{\delta}{\left(\alpha-S_c\right)}\left(e^{\left(\alpha-S_c\right)\eta}-1\right) - \frac{\delta}{\left(\alpha+S_c\right)}\left(e^{-\left(\alpha+S_c\right)\eta}-1\right) \end{pmatrix} + c_{19}e^{-\alpha\eta}$$

i.e.

$$r_2(\eta)=\begin{pmatrix} Z_{30}\eta e^{-\alpha\eta}-\dfrac{Z_{31}}{b}\left(e^{-(\alpha+b)\eta}-e^{-\alpha\eta}\right)+\dfrac{Z_{32}}{(\alpha-2b)}\left(e^{-2b\eta}-e^{-\alpha\eta}\right)+ \\ \dfrac{Z_{33}\eta^2 e^{-\alpha\eta}}{2}-\dfrac{Z_{34}}{S_c}\left(e^{-(\alpha+S_c)\eta}-e^{-\alpha\eta}\right)+\dfrac{Z_{35}}{\alpha}\left(1-e^{-\alpha\eta}\right)- \\ \dfrac{Z_{36}}{\alpha}\left(e^{-2\alpha\eta}-e^{-\alpha\eta}\right)+\dfrac{Z_{37}}{(\alpha-b)}\left(e^{-b\eta}-e^{-\alpha\eta}\right)+ \\ \dfrac{\delta}{(\alpha-S_c)}\left(e^{-S_c\eta}-e^{-\alpha\eta}\right)-\dfrac{\delta}{(\alpha+S_c)}\left(e^{-(2\alpha+S_c)\eta}-e^{-\alpha\eta}\right) \end{pmatrix}+c_{19}e^{-\alpha\eta} \tag{3.329}$$

Simplifying (3.329) in terms of similar power gives

$$\begin{aligned} r_2(\eta)=&\,Z_{30}\eta e^{-\alpha\eta}-\frac{Z_{31}}{b}e^{-(\alpha+b)\eta}+Z_0e^{-\alpha\eta}+\frac{Z_{32}}{(\alpha-2b)}e^{-2b\eta}+\frac{Z_{33}}{2}\eta^2e^{-\alpha\eta}- \\ &\frac{Z_{34}}{S_c}e^{-(\alpha+S_c)\eta}+\frac{Z_{35}}{\alpha}-\frac{Z_{36}}{\alpha}e^{-2\alpha\eta}+\frac{Z_{37}}{(\alpha-b)}e^{-b\eta}+\frac{\delta}{(\alpha-S_c)}e^{-S_c\eta} \\ &-\frac{\delta}{(\alpha+S_c)}e^{-(2\alpha+S_c)\eta}+c_{19}e^{-\alpha\eta} \end{aligned} \tag{3.330}$$

Where

$$Z_0=\left(\frac{Z_{31}}{b}-\frac{Z_{32}}{(\alpha-2b)}+\frac{Z_{34}}{S_c}-\frac{Z_{35}}{\alpha}+\frac{Z_{36}}{\alpha}-\frac{Z_{37}}{(\alpha-b)}-\frac{\delta}{(\alpha-S_c)}+\frac{\delta}{(\alpha+S_c)}\right)$$

Recalling that $r_2=\theta_1'$

Then equation (3.330) becomes

$$\theta_1'(\eta) = Z_{30}\eta e^{-\alpha\eta} - \frac{Z_{31}}{b} e^{-(\alpha+b)\eta} + Z_0 e^{-\alpha\eta} + \frac{Z_{32}}{(\alpha-2b)} e^{-2b\eta} + \frac{Z_{33}}{2}\eta^2 e^{-\alpha\eta} -$$
$$\frac{Z_{34}}{S_c} e^{-(\alpha+S_c)\eta} + \frac{Z_{35}}{\alpha} - \frac{Z_{36}}{\alpha} e^{-2\alpha\eta} + \frac{Z_{37}}{(\alpha-b)} e^{-b\eta} + \frac{\delta}{(\alpha-S_c)} e^{-S_c\eta} \qquad (3.331)$$
$$-\frac{\delta}{(\alpha+S_c)} e^{-(2\alpha+S_c)\eta} + c_{19} e^{-\alpha\eta}$$

Integrating both sides of (3.331) gives

$$\theta_1(\eta) = Z_{30}\left(-\frac{1}{b}\eta e^{-\alpha\eta} - \frac{e^{-\alpha\eta}}{b^2}\right) - \frac{Z_0}{\alpha} e^{-\alpha\eta} + \frac{Z_{31}}{b(\alpha+b)} e^{-(\alpha+b)\eta} - \frac{Z_{32}}{2b(\alpha-2b)} e^{-2b\eta}$$
$$-\frac{Z_{33}}{2}\left(\frac{\eta^2 e^{-\alpha\eta}}{\alpha} + \frac{2}{\alpha}\left(\frac{\eta}{\alpha} e^{-\alpha\eta} + \frac{e^{-\alpha\eta}}{\alpha^2}\right)\right) + \frac{Z_{34}}{S_c(\alpha+S_c)} e^{-(\alpha+S_c)\eta} + \frac{Z_{35}}{\alpha}\eta + \frac{Z_{36}}{2\alpha^2} e^{-2\alpha\eta} \qquad (3.332)$$
$$-\frac{Z_{37}}{b(\alpha-b)} e^{-b\eta} - \frac{\delta}{(\alpha-S_c)} e^{-S_c\eta} + \frac{\delta}{(\alpha+S_c)(2\alpha+S_c)} e^{-(2\alpha+S_c)\eta} - \frac{c_{19}}{\alpha} e^{-\alpha\eta} + c_{20}$$

Simplifying (3.332) in terms of similar power gives

$$\theta_1(\eta) = -Z_{45}\eta e^{-\alpha\eta} - Z_{38} e^{-\alpha\eta} + Z_{39} e^{-(\alpha+b)\eta} - Z_{42} e^{-2b\eta}$$
$$-Z_{46}\eta^2 e^{-\alpha\eta} + Z_{41} e^{-(\alpha+S_c)\eta} + Z_{40}\eta + Z_{43} e^{-2\alpha\eta} - Z_{44} e^{-b\eta} \qquad (3.333)$$
$$+Z_{01} e^{-(2\alpha+S_c)\eta} - \frac{c_{19}}{\alpha} e^{-\alpha\eta} + c_{20}$$

where

$$Z_{38} = \left(\frac{Z_{30}}{b^2} + \frac{Z_{33}}{\alpha^3} - \frac{Z_0}{\alpha}\right)$$

$$Z_{39} = \frac{Z_{31}}{b(\alpha+b)}$$

$$Z_{40} = \frac{Z_{35}}{\alpha}$$

$$Z_{41} = \frac{Z_{34}}{S_c\left(\alpha + S_c\right)}$$

$$Z_{42} = \frac{Z_{32}}{2b\left(\alpha - 2b\right)}$$

$$Z_{43} = \frac{Z_{36}}{2\alpha^2}$$

$$Z_{44} = \frac{Z_{37}}{b\left(\alpha - b\right)}$$

$$Z_{45} = \left(\frac{Z_{30}}{b} + \frac{Z_{33}}{\alpha^2}\right)$$

$$Z_{46} = \frac{Z_{33}}{2\alpha}$$

$$Z_{01} = \frac{\delta}{\left(\alpha + S_c\right)\left(2\alpha + S_c\right)}$$

So a solution is possible if $Z_{40} \to 0$.

Inserting the boundary conditions gives

$$\theta_1\left(\infty\right) = 0 + 0 + c_{19} = 0 \qquad (3.334)$$

$$c_{20} = 0 \qquad (3.335)$$

$$\theta_1\left(0\right) = \alpha\left(Z_{01} - Z_{38} + Z_{39} + Z_{41} - Z_{42} + Z_{43} - Z_{44}\right) = c_{19} \qquad (3.336)$$

$$c_{19} = \alpha k_3 \tag{3.337}$$

where

$$k_3 = Z_{01} - Z_{38} + Z_{39} + Z_{41} + Z_{01} - Z_{42} + Z_{43} - Z_{44}$$

$$\begin{aligned}\theta_1(\eta) &= -Z_{45}\eta e^{-\alpha\eta} - Z_{38}e^{-\alpha\eta} + Z_{39}e^{-(\alpha+b)\eta} - Z_{42}e^{-2b\eta} \\ &-Z_{46}\eta^2 e^{-\alpha\eta} + Z_{41}e^{-(\alpha+S_c)\eta} + Z_{43}e^{-2\alpha\eta} - Z_{44}e^{-b\eta} + Z_{01}e^{-(2\alpha+S_c)\eta} - k_3 e^{-\alpha\eta}\end{aligned} \tag{3.338}$$

Similarly Consider (3.225)

Let $r_3 = \phi_1'$ (3.340)

Then (3.225) reduces to

$$r_3' + S_c r_3 = -\left(\begin{array}{l}\left(S_c\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\phi_0' + S_c\eta\left(f_0' + g_0'\right)\phi_0' - \frac{a}{\mathrm{R_e}}S_c\left(\phi_0 + \frac{\eta}{2}\phi_0'\right) \\ -2S_c\left(\phi_0 + \frac{\eta}{2}\phi_0'\right)\left(f_0' + g_0'\right) + N_{t1}\theta_0'' - S_c\sigma\phi_0\left(1+\left(e-2\right)\theta_0 + \theta_0^{\,2}\right)\end{array}\right) \tag{3.341}$$

Substituting (3.265), (3.276), (3.277), (3.278),(3.280) and (3.283) into (3.341) provides

$$r_3' + S_c r_3 = -\left(\begin{array}{l}\left(S_c\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\left(-S_c e^{-S_c\eta}\right) \\ +S_c\eta\left(-e^{-b\eta} - \psi e^{-b\eta}\right)\left(-S_c e^{-S_c\eta}\right) \\ -\frac{a}{\mathrm{R_e}}S_c\left(e^{-S_c\eta} + \frac{\eta}{2}\left(-S_c e^{-S_c\eta}\right)\right) \\ -2S_c\left(e^{-S_c\eta} + \frac{\eta}{2}\left(-S_c e^{-S_c\eta}\right)\right)\left(-e^{-b\eta} - \psi e^{-b\eta}\right) \\ +N_{t1}\left(\alpha^2 e^{-\alpha\eta}\right) - S_c\sigma\left(e^{-S_c\eta}\right)\left(1+\left(e-2\right)e^{-\alpha\eta} + e^{-2\alpha\eta}\right)\end{array}\right) \tag{3.342}$$

Simplifying (3.342) gives

$$r_3' + S_c r_3 = -\begin{pmatrix} \left(-S_c^{\,2}\left(\frac{1+\psi}{b}\right)\left(e^{-(S_c+b)\eta} - e^{-S_c\eta}\right) + S_c e^{-S_c\eta}\right) + S_c^{\,2}(1+\psi)\eta e^{-(S_c+b)\eta} \\ -\frac{a}{\mathrm{R_e}} S_c e^{-S_c\eta} + \frac{aS_c^{\,2}}{2\mathrm{R_e}}\eta e^{-S_c\eta} + \left(2S_c e^{-(S_c+b)\eta} - S_c^{\,2}\eta e^{-(S_c+b)\eta}\right) \\ +\left(2\psi S_c e^{-(S_c+b)\eta} - S_c^{\,2}\psi\eta e^{-(S_c+b)\eta}\right) \\ +N_{t1}\alpha^2 e^{-\alpha\eta} - S_c\sigma\left(e^{-S_c\eta} + (e-2)e^{-(S_c+\alpha)\eta} + e^{-(2\alpha+S_c)\eta}\right) \end{pmatrix}$$

$$r_3' + S_c r_3 = -\begin{pmatrix} \left(-S_c^{\,2}\left(\frac{1+\psi}{b}\right)\left(e^{-(S_c+b)\eta} - e^{-S_c\eta}\right) + S_c e^{-S_c\eta}\right) \\ +S_c^{\,2}(1+\psi)\eta e^{-(S_c+b)\eta} \\ -\frac{a}{\mathrm{R_e}} S_c e^{-S_c\eta} + \frac{aS_c^{\,2}}{2\mathrm{R_e}}\eta e^{-S_c\eta} + \begin{pmatrix} 2S_c e^{-(S_c+b)\eta} \\ -S_c^{\,2}\eta e^{-(S_c+b)\eta} \end{pmatrix} \\ +\left(2\psi S_c e^{-(S_c+b)\eta} - S_c^{\,2}\psi\eta e^{-(S_c+b)\eta}\right) \\ +N_{t1}\alpha^2 e^{-\alpha\eta} - S_c\sigma\left(e^{-S_c\eta} + (e-2)e^{-(S_c+\alpha)\eta} + e^{-(2\alpha+S_c)\eta}\right) \end{pmatrix} \tag{3.343}$$

Further simplification of (3.343) gives

$$r_3' + S_c r_3 = \begin{pmatrix} \left(\frac{aS_c}{R_e} + S_c\sigma - S_c^{\,2}\left(\frac{1+\psi}{b}\right) - S_c\right)e^{-S_c\eta} + \left(S_c^{\,2}\left(\frac{1+\psi}{b}\right) - 2S_c(1+\psi)\right)e^{-(b+S_c)\eta} \\ \left(S_c^{\,2}(1+\psi) - S_c^{\,2}(1+\psi)\right)\eta e^{-(b+S_c)\eta} - \frac{aS_c^{\,2}}{2R_e}\eta e^{-S_c\eta} - \alpha^2 N_{t1} e^{-\alpha\eta} \\ +S_c\sigma(e-2)e^{-(S_c+\alpha)\eta} + S_c\sigma e^{-(2\alpha+S_c)\eta} \\ \end{pmatrix}$$

$$r_3' + S_c r_3 = Z_{47}e^{-S_c\eta} + Z_{48}e^{-(b+S_c)\eta} - Z_{49}\eta e^{-S_c\eta} - Z_{50}e^{-\alpha\eta} + Z_{51}e^{-(S_c+\alpha)\eta} + S_c\sigma e^{-(2\alpha+S_c)\eta} \tag{3.344}$$

where

$$Z_{47} = \frac{aS_c}{R_e} + S_c\sigma - S_c^{\,2}\left(\frac{1+\psi}{b}\right) - S_c$$

$$Z_{48} = S_c{}^2\left(\frac{1+\psi}{b}\right) - 2S_c\left(1+\psi\right)$$

$$Z_{49} = \frac{aS_c{}^2}{2R_e}$$

$$Z_{50} = \alpha^2 N_{t1}$$

$$Z_{51} = S_c\sigma\left(e-2\right)$$

Similarly, the solution for (3.344) is given by

$$r_3\left(\eta\right) = e^{-\int S_c d\eta}\int_0^\eta e^{\int S_c dx}\begin{pmatrix} Z_{47}e^{-S_c x} + Z_{48}e^{-\left(b+S_c\right)x} - Z_{49}xe^{-S_c x} \\ -Z_{50}e^{-\alpha x} + Z_{51}e^{-\left(S_c+\alpha\right)x} + S_c\sigma e^{-\left(2\alpha+S_c\right)\eta} \end{pmatrix}dx + c_{22}e^{-\int S_c d\eta} \tag{3.345}$$

Simplifying (3.345) gives

$$r_3\left(\eta\right) = e^{-S_c\eta}\left(Z_{47}x - \frac{Z_{48}}{b}e^{-bx} - \frac{Z_{49}}{2}x^2 - \frac{Z_{50}x^2}{S_c-\alpha}e^{\left(S_c-\alpha\right)x} - \frac{Z_{51}}{\alpha}e^{-\alpha x} - \frac{S_c\sigma}{2\alpha}e^{-2\alpha x}\right)\Bigg|_0^\eta + c_{22}e^{-S_c\eta}$$

$$r_3\left(\eta\right) = e^{-S_c\eta}\begin{pmatrix} Z_{47}\eta - \frac{Z_{48}}{b}\left(e^{-b\eta}-1\right) - \frac{Z_{49}}{2}\eta^2 - \frac{Z_{50}}{S_c-\alpha}\left(e^{\left(S_c-\alpha\right)\eta}-1\right) \\ -\frac{Z_{51}}{\alpha}\left(e^{-\alpha\eta}-1\right) - \frac{S_c\sigma}{2\alpha}\left(e^{-2\alpha\eta}-1\right) \end{pmatrix} + c_{22}e^{-S_c\eta} \tag{3.346}$$

Simplifying (3.346) in terms of similar power gives

$$r_3\left(\eta\right) = Z_{47}\eta e^{-S_c\eta} - \frac{Z_{48}}{b}e^{-\left(b+S_c\right)\eta} - \frac{Z_{49}}{2}\eta^2 e^{-S_c\eta} - \frac{Z_{50}}{S_c-\alpha}e^{-\alpha\eta} - \frac{Z_{51}}{\alpha}e^{-\left(S_c+\alpha\right)\eta}$$
$$+\left(\frac{Z_{48}}{b} + \frac{Z_{50}}{S_c-\alpha} + \frac{Z_{51}}{\alpha} + Z_{02}\right)e^{-S_c\eta} - Z_{02}e^{-\left(2\alpha+S_c\right)\eta} + c_{22}e^{-S_c\eta}$$

$$r_3(\eta) = Z_{47}\eta e^{-S_c\eta} - \frac{Z_{48}}{b}e^{-(b+S_c)\eta} - \frac{Z_{49}}{2}\eta^2 e^{-S_c\eta} - \frac{Z_{50}}{S_c-\alpha}e^{-\alpha\eta} - \frac{Z_{51}}{\alpha}e^{-(S_c+\alpha)\eta}$$
$$Z_{52}e^{-S_c\eta} - Z_{02}e^{-(2\alpha+S_c)\eta} + c_{22}e^{-S_c\eta} \tag{3.347}$$

Where $Z_{52} = Z_{02} + \frac{Z_{48}}{b} + \frac{Z_{50}}{S_c-\alpha} + \frac{Z_{50}}{\alpha}$

$$Z_{02} = \frac{S_c\sigma}{2\alpha}$$

Recalling that $r_3 = \phi_1'$

Then equation (3.347) becomes

$$\phi_1' = Z_{47}\eta e^{-S_c\eta} - \frac{Z_{48}}{b}e^{-(b+S_c)\eta} - \frac{Z_{49}}{2}\eta^2 e^{-S_c\eta} - \frac{Z_{50}}{S_c-\alpha}e^{-\alpha\eta} - \frac{Z_{51}}{\alpha}e^{-(S_c+\alpha)\eta}$$
$$+Z_{52}e^{-S_c\eta} - Z_{02}e^{-(2\alpha+S_c)\eta} + c_{22}e^{-S_c\eta} \tag{3.348}$$

Integrating both sides of (3.348) gives

$$\phi_1(\eta) = Z_{47}\left(-\frac{\eta}{S_c}e^{-S_c\eta} - \frac{e^{-S_c\eta}}{S_c{}^2}\right) + \frac{Z_{48}}{b(b+S_c)}e^{-(b+S_c)\eta}$$
$$-\frac{Z_{49}}{2}\left(-\frac{\eta^2}{S_c}e^{-S_c\eta} + \frac{2}{S_c}\left(-\frac{\eta}{S_c}e^{-S_c\eta} - \frac{e^{-S_c\eta}}{S_c{}^2}\right)\right) + \frac{Z_{50}}{\alpha(S_c-\alpha)}e^{-\alpha\eta} \tag{3.349}$$
$$+\frac{Z_{51}}{\alpha(\alpha+S_c)}e^{-(S_c+\alpha)\eta} + \frac{Z_{02}}{2\alpha+S_c}e^{-(2\alpha+S_c)\eta} - \frac{Z_{52}}{S_c}e^{-S_c\eta} - \frac{c_{22}}{S_c}e^{-S_c\eta} + c_{23}$$

Simplifying (3.349) in terms of similar power gives

$$\phi_1(\eta)=\left(\frac{Z_{49}}{S_c^{\,3}}-\frac{Z_{47}}{S_c^{\,2}}-\frac{Z_{52}}{S_c}\right)e^{-S_c\eta}+\left(\frac{Z_{49}}{S_c^{\,2}}-\frac{Z_{47}}{S_c}\right)\eta e^{-S_c\eta}+\frac{Z_{49}}{2S_c}\eta^2e^{-S_c\eta}$$
$$+\frac{Z_{48}}{b(b+S_c)}e^{-(b+S_c)\eta}+\frac{Z_{50}}{\alpha(\alpha-S_c)}e^{-\alpha\eta}+\frac{Z_{51}}{\alpha(\alpha-S_c)}e^{-(S_c+\alpha)\eta}$$
$$+\frac{Z_{02}}{2\alpha+S_c}e^{-(2\alpha+S_c)\eta}-\frac{c_{22}}{S_c}e^{-S_c\eta}+c_{23}$$

$$\phi_1(\eta)=Z_{53}e^{-S_c\eta}+Z_{54}\eta e^{-S_c\eta}+Z_{55}\eta^2e^{-S_c\eta}+Z_{56}e^{-(b+S_c)\eta}+Z_{57}e^{-\alpha\eta}$$
$$+Z_{58}e^{-(S_c+\alpha)\eta}+Z_{03}e^{-(2\alpha+S_c)\eta}-\frac{c_{22}}{S_c}e^{-S_c\eta}+c_{23} \qquad (3.350)$$

Inserting the boundary conditions gives

$$\phi_1(\infty)=0+0+c_{23}=0$$

$$c_{23}=0 \qquad (3.351)$$

$$\phi_1(0)=\left(Z_{53}+Z_{56}+Z_{57}+Z_{58}\right)-\frac{c_{22}}{S_c}=0$$

$$c_{22}=S_ck_4 \qquad (3.352)$$

where

$$Z_{53}=\frac{Z_{49}}{S_c^{\,3}}-\frac{Z_{47}}{S_c^{\,2}}-\frac{Z_{52}}{S_c}$$

$$Z_{54}=\frac{Z_{49}}{S_c^{\,2}}-\frac{Z_{47}}{S_c}$$

$$Z_{55}=\frac{Z_{49}}{2}$$

$$Z_{56}=\frac{Z_{48}}{b\left(b+S_c\right)}$$

$$Z_{57}=\frac{Z_{50}}{\alpha\left(\alpha-S_c\right)}$$

$$Z_{58}=\frac{Z_{51}}{\alpha\left(\alpha-S_c\right)}$$

$$k_4=Z_{53}+Z_{56}+Z_{57}+Z_{58}$$

$$Z_{03}=\frac{Z_{02}}{2\alpha+S_c}$$

$$\begin{aligned}\phi_1\left(\eta\right)&=Z_{53}e^{-S_c\eta}+Z_{54}\eta e^{-S_c\eta}+Z_{55}\eta^2e^{-S_c\eta}+Z_{56}e^{-(b+S_c)\eta}+Z_{57}e^{-\alpha\eta}\\&+Z_{58}e^{-(S_c+\alpha)\eta}+Z_{03}e^{-(2\alpha+S_c)\eta}-k_4e^{-S_c\eta}\end{aligned} \tag{3.353}$$

Substituting (3.239), (3.253), (3.265), (3.276), (3.303), (3.321), (3.338), and (3.353) into (3.186) results

$$\begin{aligned}f\left(\eta\right)&=\frac{1}{b}\left(e^{-b\eta}-1\right)+\in\begin{pmatrix}Z_{11}e^{-b\eta}-Z_{12}e^{-2b\eta}+Z_{13}\eta e^{-b\eta}-Z_{14}\eta^2e^{-b\eta}-Z_{15}e^{-\alpha\eta}\\+Z_{16}e^{-S_c\eta}+\dfrac{Z_{17}}{b}e^{-b\eta}+Z_{18}\end{pmatrix}\\g\left(\eta\right)&=\frac{\psi}{b}\left(e^{-b\eta}-1\right)+\in\left(Z_{26}e^{-b\eta}+Z_{27}\eta e^{-b\eta}+Z_{28}\eta^2e^{-b\eta}-Z_{29}e^{-2b\eta}+\frac{k_1}{b}e^{-b\eta}+k_2\right)\\\theta\left(\eta\right)&=e^{-\alpha\eta}+\in\begin{pmatrix}-Z_{45}\eta e^{-\alpha\eta}-Z_{38}e^{-\alpha\eta}+Z_{39}e^{-(\alpha+b)\eta}-Z_{42}e^{-2b\eta}\\-Z_{46}\eta^2e^{-\alpha\eta}+Z_{41}e^{-(\alpha+S_c)\eta}+Z_{43}e^{-2\alpha\eta}-Z_{44}e^{-b\eta}+Z_{01}e^{-(2\alpha+S_c)\eta}-k_3e^{-\alpha\eta}\end{pmatrix}\\\phi\left(\eta\right)&=e^{-S_c\eta}+\in\begin{pmatrix}Z_{53}e^{-S_c\eta}+Z_{54}\eta e^{-S_c\eta}+Z_{55}\eta^2e^{-S_c\eta}+Z_{56}e^{-(b+S_c)\eta}+Z_{57}e^{-\alpha\eta}\\+Z_{58}e^{-(S_c+\alpha)\eta}+Z_{03}e^{-(2\alpha+S_c)\eta}-k_4e^{-S_c\eta}\end{pmatrix}\end{aligned} \tag{3.354}$$

3.2.5 Case 2: When the reaction is steady with Arrhenius chemical reaction: $a = 0$

3.2.5.1 Solution for case 2

In this case, equations (3.73) – (3.77) reduces to

$$\begin{aligned} & f''' + \frac{1}{b}\left(e^{-b\eta} - 1\right) f'' + \frac{\psi}{b}\left(e^{-b\eta} - 1\right) f'' + \eta\left(f' + g'\right) f'' - 2f'\left(f' + \frac{\eta}{2} f''\right) \\ & -2g'\left(f' + \frac{\eta}{2} f''\right) - \Omega f'^2 - \left(M + \gamma\right) f' + G_{r\theta}\left(1 + \in \theta\right) + G_{r\phi}\phi = 0 \end{aligned} \tag{3.355}$$

$$\begin{aligned} & g''' + \frac{1}{b}\left(e^{-b\eta} - 1\right) g'' + \frac{\psi}{b}\left(e^{-b\eta} - 1\right) g'' + \eta\left(f' + g'\right) g'' - 2f'\left(g' + \frac{\eta}{2} g''\right) \\ & -2g'\left(g' + \frac{\eta}{2} g''\right) - \Omega g'^2 - \left(M + \gamma\right) g' = 0 \end{aligned} \tag{3.356}$$

$$\begin{aligned} & R_1\theta'' + \frac{1}{b}\left(e^{-b\eta} - 1\right)\theta' + \frac{\psi}{b}\left(e^{-b\eta} - 1\right)\theta' + \eta\left(f' + g'\right)\theta' \\ & -2\left(\left(\frac{1}{\in} + \theta\right) + \frac{\eta}{2}\theta'\right)\left(f' + g'\right) + E_c M\left(f'^2 + g'^2\right) + N_b\theta'\phi' \\ & + N_t\theta'^2 + Q_h\left(1 + \in \theta\right) + \delta\phi\left(1 + \left(e - 2\right)\theta + \theta^2\right) = 0 \end{aligned} \tag{3.357}$$

$$\begin{aligned} & \phi'' + \frac{S_c}{b}\left(e^{-b\eta} - 1\right)\phi' + \frac{S_c\psi}{b}\left(e^{-b\eta} - 1\right)\phi' + S_c\eta\left(f' + g'\right)\phi' \\ & -2S_c\left(\phi + \frac{\eta}{2}\phi'\right)\left(f' + g'\right) + N_{t1}\theta'' - S_c\sigma\phi\left(1 + \left(e - 2\right)\theta + \theta^2\right) = 0 \end{aligned} \tag{3.358}$$

Similarly, rewriting equations (3.355) – (3.358) in the form:

$$\begin{aligned} & f''' + bf'' + \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta} - 1\right) - b\right) f'' + \eta\left(f' + g'\right) f'' - 2f'\left(f' + \frac{\eta}{2} f''\right) \\ & -2g'\left(f' + \frac{\eta}{2} f''\right) - \Omega f'^2 - \left(M + \gamma\right) f' + G_{r\theta}\left(1 + \in \theta\right) + G_{r\phi}\phi = 0 \end{aligned} \tag{3.359}$$

$$g''' + bg'' + \left(\left(\frac{1+\psi}{b} \right) \left(e^{-b\eta} - 1 \right) - b \right) g'' + \eta \left(f' + g' \right) g''$$
$$-2 f' \left(g' + \frac{\eta}{2} g'' \right) - 2g' \left(g' + \frac{\eta}{2} g'' \right) - \Omega g'^2 - \left(M + \gamma \right) g' = 0 \tag{3.360}$$

$$R_1 \theta'' + \theta' + \left(\left(\frac{1+\psi}{b} \right) \left(e^{-b\eta} - 1 \right) - 1 \right) \theta' + \eta \left(f' + g' \right) \theta'$$
$$-2 \left(\left(\frac{1}{\in} + \theta \right) + \frac{\eta}{2} \theta' \right) \left(f' + g' \right) + E_c M \left(f'^2 + g'^2 \right) \tag{3.361}$$
$$+ N_b \theta' \phi' + N_t \theta'^2 + Q_h \left(1 + \in \theta \right) + \delta \phi \left(1 + \left(e - 2 \right) \theta + \theta^2 \right) = 0$$

$$\phi'' + S_c \phi' + S_c \left(\left(\frac{1+\psi}{b} \right) \left(e^{-b\eta} - 1 \right) - 1 \right) \phi' + S_c \eta \left(f' + g' \right) \phi'$$
$$-2 S_c \left(\phi + \frac{\eta}{2} \phi' \right) \left(f' + g' \right) + N_{t1} \theta'' - S_c \sigma \phi \left(1 + \left(e - 2 \right) \theta + \theta^2 \right) = 0 \tag{3.362}$$

Introducing an artificial parameter $\in$ in equations (3.359) – (3.362) gives

$$f''' + bf'' + \in \begin{pmatrix} \left(\left(\frac{1+\psi}{b} \right) \left(e^{-b\eta} - 1 \right) - b \right) f'' + \eta \left(f' + g' \right) f'' - 2 f' \left(f' + \frac{\eta}{2} f'' \right) \\ -2g' \left(f' + \frac{\eta}{2} f'' \right) - \Omega f'^2 - \left(M + \gamma \right) f' + G_{r\theta} \left(1 + \in \theta \right) + G_{r\phi} \phi \end{pmatrix} = 0 \tag{3.363}$$

$$g''' + bg'' + \in \begin{pmatrix} \left(\left(\frac{1+\psi}{b} \right) \left(e^{-b\eta} - 1 \right) - b \right) g'' + \eta \left(f' + g' \right) g'' - 2 f' \left(g' + \frac{\eta}{2} g'' \right) \\ -2g' \left(g' + \frac{\eta}{2} g'' \right) - \Omega g'^2 - \left(M + \gamma \right) g' \end{pmatrix} = 0 \tag{3.364}$$

$$R_1 \theta'' + \theta' + \in \begin{pmatrix} \left(\left(\frac{1+\psi}{b} \right) \left(e^{-b\eta} - 1 \right) - 1 \right) \theta' + \eta \left(f' + g' \right) \theta' - \\ 2 \left(\left(\frac{1}{\in} + \theta \right) + \frac{\eta}{2} \theta' \right) \left(f' + g' \right) + E_c M \left(f'^2 + g'^2 \right) + N_b \theta' \phi' + N_t \theta'^2 + \\ Q_h \left(1 + \in \theta \right) + \delta \phi \left(1 + \left(e - 2 \right) \theta + \theta^2 \right) \end{pmatrix} = 0 \tag{3.365}$$

$$\phi'' + S_c\phi' + \in \begin{pmatrix} S_c\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\phi' + S_c\eta\left(f'+g'\right)\phi' \\ -2S_c\left(\phi+\frac{\eta}{2}\phi'\right)\left(f'+g'\right)+N_{t1}\theta'' - S_c\sigma\phi\left(1+\left(e-2\right)\theta+\theta^2\right) \end{pmatrix} = 0 \qquad (3.366)$$

Similarly, Simplifying and processing interms like powers of $\in$ gives the order zero and one equations with their respective boundary conditions given below

$\in^0$:

$$\begin{aligned} &f_0''' + bf_0'' = 0 \\ &f_0(0) = 0, \quad f_0'(0) = -1, \quad f_0'(\eta\to\infty)\to 0 \end{aligned} \qquad (3.368)$$

$$\begin{aligned} &g_0''' + bg_0'' = 0 \\ &g_0(0) = 0, \quad g_0'(0) = -\psi, \quad g_0'(\eta\to\infty)\to 0 \end{aligned} \qquad (3.369)$$

$$\begin{aligned} &\theta_0'' + \frac{1}{R_1}\theta_0' = 0 \\ &\theta_0(0) = 1, \quad \theta_0(\eta\to\infty)\to 0 \end{aligned} \qquad (3.370)$$

$$\begin{aligned} &\phi_0'' + S_c\phi_0' = 0 \\ &\phi_0(0) = 1, \quad \phi_0(\eta\to\infty)\to 0 \end{aligned} \qquad (3.371)$$

$\in^1$:

$$\begin{aligned} &f_1''' + bf_1'' + \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-b\right)f_0'' + \eta\left(f_0'+g_0'\right)f_0'' - \\ &2f_0'\left(f_0'+\frac{\eta}{2}f_0''\right) - 2g_0'\left(f_0'+\frac{\eta}{2}f_0''\right) - \Omega f_0'^2 - \left(M+\gamma\right)f_0' \\ &+G_{r\theta}\left(1+\in\theta_0\right) + G_{r\phi}\phi_0 = 0 \\ &f_1(0) = 0, \quad f_1'(0) = 0, \quad f_1'(\eta\to\infty)\to 0 \end{aligned} \qquad (3.372)$$

$$g_1''' + b g_1'' + \left(\left(\frac{1+\psi}{b} \right) \left(e^{-b\eta} - 1 \right) - b \right) g_0'' + \eta \left(f_0' + g_0' \right) g_0''$$

$$-2 f_0' \left(g_0' + \frac{\eta}{2} g_0'' \right) - 2 g_0' \left(g_0' + \frac{\eta}{2} g_0'' \right) - \Omega g_0'^2 - \left(M + \gamma \right) g_0' = 0 \qquad (3.373)$$

$$g_1(0) = 0, \quad g_1'(0) = 0, \quad g_1'(\eta \to \infty) \to 0$$

$$\theta_1'' + \frac{1}{R_1} \theta_1' + \frac{1}{R_1} \begin{pmatrix} \left(\left(\frac{1+\psi}{b} \right) \left(e^{-b\eta} - 1 \right) - 1 \right) \theta_0' + \eta \left(f_0' + g_0' \right) \theta_0' - \\ -2 \left(\left(\frac{1}{\in} + \theta_0 \right) + \frac{\eta}{2} \theta_0' \right) \left(f_0' + g_0' \right) + E_c M \left(f_0'^2 + g_0'^2 \right) \\ + N_b \theta_0' \phi_0' + N_t \theta_0'^2 + Q_h \left(1 + \in \theta_0 \right) + \delta \phi_0 \left(1 + (e-2) \theta_0 + \theta_0^2 \right) \end{pmatrix} = 0$$

$$\theta_1(0) = 0, \quad \theta_1(\eta \to \infty) \to 0$$

$$\phi_1'' + S_c \phi_1' + S_c \left(\left(\frac{1+\psi}{b} \right) \left(e^{-b\eta} - 1 \right) - 1 \right) \phi_0' + S_c \eta \left(f_0' + g_0' \right) \phi_0' -$$

(3.374) $$2 S_c \left(\phi_0 + \frac{\eta}{2} \phi_0' \right) \left(f_0' + g_0' \right) + N_{t1} \theta_0'' - S_c \sigma \phi_0 \left(1 + (e-2) \theta_0 + \theta_0^2 \right) = 0$$

$$\phi_1(0) = 0, \quad \phi_1(\eta \to \infty) \to 0$$

(3.375)

Similarly, following the same processes of case one above gives

$$f(\eta) = \frac{1}{b} \left(e^{-b\eta} - 1 \right) + \in \begin{pmatrix} Z_{69} e^{-b\eta} - Z_{70} e^{-2b\eta} + Z_{71} \eta e^{-b\eta} - Z_{72} e^{-\alpha\eta} \\ + Z_{73} e^{-S_c \eta} + \frac{Z_{74}}{b} e^{-b\eta} + Z_{75} \end{pmatrix}$$

$$g(\eta) = \frac{\psi}{b} \left(e^{-b\eta} - 1 \right) + \in \left(Z_{81} e^{-b\eta} + Z_{82} \eta e^{-b\eta} Z_{83} e^{-2b\eta} + \frac{Z_{84}}{b} e^{-b\eta} + Z_{85} \right)$$

$$\theta(\eta) = e^{-\alpha\eta} + \in \begin{pmatrix} -Z_{100} \eta e^{-\alpha\eta} - Z_{93} e^{-\alpha\eta} + Z_{94} e^{-(\alpha+b)\eta} - Z_{97} e^{-2b\eta} \\ + Z_{96} e^{-(\alpha+S_c)\eta} + Z_{98} e^{-2\alpha\eta} - Z_{99} e^{-b\eta} - Z_{101} e^{-\alpha\eta} \end{pmatrix}$$

$$\phi(\eta) = e^{-S_c \eta} + \in \begin{pmatrix} Z_{107} e^{-S_c \eta} + Z_{108} \eta e^{-S_c \eta} + Z_{109} e^{-(b+S_c)\eta} + Z_{110} e^{-\alpha\eta} \\ + Z_{111} e^{-(S_c+\alpha)\eta} - Z_{112} e^{-S_c \eta} \end{pmatrix} \qquad (3.376)$$

where

$$Z_{61} = \left(M + \gamma\right) - b\left(\frac{1+\psi}{b}\right) - b^2$$

$$Z_{62} = b\left(\frac{1+\psi}{b}\right) - 2\left(1+\psi\right)$$

$$Z_{63} = \left(\frac{G_{r\theta} \in}{b-\alpha} + \frac{G_{r\theta}}{b} + \frac{Z_2}{b} + \frac{G_{r\phi}}{b-S_c}\right)$$

$$Z_{64} = \frac{Z_{61}}{b^2} - \frac{Z_{63}}{b}$$

$$Z_{65} = \frac{Z_{62}}{2b^2}$$

$$Z_{66} = \frac{Z_{61}}{b}$$

$$Z_{67} = \frac{G_{r\theta} \in}{\alpha\left(b-\alpha\right)}$$

$$Z_{68} = \frac{G_{r\phi}}{S_c\left(b-S_c\right)}$$

$$Z_{69} = -\frac{Z_{64}}{b} - \frac{Z_{66}}{b^2}$$

$$Z_{70} = \frac{Z_{65}}{2b}$$

$$Z_{71} = -\frac{Z_{66}}{b}$$

$$Z_{72} = \frac{Z_{67}}{\alpha}$$

$$Z_{73} = \frac{Z_{68}}{S_c}$$

$$Z_{74} = Z_{64} + Z_{65} + Z_{67} + Z_{68}$$

$$Z_{75} = Z_{70} + Z_{72} - Z_{69} + Z_{73} - \frac{Z_{74}}{b}$$

$$Z_{76} = \frac{a\psi}{R_e} - \left(\psi + \psi^2\right) - b^2\psi + \left(M + \gamma\right)\psi$$

$$Z_{77} = \left(\psi + \psi^2\right) - \Omega\psi^2$$

$$Z_{78} = \frac{Z_{76}}{b^2} - \frac{Z_{77}}{b^2}$$

$$Z_{79} = \frac{Z_{76}}{b}$$

$$Z_{80} = \frac{Z_{77}}{2b^2}$$

$$Z_{81} = -\frac{Z_{78}}{b} - \frac{Z_{79}}{b^2}$$

$$Z_{82} = -\frac{Z_{79}}{b}$$

$$Z_{83} = \frac{Z_{80}}{2b}$$

$$Z_{84} = Z_{78} + Z_{80}$$

$$Z_{86} = -\frac{1}{R_1}\left(Q_h \in + \frac{1+\psi}{b}\alpha + \alpha\right)$$

$$Z_{87} = -\frac{1}{R_1}\left(2\left(1+\psi\right) - \frac{1+\psi}{b}\alpha\right)$$

$$Z_{88} = -\frac{1}{R_1}E_c M\left(1+\psi^2\right)$$

$$Z_{89} = -\frac{1}{R_1}\left(\delta\left(e-2\right) + N_b S_c \alpha\right)$$

$$Z_{90} = -\frac{1}{R_1}\left(Q_h\right)$$

$$Z_{91} = -\frac{1}{R_1}N_t \alpha^2$$

$$Z_{92} = -2\frac{1}{R_1}\left(1+\psi\right)\frac{1}{\in}$$

$$Z_0 = \left(\frac{Z_{87}}{b} - \frac{Z_{88}}{\left(\alpha - 2b\right)} + \frac{Z_{89}}{S_c} - \frac{Z_{90}}{\alpha} + \frac{Z_{91}}{\alpha} - \frac{Z_{92}}{\left(\alpha - b\right)} - \frac{\delta}{\left(\alpha - S_c\right)}\right)$$

$$Z_{93} = \left(\frac{Z_{86}}{b^2} - \frac{Z_0}{\alpha}\right)$$

$$Z_{94} = \frac{Z_{87}}{b\left(\alpha + b\right)}$$

$$Z_{95} = \frac{Z_{90}}{\alpha}$$

$$Z_{96} = \frac{Z_{89}}{S_c\left(\alpha + S_c\right)}$$

$$Z_{97} = \frac{Z_{88}}{2b\left(\alpha - 2b\right)}$$

$$Z_{98} = \frac{Z_{91}}{2\alpha^2}$$

$$Z_{99} = \frac{Z_{92}}{b\left(\alpha - b\right)}$$

$$Z_{100} = \left(\frac{Z_{30}}{b}\right)$$

$$Z_{101} = -Z_{93} + Z_{94} + Z_{96} - Z_{97} + Z_{98} - Z_{99}$$

$$Z_{102} = S_c\sigma - S_c^{\,2}\left(\frac{1+\psi}{b}\right) - S_c$$

$$Z_{103} = S_c^{\,2}\left(\frac{1+\psi}{b}\right) - 2S_c\left(1+\psi\right)$$

$$Z_{104} = \alpha^2 N_{t1}$$

$$Z_{105} = S_c\sigma\left(e-2\right)$$

$$Z_{106} = \frac{Z_{103}}{b} + \frac{Z_{104}}{S_c - \alpha} + \frac{Z_{105}}{\alpha}$$

$$Z_{107} = -\frac{Z_{102}}{S_c^{\ 2}} - \frac{Z_{106}}{S_c}$$

$$Z_{108} = -\frac{Z_{102}}{S_c}$$

$$Z_{109} = \frac{Z_{103}}{b\left(b+S_c\right)}$$

$$Z_{110} = \frac{Z_{104}}{\alpha\left(\alpha-S_c\right)}$$

$$Z_{111} = \frac{Z_{105}}{\alpha\left(\alpha-S_c\right)}$$

$$Z_{112} = Z_{107} + Z_{109} + Z_{110} + Z_{111}$$

3.2.6 Case 3: When the reaction is unsteady with chemical reaction of Constant rate $\frac{\partial}{\partial t}\bullet(\ )\neq 0$

3.2.6.1 Solution for case 3

In this case, equations (3.73) – (3.77) reduces to

$$\begin{aligned} & f''' + \frac{1}{b}\left(e^{-b\eta}-1\right)f'' + \frac{\psi}{b}\left(e^{-b\eta}-1\right)f'' + \eta\left(f'+g'\right)f'' - \frac{a}{R_e}\left(f' + \frac{\eta}{2}f''\right) - \\ & 2f'\left(f'+\frac{\eta}{2}f''\right) - 2g'\left(f'+\frac{\eta}{2}f''\right) - \Omega f'^2 - \left(M+\gamma\right)f' + G_{r\theta}\left(1+\in\theta\right) + G_{r\phi}\phi = 0 \end{aligned} \qquad (3.377)$$

$$g''' + \frac{1}{b}\left(e^{-b\eta} - 1\right)g'' + \frac{\psi}{b}\left(e^{-b\eta} - 1\right)g'' + \eta\left(f' + g'\right)g'' - \frac{a}{R_e}\left(g' + \frac{\eta}{2}g''\right) -$$
$$2f'\left(g' + \frac{\eta}{2}g''\right) - 2g'\left(g' + \frac{\eta}{2}g''\right) - \Omega g'^2 - \left(M + \gamma\right)g' = 0 \tag{3.378}$$

$$R_1\theta'' + \frac{1}{b}\left(e^{-b\eta} - 1\right)\theta' + \frac{\psi}{b}\left(e^{-b\eta} - 1\right)\theta' + \eta\left(f' + g'\right)\theta' - \frac{a}{R_e}\left(\left(\frac{1}{\in} + \theta\right) + \frac{\eta}{2}\theta'\right) -$$
$$2\left(\left(\frac{1}{\in} + \theta\right) + \frac{\eta}{2}\theta'\right)\left(f' + g'\right) + E_c M\left(f'^2 + g'^2\right) + N_b\theta'\phi' + N_t\theta'^2 + Q_h\left(1 + \in\theta\right) + \tag{3.379}$$
$$\delta\phi = 0$$

$$\phi'' + \frac{S_c}{b}\left(e^{-b\eta} - 1\right)\phi' + \frac{S_c\psi}{b}\left(e^{-b\eta} - 1\right)\phi' + S_c\eta\left(f' + g'\right)\phi' - \frac{a}{R_e}S_c\left(\phi + \frac{\eta}{2}\phi'\right) -$$
$$2S_c\left(\phi + \frac{\eta}{2}\phi'\right)\left(f' + g'\right) + N_{t1}\theta'' - S_c\sigma\phi = 0 \tag{3.380}$$

Again, Rewriting equations (3.377) – (3.380) in the form:

$$f''' + bf'' + \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta} - 1\right) - b\right)f'' + \eta\left(f' + g'\right)f'' - \frac{a}{R_e}\left(f' + \frac{\eta}{2}f''\right) -$$
$$2f'\left(f' + \frac{\eta}{2}f''\right) - 2g'\left(f' + \frac{\eta}{2}f''\right) - \Omega f'^2 - \left(M + \gamma\right)f' + G_{r\theta}\left(1 + \in\theta\right) + G_{r\phi}\phi = 0 \tag{3.381}$$

$$g''' + bg'' + \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta} - 1\right) - b\right)g'' + \eta\left(f' + g'\right)g'' - \frac{a}{R_e}\left(g' + \frac{\eta}{2}g''\right) -$$
$$2f'\left(g' + \frac{\eta}{2}g''\right) - 2g'\left(g' + \frac{\eta}{2}g''\right) - \Omega g'^2 - \left(M + \gamma\right)g' = 0 \tag{3.382}$$

$$R_1\theta'' + \theta' + \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta} - 1\right) - 1\right)\theta' + \eta\left(f' + g'\right)\theta' - \frac{a}{R_e}\left(\left(\frac{1}{\in} + \theta\right) + \frac{\eta}{2}\theta'\right) -$$
$$2\left(\left(\frac{1}{\in} + \theta\right) + \frac{\eta}{2}\theta'\right)\left(f' + g'\right) + E_c M\left(f'^2 + g'^2\right) + N_b\theta'\phi' + N_t\theta'^2 + Q_h\left(1 + \in\theta\right) + \tag{3.383}$$
$$\delta\phi = 0$$

$$\phi'' + S_c\phi' + S_c\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\phi' + S_c\eta\left(f'+g'\right)\phi' - \frac{a}{R_e}S_c\left(\phi+\frac{\eta}{2}\phi'\right) - 2S_c\left(\phi+\frac{\eta}{2}\phi'\right)\left(f'+g'\right) + N_{t1}\theta'' - S_c\sigma\phi = 0 \tag{3.384}$$

Introducing an artificial parameter $\in$ in equations (3.381) – (3.384) gives

$$f''' + bf'' + \in\left(\begin{array}{l}\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-b\right)f'' + \eta\left(f'+g'\right)f'' - \frac{a}{R_e}\left(f'+\frac{\eta}{2}f''\right) - \\ 2f'\left(f'+\frac{\eta}{2}f''\right) - 2g'\left(f'+\frac{\eta}{2}f''\right) - \Omega f'^2 - \left(M+\gamma\right)f' + \\ G_{r\theta}\left(1+\in\theta\right) + G_{r\phi}\phi\end{array}\right) = 0 \tag{3.385}$$

$$g''' + bg'' + \in\left(\begin{array}{l}\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-b\right)g'' + \eta\left(f'+g'\right)g'' - \frac{a}{R_e}\left(g'+\frac{\eta}{2}g''\right) - \\ 2f'\left(g'+\frac{\eta}{2}g''\right) - 2g'\left(g'+\frac{\eta}{2}g''\right) - \Omega g'^2 - \left(M+\gamma\right)g'\end{array}\right) = 0 \tag{3.386}$$

$$R_1\theta'' + \theta' + \in\left(\begin{array}{l}\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\theta' + \eta\left(f'+g'\right)\theta' - \frac{a}{R_e}\left(\left(\frac{1}{\in}+\theta\right)+\frac{\eta}{2}\theta'\right) - \\ 2\left(\left(\frac{1}{\in}+\theta\right)+\frac{\eta}{2}\theta'\right)\left(f'+g'\right) + E_cM\left(f'^2+g'^2\right) + N_b\theta'\phi' + N_t\theta'^2 + \\ Q_h\left(1+\in\theta\right) + \delta\phi\end{array}\right) = 0 \tag{3.387}$$

$$\phi'' + S_c\phi' + \in\left(\begin{array}{l}S_c\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\phi' + S_c\eta\left(f'+g'\right)\phi' - \frac{a}{R_e}S_c\left(\phi+\frac{\eta}{2}\phi'\right) - \\ 2S_c\left(\phi+\frac{\eta}{2}\phi'\right)\left(f'+g'\right) + N_{t1}\theta'' - S_c\sigma\phi\end{array}\right) = 0 \tag{3.388}$$

Similarly, Simplifying and processing interms of like powers of $\in$ gives the order zero and one equations with their respective boundary conditions given below

$\in^0$:

$$f_0''' + bf_0'' = 0$$
$$f_0(0) = 0, \quad f_0'(0) = -1, \quad f_0'(\eta \to \infty) \to 0 \tag{3.389}$$

$$g_0''' + bg_0'' = 0$$
$$g_0(0) = 0, \quad g_0'(0) = -\psi, \quad g_0'(\eta \to \infty) \to 0 \tag{3.390}$$

$$\theta_0'' + \frac{1}{R_1}\theta_0' = 0$$
$$\theta_0(0) = 1, \quad \theta_0(\eta \to \infty) \to 0 \tag{3.391}$$

$$\phi_0'' + S_c\phi_0' = 0$$
$$\phi_0(0) = 1, \quad \phi_0(\eta \to \infty) \to 0 \tag{3.392}$$

$\in^1$:

$$f_1''' + bf_1'' + \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta} - 1\right) - b\right) f_0'' + \eta\left(f_0' + g_0'\right) f_0'' - \frac{a}{R_e}\left(f_0' + \frac{\eta}{2} f_0''\right) -$$
$$2 f_0'\left(f_0' + \frac{\eta}{2} f_0''\right) - 2g_0'\left(f_0' + \frac{\eta}{2} f_0''\right) - \Omega f_0'^2 - \left(M + \gamma\right) f_0' + \tag{3.393}$$
$$G_{r\theta}\left(1 + \in \theta_0\right) + G_{r\phi}\phi_0 = 0$$
$$f_1(0) = 0, \quad f_1'(0) = 0, \quad f_1'(\eta \to \infty) \to 0$$

$$g_1''' + bg_1'' + \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta} - 1\right) - b\right) g_0'' + \eta\left(f_0' + g_0'\right) g_0'' - \frac{a}{R_e}\left(g_0' + \frac{\eta}{2} g_0''\right) -$$
$$2 f_0'\left(g_0' + \frac{\eta}{2} g_0''\right) - 2g_0'\left(g_0' + \frac{\eta}{2} g_0''\right) - \Omega g_0'^2 - \left(M + \gamma\right) g_0' = 0 \tag{3.394}$$
$$g_1(0) = 0, \quad g_1'(0) = 0, \quad g_1'(\eta \to \infty) \to 0$$

$$\theta_1''+\frac{1}{R_1}\theta_1'+\frac{1}{R_1}\begin{pmatrix}\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\theta_0'+\eta\left(f_0'+g_0'\right)\theta_0'-\\ \frac{a}{R_e}\left(\left(\frac{1}{\in}+\theta_0\right)+\frac{\eta}{2}\theta_0'\right)-2\left(\left(\frac{1}{\in}+\theta_0\right)+\frac{\eta}{2}\theta_0'\right)\left(f_0'+g_0'\right)+\\ E_cM\left(f_0'^2+g_0'^2\right)+N_b\theta_0'\phi_0'+N_t\theta_0'^2+Q_h\left(1+\in\theta_0\right)+\\ \delta\phi_0\end{pmatrix}=0$$

$$\theta_1(0)=0,\quad \theta_1(\eta\to\infty)\to 0$$

$$\phi_1''+S_c\phi_1'+S_c\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\phi_0'+S_c\eta\left(f_0'+g_0'\right)\phi_0'-\frac{a}{R_e}S_c\left(\phi_0+\frac{\eta}{2}\phi_0'\right)-$$

(3.395) $$2S_c\left(\phi_0+\frac{\eta}{2}\phi_0'\right)\left(f_0'+g_0'\right)+N_{t1}\theta_0''-S_c\sigma\phi_0=0$$

$$\phi_1(0)=0,\quad \phi_1(\eta\to\infty)\to 0$$

(3.396)

Similarly, following the same processes of case one above gives

$$f(\eta)=\frac{1}{b}\left(e^{-b\eta}-1\right)+\in\begin{pmatrix}Z_{123}e^{-b\eta}-Z_{124}e^{-2b\eta}+Z_{125}\eta e^{-b\eta}-Z_{126}\eta^2e^{-b\eta}\\ -Z_{127}e^{-\alpha\eta}+Z_{128}e^{-S_c\eta}+\frac{Z_{129}}{b}e^{-b\eta}+Z_{130}\end{pmatrix}$$

$$g(\eta)=\frac{\psi}{b}\left(e^{-b\eta}-1\right)+\in\begin{pmatrix}Z_{138}e^{-b\eta}+Z_{139}\eta e^{-b\eta}+Z_{140}\eta^2e^{-b\eta}-Z_{141}e^{-2b\eta}+\\ \frac{Z_{142}}{b}e^{-b\eta}+Z_{143}\end{pmatrix}$$

$$\theta(\eta)=e^{-\alpha\eta}+\in\begin{pmatrix}-Z_{160}\eta e^{-\alpha\eta}-Z_{153}e^{-\alpha\eta}+Z_{154}e^{-(\alpha+b)\eta}-Z_{157}e^{-2b\eta}\\ -Z_{161}\eta^2e^{-\alpha\eta}+Z_{156}e^{-(\alpha+S_c)\eta}+Z_{158}e^{-2\alpha\eta}-Z_{159}e^{-b\eta}-\\ Z_{162}e^{-\alpha\eta}\end{pmatrix} \quad (3.397)$$

$$\phi(\eta)=e^{-S_c\eta}+\in\begin{pmatrix}Z_{168}e^{-S_c\eta}+Z_{169}\eta e^{-S_c\eta}+Z_{170}\eta^2e^{-S_c\eta}+Z_{171}e^{-(b+S_c)\eta}+\\ Z_{172}e^{-\alpha\eta}-Z_{173}e^{-S_c\eta}\end{pmatrix}$$

where

$$Z_{113} = \frac{a}{R_e} - b\left(\frac{1+\psi}{b}\right) - b^2 + \left(M + \gamma\right)$$

$$Z_{114} = b\left(\frac{1+\psi}{b}\right) - 2\left(1+\psi\right)$$

$$Z_{115} = \frac{ab}{2R_e}$$

$$Z_{116} = \left(\frac{G_{r\theta} \in}{b-\alpha} + \frac{G_{r\theta}}{b} + \frac{Z_2}{b} + \frac{G_{r\phi}}{b-S_c}\right)$$

$$Z_{117} = \frac{Z_{113}}{b^2} - \frac{Z_{115}}{b^3} - \frac{Z_{116}}{b}$$

$$Z_{118} = \frac{Z_{114}}{2b^2}$$

$$Z_{119} = \frac{Z_{113}}{b} - \frac{Z_{115}}{b^2}$$

$$Z_{120} = \frac{Z_{115}}{2b}$$

$$Z_{121} = \frac{G_{r\theta} \in}{\alpha\left(b-\alpha\right)}$$

$$Z_{122} = \frac{G_{r\phi}}{S_c\left(b-S_c\right)}$$

$$Z_{123} = \frac{2Z_{120}}{b^3} - \frac{Z_{117}}{b} - \frac{Z_{119}}{b^2}$$

$$Z_{124} = \frac{Z_{118}}{2b}$$

$$Z_{125} = \frac{2Z_{120}}{b^2} - \frac{Z_{119}}{b}$$

$$Z_{126} = \frac{Z_{120}}{b}$$

$$Z_{127} = \frac{Z_{121}}{\alpha}$$

$$Z_{128} = \frac{Z_{122}}{S_c}$$

$$Z_{129} = Z_{117} + Z_{118} + Z_{121} + Z_{122}$$

$$Z_{130} = Z_{124} + Z_{127} - Z_{123} + Z_{128} - \frac{Z_{129}}{b}$$

$$Z_{131} = \frac{a\psi}{R_e} - \left(\psi + \psi^2\right) - b^2\psi + \left(M + \gamma\right)\psi$$

$$Z_{132} = \left(\psi + \psi^2\right) - \Omega\psi^2$$

$$Z_{133} = \frac{ab\psi}{2R_e}$$

$$Z_{134} = \frac{Z_{131}}{b^2} - \frac{2Z_{133}}{2b^3} - \frac{Z_{132}}{b^2}$$

$$Z_{135} = \frac{Z_{131}}{b} - \frac{2Z_{133}}{2b^2}$$

$$Z_{136} = \frac{Z_{133}}{2b}$$

$$Z_{137} = \frac{Z_{132}}{2b^2}$$

$$Z_{138} = \frac{2Z_{136}}{b^3} - \frac{Z_{134}}{b} - \frac{Z_{135}}{b^2}$$

$$Z_{139} = \frac{2Z_{136}}{b^2} - \frac{Z_{135}}{b}$$

$$Z_{140} = \frac{Z_{136}}{b}$$

$$Z_{141} = \frac{Z_{137}}{2b}$$

$$Z_{142} = Z_{134} + Z_{137}$$

$$Z_{143} = Z_{141} - Z_{138} - \frac{Z_{142}}{b}$$

$$Z_{144} = -\frac{1}{R_1}\left(Q_h \in + \frac{1+\psi}{b}\alpha + \alpha - \frac{a}{R_e}\right)$$

$$Z_{145} = -\frac{1}{R_1}\left(2\left(1+\psi\right) - \frac{1+\psi}{b}\alpha\right)$$

$$Z_{146} = -\frac{1}{R_1}E_c M\left(1+\psi^2\right)$$

$$Z_{147} = -\frac{1}{R_1}\frac{a}{2R_e}\alpha$$

$$Z_{148} = -\frac{1}{R_1}\left(N_b S_c \alpha\right)$$

$$Z_{149} = -\frac{1}{R_1}\left(Q_h - \frac{a}{R_e}\frac{1}{\in}\right)$$

$$Z_{150} = -\frac{1}{R_1} N_t \alpha^2$$

$$Z_{151} = -2\frac{1}{R_1}(1+\psi)\frac{1}{\in}$$

$$Z_0 = \left(\frac{Z_{145}}{b} - \frac{Z_{146}}{(\alpha-2b)} + \frac{Z_{148}}{S_c} - \frac{Z_{149}}{\alpha} + \frac{Z_{150}}{\alpha} - \frac{Z_{151}}{(\alpha-b)} - \frac{\delta}{(\alpha-S_c)}\right)$$

$$Z_{153} = \left(\frac{Z_{144}}{b^2} + \frac{Z_{147}}{\alpha^3} - \frac{Z_{152}}{\alpha}\right)$$

$$Z_{154} = \frac{Z_{145}}{b(\alpha+b)}$$

$$Z_{155} = \frac{Z_{149}}{\alpha}$$

$$Z_{156} = \frac{Z_{148}}{S_c(\alpha+S_c)}$$

$$Z_{157} = \frac{Z_{146}}{2b(\alpha-2b)}$$

$$Z_{158} = \frac{Z_{150}}{2\alpha^2}$$

$$Z_{159} = \frac{Z_{151}}{b(\alpha-b)}$$

$$Z_{160} = \left(\frac{Z_{144}}{b} + \frac{Z_{147}}{\alpha^2} \right)$$

$$Z_{161} = \frac{Z_{147}}{2\alpha}$$

$$Z_{162} = -Z_{153} + Z_{154} + Z_{156} - Z_{157} + Z_{158} - Z_{159}$$

$$Z_{163} = \frac{aS_c}{R_e} + S_c\sigma - S_c{}^2 \left(\frac{1+\psi}{b} \right) - S_c$$

$$Z_{164} = S_c{}^2 \left(\frac{1+\psi}{b} \right) - 2S_c (1+\psi)$$

$$Z_{165} = \frac{aS_c{}^2}{2R_e}$$

$$Z_{166} = \alpha^2 N_{t1}$$

$$Z_{167} = \frac{Z_{164}}{b} + \frac{Z_{166}}{S_c - \alpha} + \frac{Z_{166}}{\alpha}$$

$$Z_{168} = \frac{Z_{165}}{S_c{}^3} - \frac{Z_{163}}{S_c{}^2} - \frac{Z_{167}}{S_c}$$

$$Z_{169} = \frac{Z_{165}}{S_c{}^2} - \frac{Z_{163}}{S_c}$$

$$Z_{170} = \frac{Z_{165}}{2}$$

$$Z_{171} = \frac{Z_{164}}{b(b+S_c)}$$

$$Z_{172}=\frac{Z_{166}}{\alpha\left(\alpha-S_c\right)}$$

$$Z_{173}=Z_{168}+Z_{171}+Z_{172}$$

3.2.7 Case 4: When the reaction is steady with chemical reaction of constant rate: $a=0$

3.2.7.1 Solution for case 4

In this case, equations (3.73) – (3.77) reduces to

$$\begin{aligned}&f'''+\frac{1}{b}\left(e^{-b\eta}-1\right)f''+\frac{\psi}{b}\left(e^{-b\eta}-1\right)f''+\eta\left(f'+g'\right)f''-2f'\left(f'+\frac{\eta}{2}f''\right)\\&-2g'\left(f'+\frac{\eta}{2}f''\right)-\Omega f'^2-\left(M+\gamma\right)f'+G_{r\theta}\left(1+\in\theta\right)+G_{r\phi}\phi=0\end{aligned}\tag{3.398}$$

$$\begin{aligned}&g'''+\frac{1}{b}\left(e^{-b\eta}-1\right)g''+\frac{\psi}{b}\left(e^{-b\eta}-1\right)g''+\eta\left(f'+g'\right)g''-2f'\left(g'+\frac{\eta}{2}g''\right)\\&-2g'\left(g'+\frac{\eta}{2}g''\right)-\Omega g'^2-\left(M+\gamma\right)g'=0\end{aligned}\tag{3.399}$$

$$\begin{aligned}&R_1\theta''+\frac{1}{b}\left(e^{-b\eta}-1\right)\theta'+\frac{\psi}{b}\left(e^{-b\eta}-1\right)\theta'+\eta\left(f'+g'\right)\theta'\\&-2\left(\left(\frac{1}{\in}+\theta\right)+\frac{\eta}{2}\theta'\right)\left(f'+g'\right)+E_cM\left(f'^2+g'^2\right)+N_b\theta'\phi'\\&+N_t\theta'^2+Q_h\left(1+\in\theta\right)+\delta\phi=0\end{aligned}\tag{3.400}$$

$$\begin{aligned}&\phi''+\frac{S_c}{b}\left(e^{-b\eta}-1\right)\phi'+\frac{S_c\psi}{b}\left(e^{-b\eta}-1\right)\phi'+S_c\eta\left(f'+g'\right)\phi'\\&-2S_c\left(\phi+\frac{\eta}{2}\phi'\right)\left(f'+g'\right)+N_{t1}\theta''-S_c\sigma\phi=0\end{aligned}\tag{3.401}$$

Again, rewriting equations (3.398) – (3.401) in the form:

$$f''' + bf'' + \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-b\right)f'' + \eta\left(f'+g'\right)f'' - 2f'\left(f'+\frac{\eta}{2}f''\right)$$
$$-2g'\left(f'+\frac{\eta}{2}f''\right) - \Omega f'^2 - \left(M+\gamma\right)f' + G_{r\theta}\left(1+\in\theta\right) + G_{r\phi}\phi = 0 \tag{3.402}$$

$$g''' + bg'' + \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-b\right)g'' + \eta\left(f'+g'\right)g''$$
$$-2f'\left(g'+\frac{\eta}{2}g''\right) - 2g'\left(g'+\frac{\eta}{2}g''\right) - \Omega g'^2 - \left(M+\gamma\right)g' = 0 \tag{3.403}$$

$$R_1\theta'' + \theta' + \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\theta' + \eta\left(f'+g'\right)\theta'$$
$$-2\left(\left(\frac{1}{\in}+\theta\right)+\frac{\eta}{2}\theta'\right)\left(f'+g'\right) + E_c M\left(f'^2+g'^2\right) \tag{3.404}$$
$$+N_b\theta'\phi' + N_t\theta'^2 + Q_h\left(1+\in\theta\right) + \delta\phi = 0$$

$$\phi'' + S_c\phi' + S_c\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\phi' + S_c\eta\left(f'+g'\right)\phi'$$
$$-2S_c\left(\phi+\frac{\eta}{2}\phi'\right)\left(f'+g'\right) + N_{t1}\theta'' - S_c\sigma\phi = 0 \tag{3.405}$$

Introducing an artificial parameter $\in$ in equations (3.402) – (3.405) gives

$$f''' + bf'' + \in\left(\begin{array}{l}\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-b\right)f'' + \eta\left(f'+g'\right)f'' - 2f'\left(f'+\frac{\eta}{2}f''\right)\\ -2g'\left(f'+\frac{\eta}{2}f''\right) - \Omega f'^2 - \left(M+\gamma\right)f' + G_{r\theta}\left(1+\in\theta\right) + G_{r\phi}\phi\end{array}\right) = 0 \tag{3.406}$$

$$g''' + bg'' + \in\left(\begin{array}{l}\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-b\right)g'' + \eta\left(f'+g'\right)g'' - 2f'\left(g'+\frac{\eta}{2}g''\right)\\ -2g'\left(g'+\frac{\eta}{2}g''\right) - \Omega g'^2 - \left(M+\gamma\right)g'\end{array}\right) = 0 \tag{3.407}$$

$$R_1\theta''+\theta'+\in\begin{pmatrix}\left(\left(\left(\dfrac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\theta'+\eta\left(f'+g'\right)\theta'-\right.\\ 2\left(\left(\dfrac{1}{\in}+\theta\right)+\dfrac{\eta}{2}\theta'\right)\left(f'+g'\right)+E_cM\left(f'^2+g'^2\right)+N_b\theta'\phi'+N_t\theta'^2+\\ Q_h\left(1+\in\theta\right)+\delta\phi\end{pmatrix}=0 \quad (3.408)$$

$$\phi''+S_c\phi'+\in\begin{pmatrix}S_c\left(\left(\dfrac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\phi'+S_c\eta\left(f'+g'\right)\phi'\\ -2S_c\left(\phi+\dfrac{\eta}{2}\phi'\right)\left(f'+g'\right)+N_{t1}\theta''-S_c\sigma\phi\end{pmatrix}=0 \quad (3.409)$$

Similarly, Simplifying, processing and collecting like powers of $\in$ gives the order zero and one equations with their respective boundary conditions given below:

$\in^0$:

$$\begin{aligned}&f_0'''+bf_0''=0\\&f_0(0)=0,\quad f_0'(0)=-1,\quad f_0'(\eta\to\infty)\to0\end{aligned} \quad (3.410)$$

$$\begin{aligned}&g_0'''+bg_0''=0\\&g_0(0)=0,\quad g_0'(0)=-\psi,\quad g_0'(\eta\to\infty)\to0\end{aligned} \quad (3.411)$$

$$\begin{aligned}&\theta_0''+\frac{1}{R_1}\theta_0'=0\\&\theta_0(0)=1,\quad \theta_0(\eta\to\infty)\to0\end{aligned} \quad (3.412)$$

$$\begin{aligned}&\phi_0''+S_c\phi_0'=0\\&\phi_0(0)=1,\quad \phi_0(\eta\to\infty)\to0\end{aligned} \quad (3.413)$$

$\in^1$:

$$f_1''' + bf_1'' + \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-b\right) f_0'' + \eta\left(f_0' + g_0'\right) f_0'' -$$
$$2 f_0'\left(f_0' + \frac{\eta}{2} f_0''\right) - 2g_0'\left(f_0' + \frac{\eta}{2} f_0''\right) - \Omega f_0'^2 - \left(M+\gamma\right) f_0' \qquad (3.414)$$
$$+G_{r\theta}\left(1+\in\theta_0\right) + G_{r\phi}\phi_0 = 0$$
$$f_1(0)=0, \quad f_1'(0)=0, \quad f_1'(\eta\to\infty)\to 0$$

$$f_1''' + bf_1'' + \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-b\right) f_0'' + \eta\left(f_0' + g_0'\right) f_0'' -$$
$$2 f_0'\left(f_0' + \frac{\eta}{2} f_0''\right) - 2g_0'\left(f_0' + \frac{\eta}{2} f_0''\right) - \Omega f_0'^2 - \left(M+\gamma\right) f_0' \qquad (3.415)$$
$$+G_{r\theta}\left(1+\in\theta_0\right) + G_{r\phi}\phi_0 = 0$$
$$f_1(0)=0, \quad f_1'(0)=0, \quad f_1'(\eta\to\infty)\to 0$$

$$\theta_1'' + \frac{1}{R_1}\theta_1' + \frac{1}{R_1}\left(\begin{array}{l} \left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\theta_0' + \eta\left(f_0'+g_0'\right)\theta_0' - \\ -2\left(\left(\frac{1}{\in}+\theta_0\right)+\frac{\eta}{2}\theta_0'\right)\left(f_0'+g_0'\right) + E_c M\left(f_0'^2 + g_0'^2\right) \\ +N_b\theta_0'\phi_0' + N_t\theta_0'^2 + Q_h\left(1+\in\theta_0\right) + \delta\phi_0 \end{array}\right) = 0 \qquad (3.416)$$
$$\theta_1(0)=0, \quad \theta_1(\eta\to\infty)\to 0$$

$$\phi_1'' + S_c\phi_1' + S_c\left(\left(\frac{1+\psi}{b}\right)\left(e^{-b\eta}-1\right)-1\right)\phi_0' + S_c\eta\left(f_0'+g_0'\right)\phi_0' -$$
$$2S_c\left(\phi_0 + \frac{\eta}{2}\phi_0'\right)\left(f_0'+g_0'\right) + N_{t1}\theta_0'' - S_c\sigma\phi_0 = 0 \qquad (3.417)$$
$$\phi_1(0)=0, \quad \phi_1(\eta\to\infty)\to 0$$

Similarly, following the same processes of case one above gives

$$\left.\begin{array}{l}
f(\eta)=\frac{1}{b}\left(e^{-b\eta}-1\right)+\in\begin{pmatrix} Z_{189}e^{-b\eta}-Z_{190}e^{-2b\eta}+Z_{191}\eta e^{-b\eta}-Z_{192}e^{-\alpha\eta} \\ +Z_{193}e^{-S_c\eta}+\frac{Z_{194}}{b}e^{-b\eta}+Z_{195} \end{pmatrix} \\
g(\eta)=\frac{\psi}{b}\left(e^{-b\eta}-1\right)+\in\left(Z_{201}e^{-b\eta}+Z_{202}\eta e^{-b\eta}-Z_{203}e^{-2b\eta}+\frac{Z_{204}}{b}e^{-b\eta}+Z_{205}\right) \\
\theta(\eta)=e^{-\alpha\eta}+\in\begin{pmatrix} -Z_{220}\eta e^{-\alpha\eta}-Z_{213}e^{-\alpha\eta}+Z_{214}e^{-(\alpha+b)\eta}-Z_{217}e^{-2b\eta} \\ +Z_{216}e^{-(\alpha+S_c)\eta}+Z_{218}e^{-2\alpha\eta}-Z_{219}e^{-b\eta}-Z_{221}e^{-\alpha\eta} \end{pmatrix} \\
\phi(\eta)=e^{-S_c\eta}+\in\begin{pmatrix} Z_{227}e^{-S_c\eta}+Z_{228}\eta e^{-S_c\eta}+Z_{229}e^{-(b+S_c)\eta}+Z_{230}e^{-\alpha\eta} \\ +Z_{231}e^{-(S_c+\alpha)\eta}-Z_{232}e^{-S_c\eta} \end{pmatrix}
\end{array}\right\} \qquad (3.418)$$

where

$$Z_{181}=(M+\gamma)-b\left(\frac{1+\psi}{b}\right)-b^2$$

$$Z_{182}=b\left(\frac{1+\psi}{b}\right)-2(1+\psi)$$

$$Z_{183}=\left(\frac{G_{r\theta}\in}{b-\alpha}+\frac{G_{r\theta}}{b}+\frac{Z_2}{b}+\frac{G_{r\phi}}{b-S_c}\right)$$

$$Z_{184}=\frac{Z_{181}}{b^2}-\frac{Z_{183}}{b}$$

$$Z_{185}=\frac{Z_{182}}{2b^2}$$

$$Z_{186}=\frac{Z_{181}}{b}$$

$$Z_{187} = \frac{G_{r\theta} \in}{\alpha(b-\alpha)}$$

$$Z_{188} = \frac{G_{r\phi}}{S_c(b-S_c)}$$

$$Z_{189} = -\frac{Z_{184}}{b} - \frac{Z_{186}}{b^2}$$

$$Z_{190} = \frac{Z_{185}}{2b}$$

$$Z_{191} = -\frac{Z_{186}}{b}$$

$$Z_{192} = \frac{Z_{187}}{\alpha}$$

$$Z_{193} = \frac{Z_{188}}{S_c}$$

$$Z_{194} = Z_{184} + Z_{185} + Z_{187} + Z_{188}$$

$$Z_{195} = Z_{190} + Z_{192} - Z_{189} + Z_{193} - \frac{Z_{194}}{b}$$

$$Z_{196} = \frac{a\psi}{R_e} - (\psi + \psi^2) - b^2\psi + (M + \gamma)\psi$$

$$Z_{197} = (\psi + \psi^2) - \Omega\psi^2$$

$$Z_{198} = \frac{Z_{196}}{b^2} - \frac{Z_{197}}{b^2}$$

$$Z_{199} = \frac{Z_{196}}{b}$$

$$Z_{200} = \frac{Z_{197}}{2b^2}$$

$$Z_{201} = -\frac{Z_{198}}{b} - \frac{Z_{199}}{b^2}$$

$$Z_{202} = -\frac{Z_{199}}{b}$$

$$Z_{203} = \frac{Z_{200}}{2b}$$

$$Z_{204} = Z_{198} + Z_{200}$$

$$Z_{205} = Z_{203} - Z_{201} - \frac{Z_{204}}{b}$$

$$Z_{206} = -\frac{1}{R_1}\left(Q_h \in + \frac{1+\psi}{b}\alpha + \alpha\right)$$

$$Z_{207} = -\frac{1}{R_1}\left(2\left(1+\psi\right) - \frac{1+\psi}{b}\alpha\right)$$

$$Z_{208} = -\frac{1}{R_1} E_c M \left(1+\psi^2\right)$$

$$Z_{209} = -\frac{1}{R_1}\left(\delta\left(e-2\right) + N_b S_c \alpha\right)$$

$$Z_{210} = -\frac{1}{R_1}\left(Q_h\right)$$

$$Z_{211} = -\frac{1}{R_1} N_t \alpha^2$$

$$Z_{212} = -2\frac{1}{R_1}\left(1+\psi\right)\frac{1}{\in}$$

$$Z_0 = \left(\frac{Z_{207}}{b} - \frac{Z_{208}}{\left(\alpha - 2b\right)} + \frac{Z_{209}}{S_c} - \frac{Z_{210}}{\alpha} + \frac{Z_{211}}{\alpha} - \frac{Z_{212}}{\left(\alpha - b\right)} - \frac{\delta}{\left(\alpha - S_c\right)} \right)$$

$$Z_{213} = \left(\frac{Z_{206}}{b^2} - \frac{Z_0}{\alpha} \right)$$

$$Z_{214} = \frac{Z_{207}}{b\left(\alpha + b\right)}$$

$$Z_{215} = \frac{Z_{210}}{\alpha}$$

$$Z_{216} = \frac{Z_{209}}{S_c\left(\alpha + S_c\right)}$$

$$Z_{217} = \frac{Z_{208}}{2b\left(\alpha - 2b\right)}$$

$$Z_{218} = \frac{Z_{211}}{2\alpha^2}$$

$$Z_{219} = \frac{Z_{212}}{b\left(\alpha - b\right)}$$

$$Z_{220} = \frac{Z_{206}}{b}$$

$$Z_{221} = -Z_{213} + Z_{214} + Z_{216} - Z_{217} + Z_{218} - Z_{219}$$

$$Z_{222} = S_c\sigma - S_c{}^2\left(\frac{1+\psi}{b}\right) - S_c$$

$$Z_{223} = S_c{}^2\left(\frac{1+\psi}{b}\right) - 2S_c\left(1+\psi\right)$$

$$Z_{224} = \alpha^2 N_{t1}$$

$$Z_{225} = S_c\sigma$$

$$Z_{226} = \frac{Z_{223}}{b} + \frac{Z_{224}}{S_c - \alpha} + \frac{Z_{225}}{\alpha}$$

$$Z_{227} = -\frac{Z_{222}}{S_c{}^2} - \frac{Z_{226}}{S_c}$$

$$Z_{228} = -\frac{Z_{222}}{S_c}$$

$$Z_{229} = \frac{Z_{223}}{b\left(b+S_c\right)}$$

$$Z_{230} = \frac{Z_{224}}{\alpha\left(\alpha - S_c\right)}$$

$$Z_{231} = \frac{Z_{225}}{\alpha\left(\alpha - S_c\right)}$$

$$Z_{232} = Z_{227} + Z_{229} + Z_{230} + Z_{231}$$

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

4.1 Analysis of Results

In the analysis, The criteria for the existence of unique solutions of the model was established, this is to show that the solutions of the model formulated depend continously on the initial and boundary conditions; that is to say the model is well posed. Also we examined the properties of solution of the model formulated, this is to show the behavior of the solution when values are assigned to some key parameters of the model. We solved the equations using iteration perturbation method where details can be found in (He, 2006); this is to see the effect of parameters involved on the concentration, temperature and velocities. Finally, we examined the effect local Reynold number R_e, prandtl number P_r, ratio parameter ψ, radiation parameter R, Frank-kamenetskii parameter δ, magnetic parameter M, thermal grashof number G_{r_θ}, activation energy parameter $\in$, porosity parameter Ω, unsteadiness parameter a, heat source Q_T, concentration grashof number G_{r_ϕ},Schmidt number S_c, constant number b, Brownian diffusion parameter N_b, thermopheresis parameter N_t, permeability parameter γ, Ecket number E_c, concentration chemical reaction parameter σ on the steady and unsteady state problems. Solutions given by equation (3.354), (3.376), (3.397) and (3.418) were computed and simulated using computer symbolic algebraic package MAPLE 17. The results obtained from the method are shown in Figures 4.1 to 4.33 for case 1 and Figures 4.34 to 4.40 for case 2, 4.41 to 4.47 for case 3 and 4.48 to 4.50 for case 1

The graphical illustrations of the velocity profiles $f(\eta)$, $g(\eta)$, $f'(\eta)$ and $g'(\eta)$, temperature profile $\theta(\eta)$ and concentration profile $\phi(\eta)$ for different physical parameters which includes, local Ryenold number R_e, prandtl number P_r, ratio parameter ψ, radiation parameter R , Frank-kamenetskii parameter δ, magnetic parameter M, thermal grashof number G_{r_θ}, activation energy parameter $\in$, porosity parameter Ω, unsteadiness parameter a, heat source Q_h, concentration grashof number G_{r_ϕ} ,schmidt number S_c, constant number b, Brownian diffusion parameter N_b, thermopheresis parameter N_t, permeability parameter γ, Ecket number E_c, concentration chemical reaction parameter σ with the aid of Maple 17 for four cases.

4.1.1 Graphs of Case 1: Transient state with Arrhenius chemical reaction

The graphical illustrations for the transient state with Arrhenius chemical reaction are presented in Figures 4.1 to 4.33. Comparison between analytical and numerical results is also presented in Table 4.1. The computations were done for the values of

$$R_e = 0.1,\ R = 10,\ \mathrm{Pr} = 0.71,\ S_c = 0.22,\ G_{r\theta} = 0.1,\ G_{r\phi} = 0.1,\ \psi = 1,\ \gamma = 1, a = -1, \delta = 0.1,$$
$$\Omega = 1,\ \in = 0.01,\ \mathrm{N}_b = 0.1,\ \mathrm{N}_t = 0.3,\ E_c = 0.1, Q = 0.2,\ \sigma = 0.1,\ \mathrm{M} = 1\ and\ b = 0.2$$

Table 4.1: Comparison between iterative perturbation method and numerical results

η	$f'(\eta)$ IPM Results	$f'(\eta)$ Numerical Results	$\left\| f'_{numer} - f'_{pertu} \right\|$
0	-1	-1	0
1	0.8486088699	0.1404504484	7.082 x10^{-1}
2	0.7000842521	0.1683682918	5.317 x10^{-1}
3	0.5600638028	0.1583546596	4.017 x10^{-1}
4	0.4323550159	0.1540526436	2.783 x10^{-1}
5	0.3192471021	0.1529084507	1.663 x10^{-1}
6	0.2218070827	0.1526534215	6.915 x10^{-2}
7	0.1401514746	0.1521795550	1.203 x10^{-2}
8	0.0736882343	0.1510442569	7.736 x10^{-2}
9	0.0213262897	0.1491157112	1.278 x10^{-1}
10	0.0183479204	0.1463876809	1.280 x10^{-1}
11	0.0469259176	0.1428990509	9.597 x10^{-2}
12	0.0660625567	0.1387001841	7.264 x10^{-2}
13	0.0773884586	0.1338294798	5.644 x10^{-2}
14	0.0824462359	0.1282728465	4.583 x10^{-2}
15	0.0826470683	0.1218691242	3.922 x10^{-2}
16	0.0792442096	0.1140906653	3.485 x10^{-2}
17	0.0733201633	0.1035584878	3.024 x10^{-2}
18	0.0657845281	0.0870266879	2.124 x10^{-2}
19	0.0372687238	0.0573798328	2.011 x10^{-2}
20	0.0486930351	0	4.869 x10^{-2}

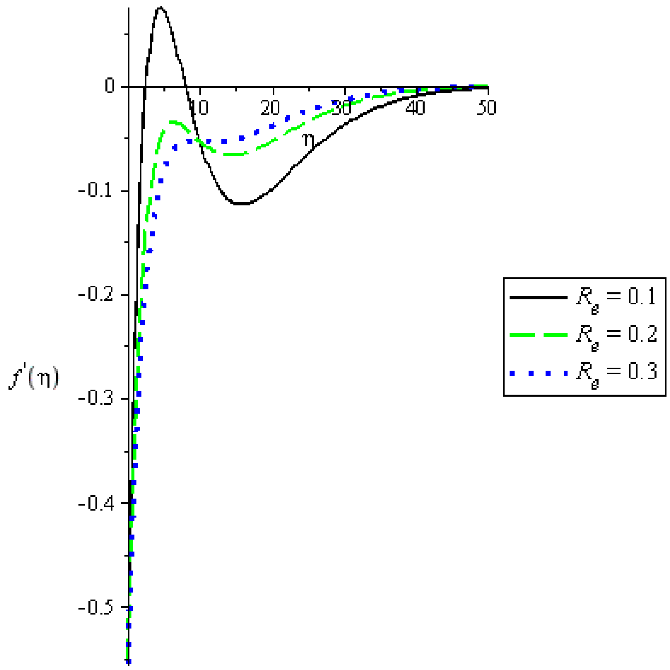


Figure 4.1: Effect of R_e **on** $f'(\eta)$

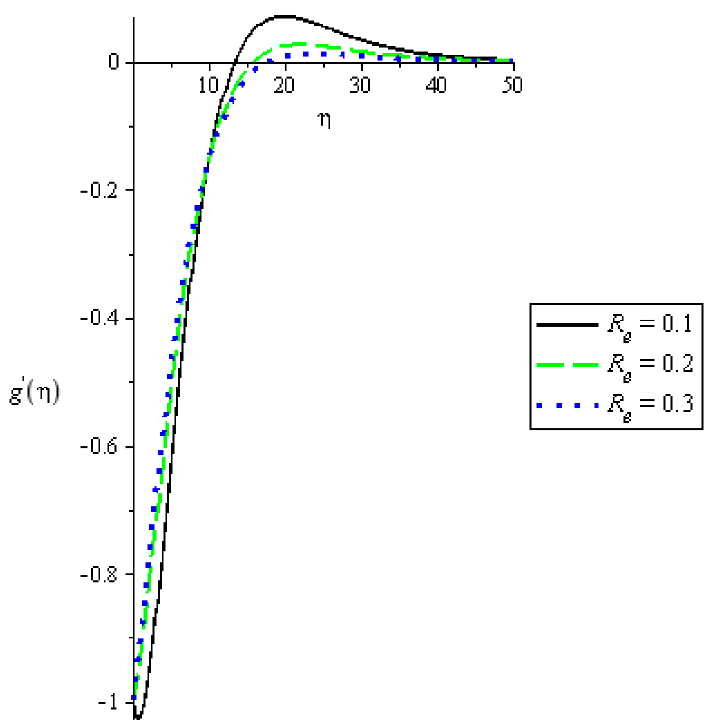


Figure 4.2: Effect of R_e **on** $g'(\eta)$

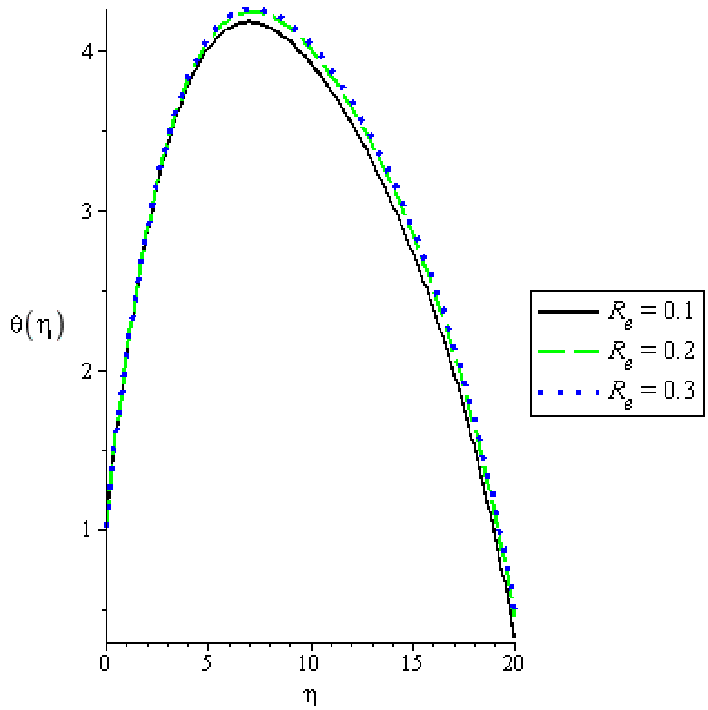


Figure 4.3: Effect of R_e on $\theta(\eta)$

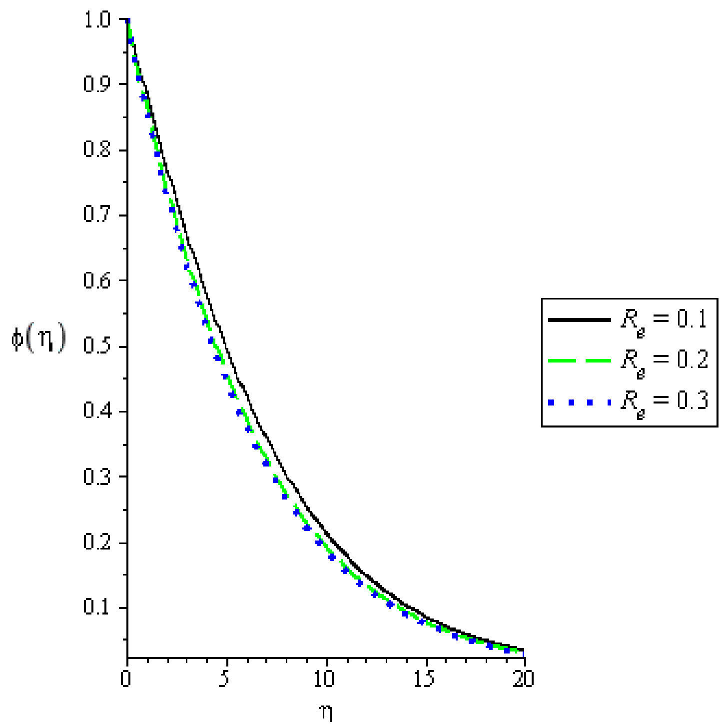


Figure 4.4: Effect of R_e on $\phi(\eta)$

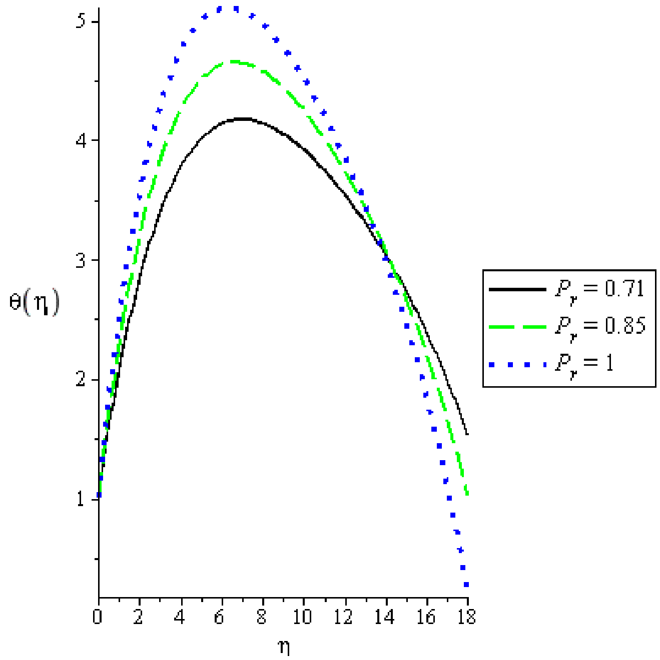


Figure 4.5: Effect of $\Pr$ **on** $\theta(\eta)$

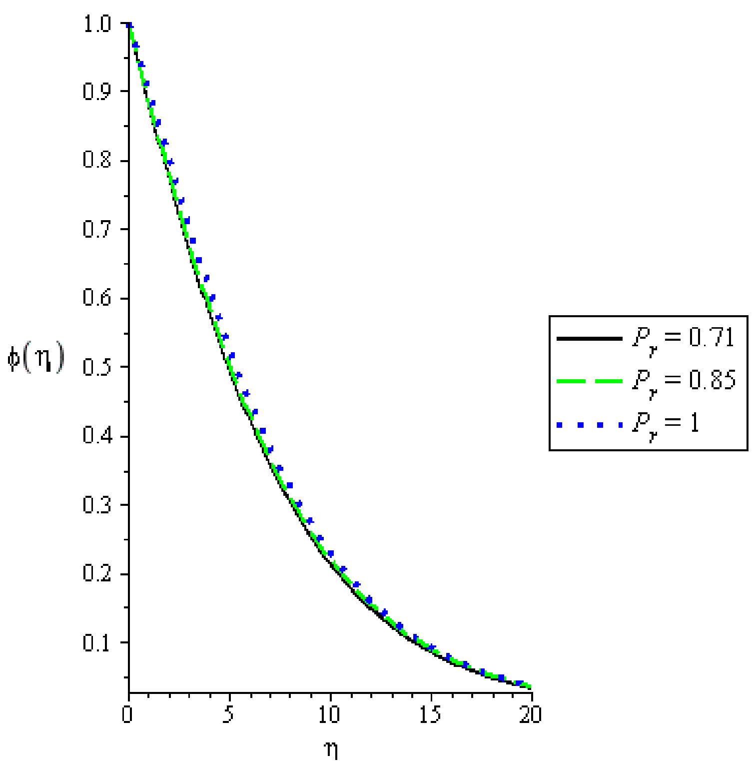


Figure 4.6: Effect of $\Pr$ **on** $\phi(\eta)$

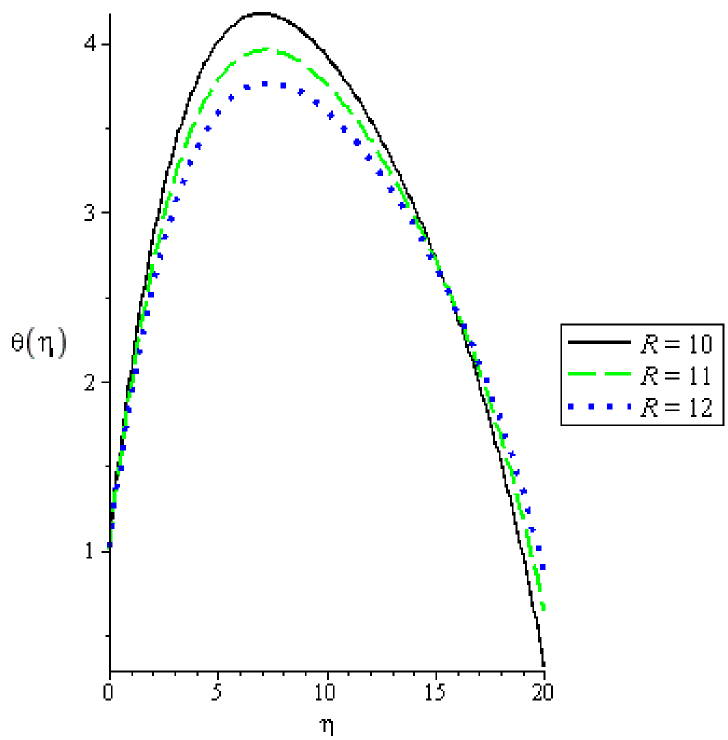


Figure 4.7: Effect of R **on** $\theta(\eta)$

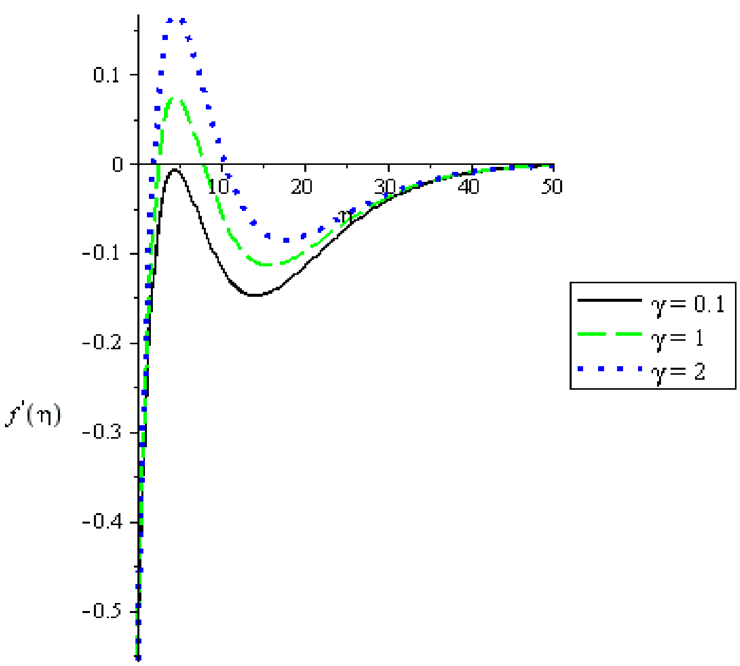


Figure 4.8: Effect of γ **on** $f'(\eta)$

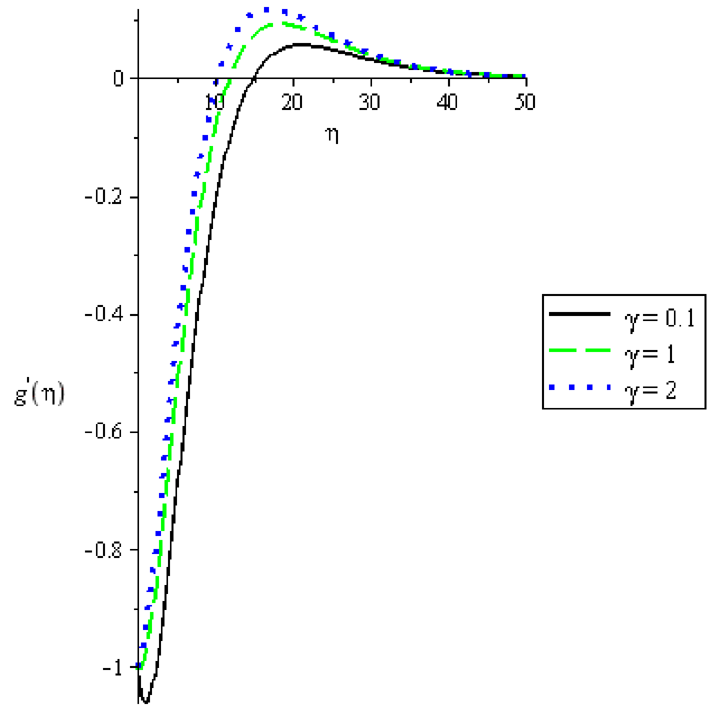


Figure 4.9: Effect of γ **on** $g'(\eta)$

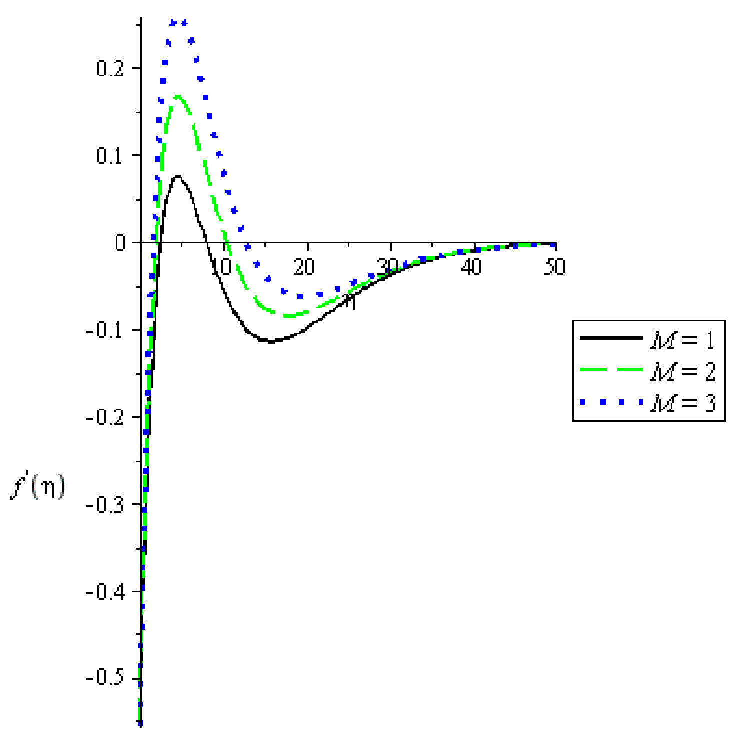


Figure 4.10: Effect of M **on** $f'(\eta)$

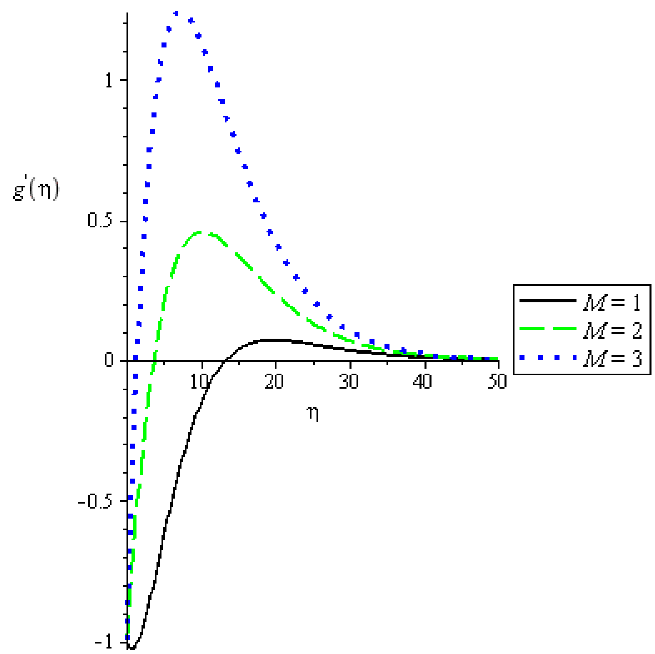


Figure 4.11: Effect of M **on** $g'(\eta)$

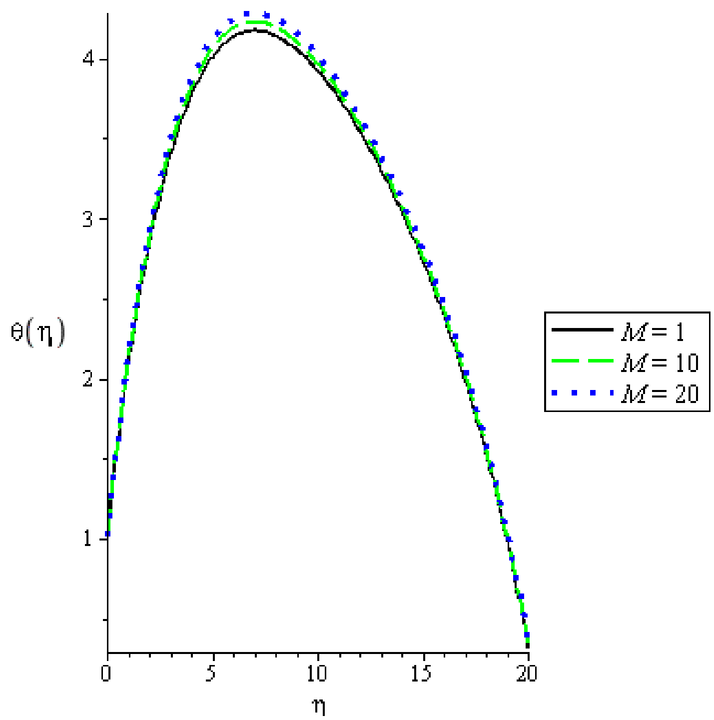


Figure 4.12: Effect of M **on** $\theta(\eta)$

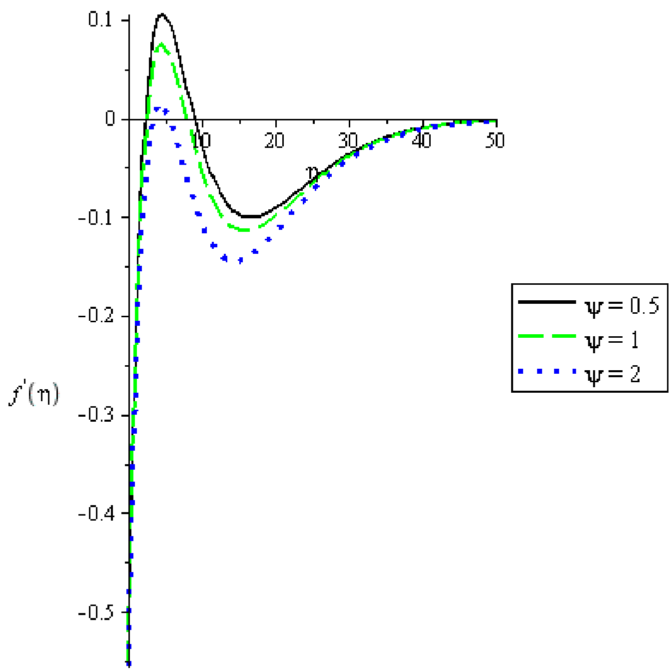


Figure 4.13: Effect of ψ **on** $f'(\eta)$

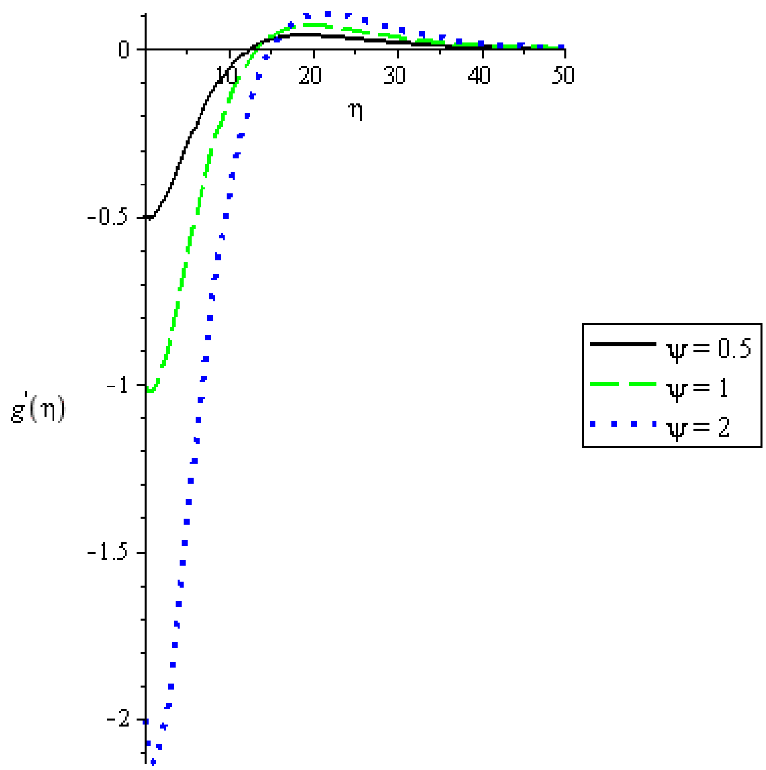


Figure 4.14: Effect of ψ **on** $g'(\eta)$

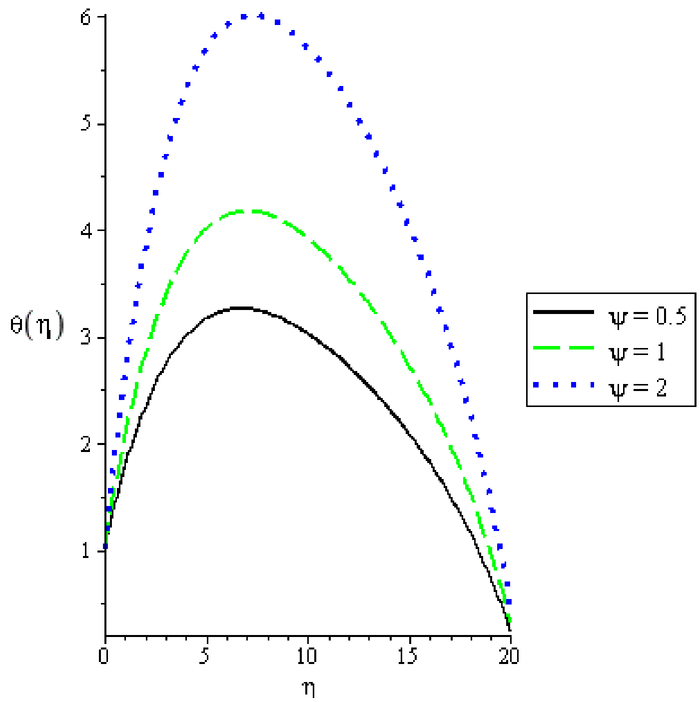


Figure 4.15: Effect of ψ **on** $\theta(\eta)$

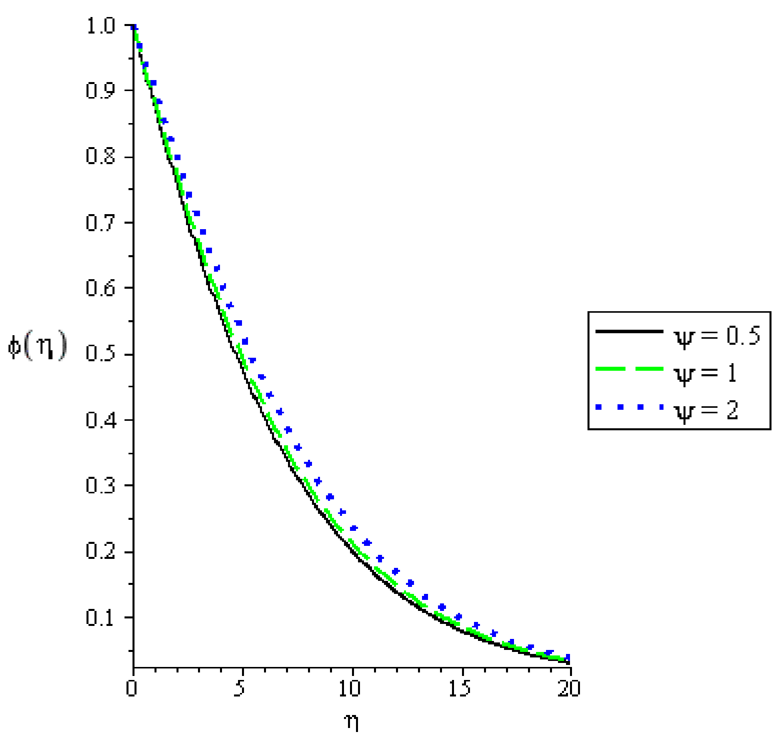


Figure 4.16: Effect of ψ **on** $\phi(\eta)$

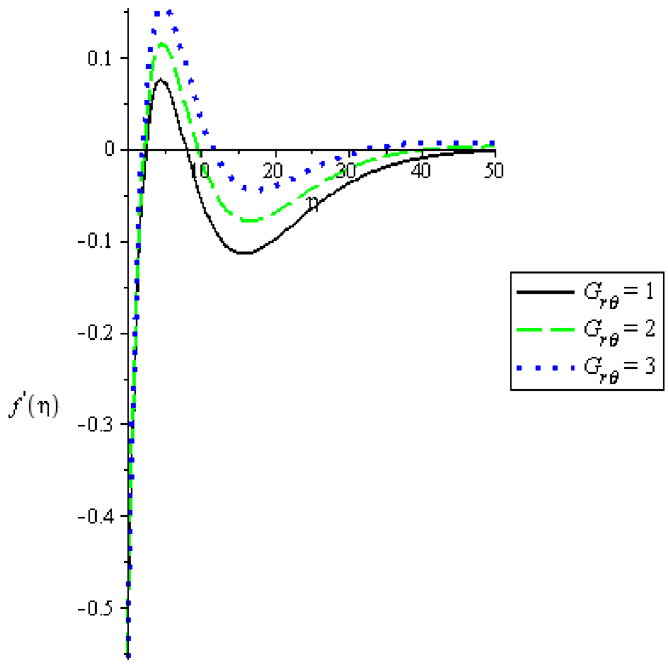


Figure 4.17: Effect of $G_{r\theta}$ on $f'(\eta)$

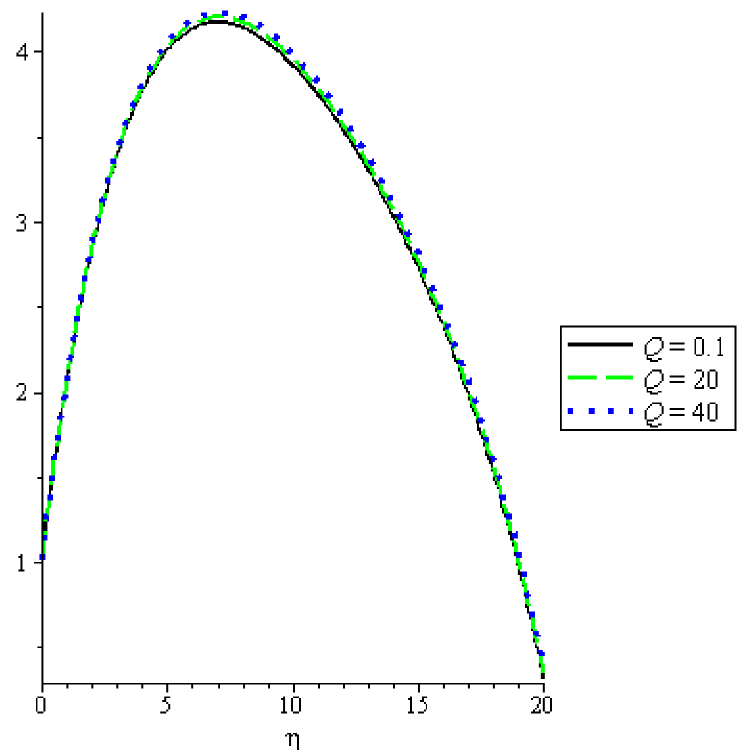


Figure 4.18: Effect of Q on $\theta(\eta)$

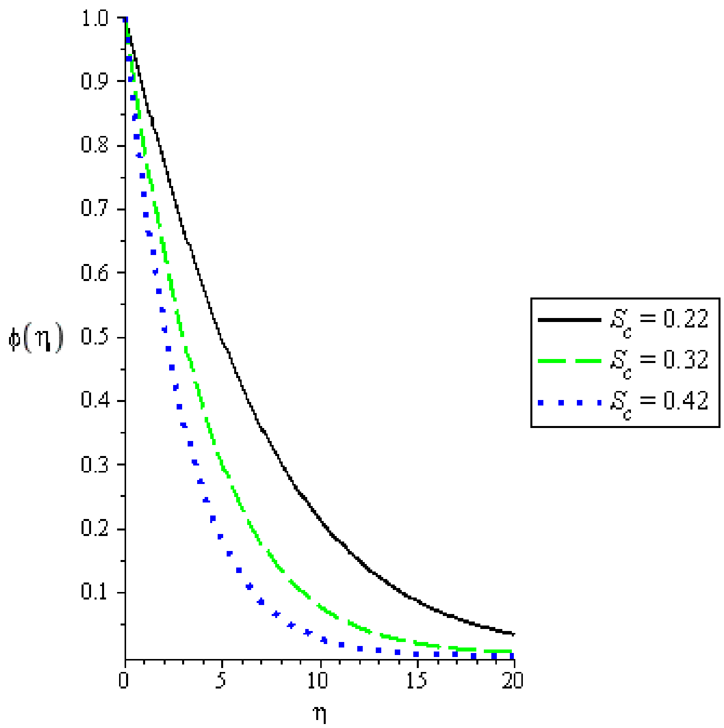


Figure 4.19: Effect of S_c on $\phi(\eta)$

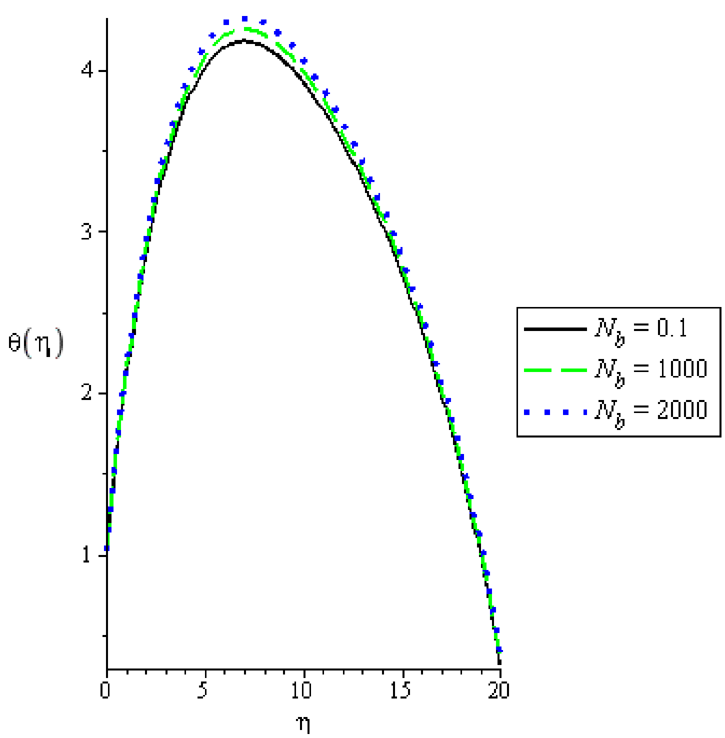


Figure 4.20: Effect of N_b on $\theta(\eta)$

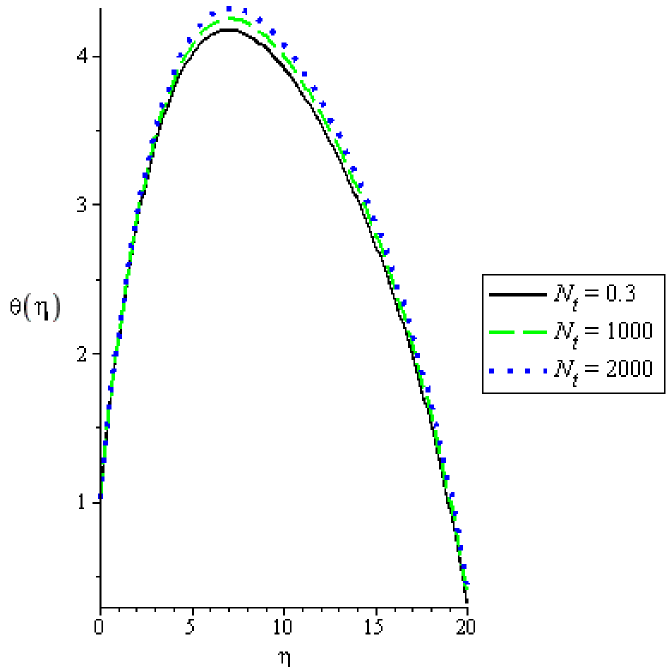


Figure 4.21: Effect of N_t on $\theta(\eta)$

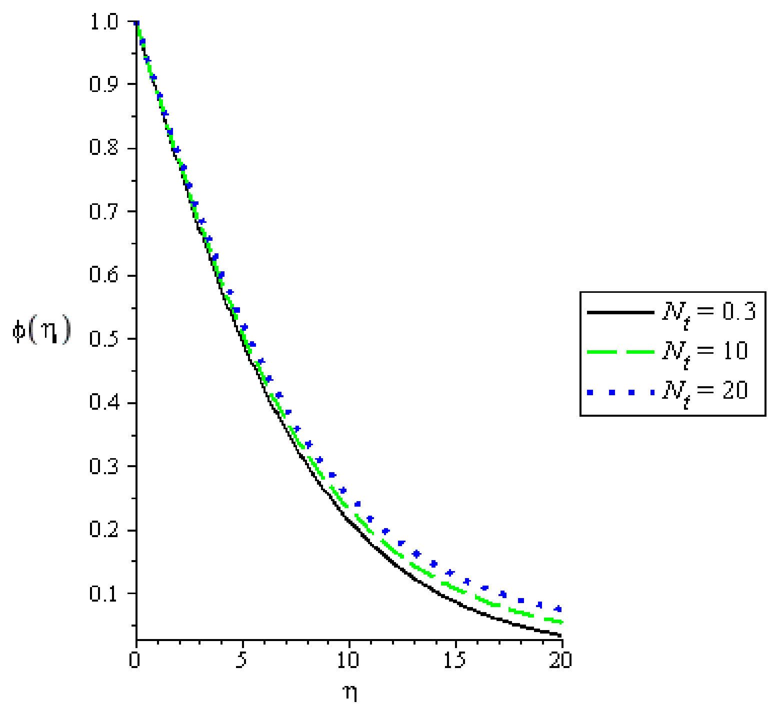


Figure 4.22: Effect of N_t on $\phi(\eta)$

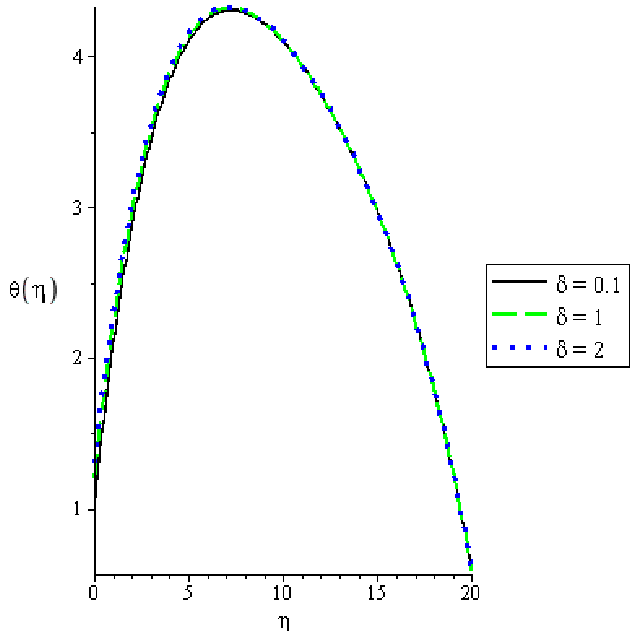


Figure 4.23: Effect of δ on $\theta(\eta)$

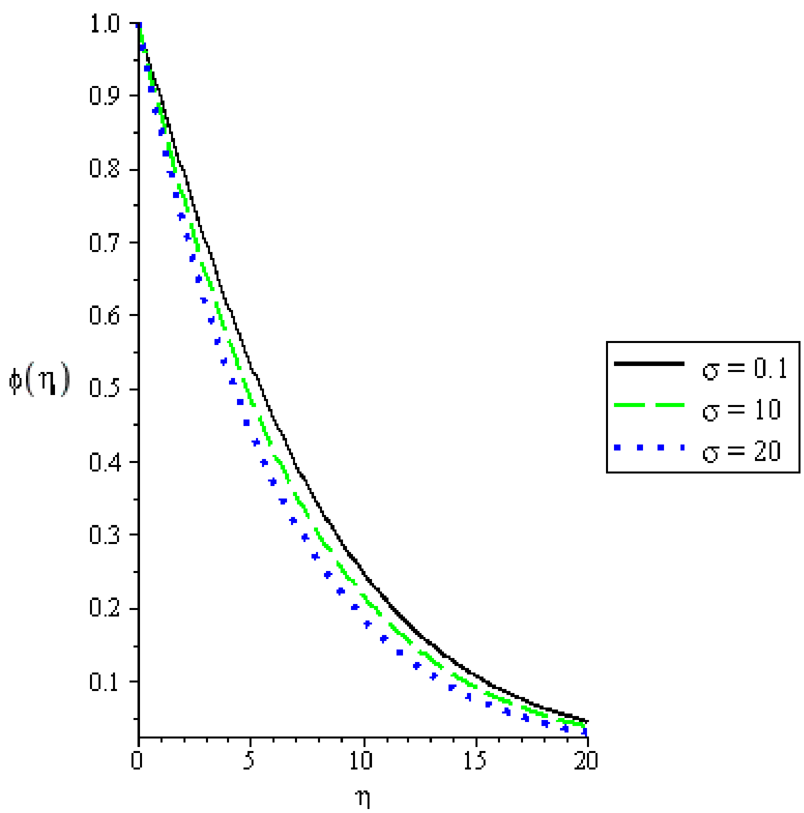


Figure 4.24: Effect of σ on $\phi(\eta)$

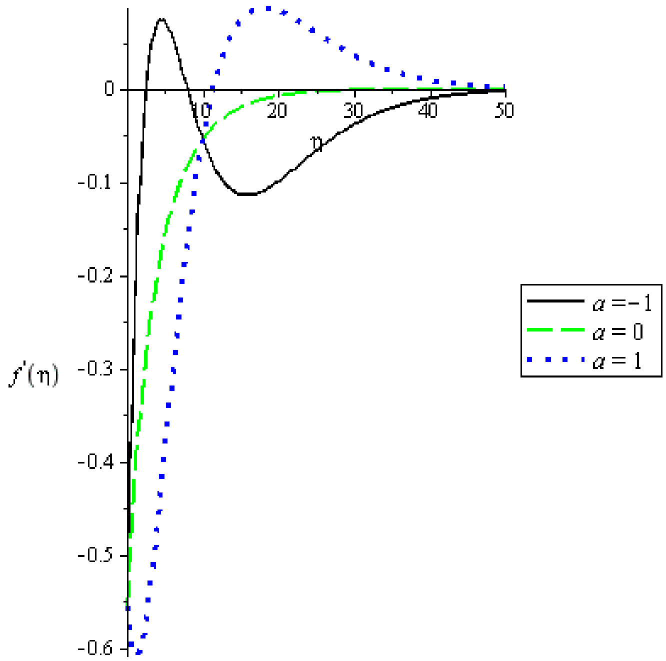


Figure 4.25: Effect of a **on** $f'(\eta)$

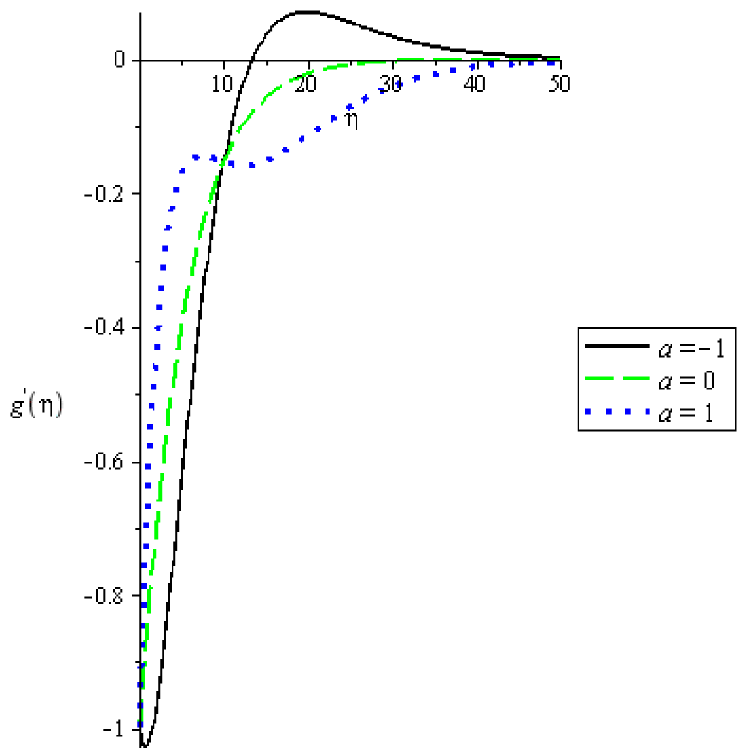


Figure 4.26: Effect of a **on** $g'(\eta)$

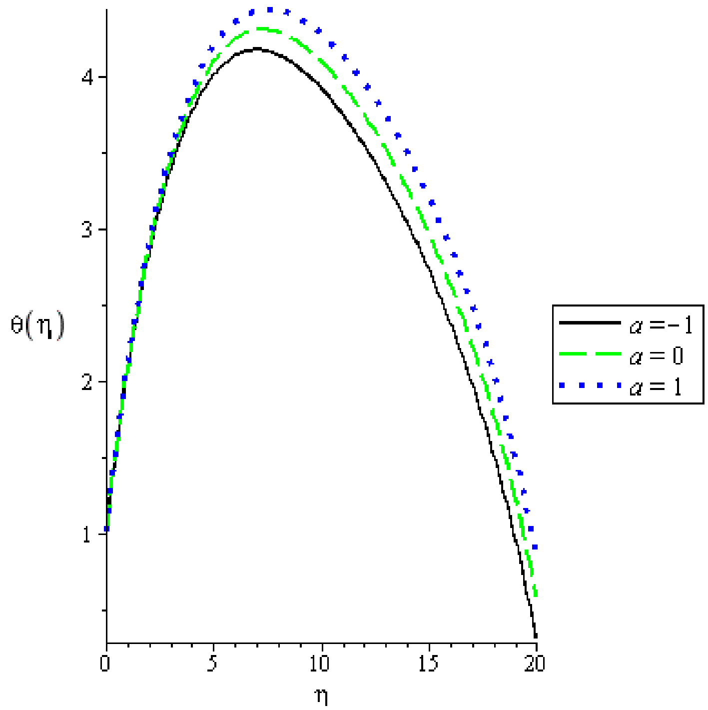


Figure 4.27: Effect of a **on** $\theta(\eta)$

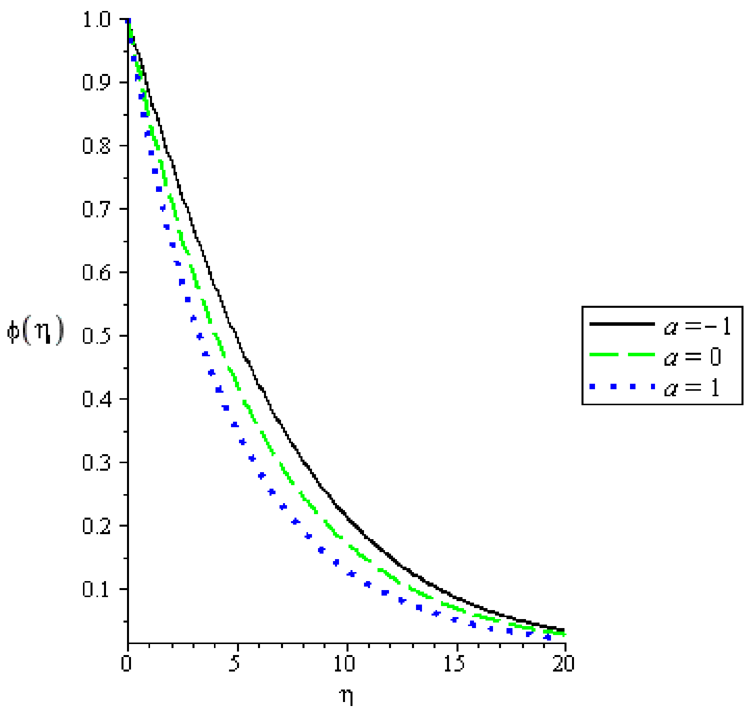


Figure 4.28: Effect of a **on** $\phi(\eta)$

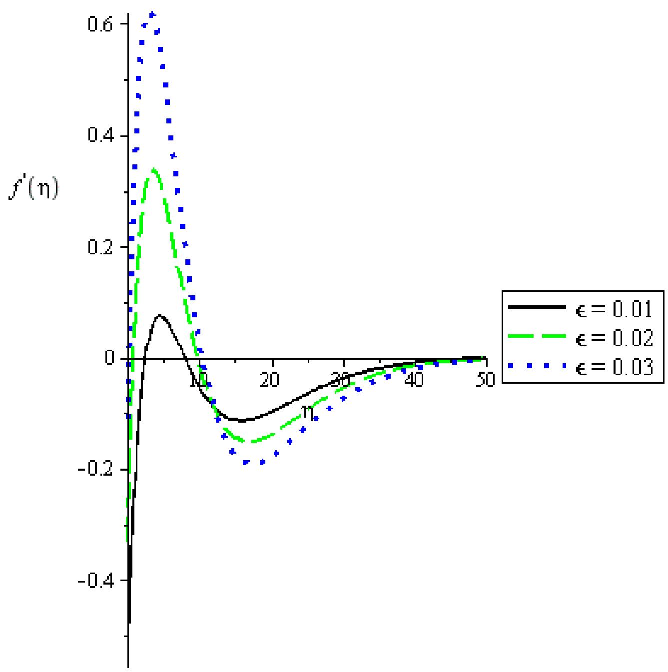


Figure 4.29: Effect of $\in$ **on** $f'(\eta)$

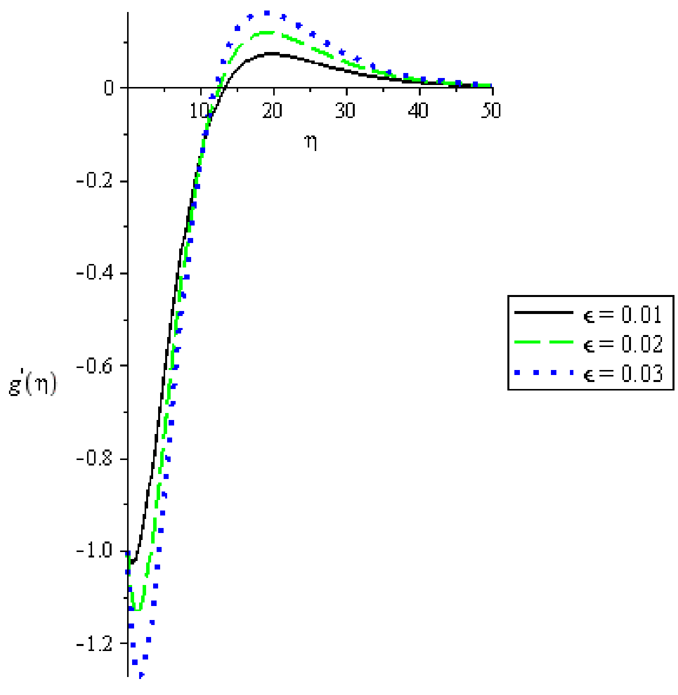


Figure 4.30: Effect of $\in$ **on** $g'(\eta)$

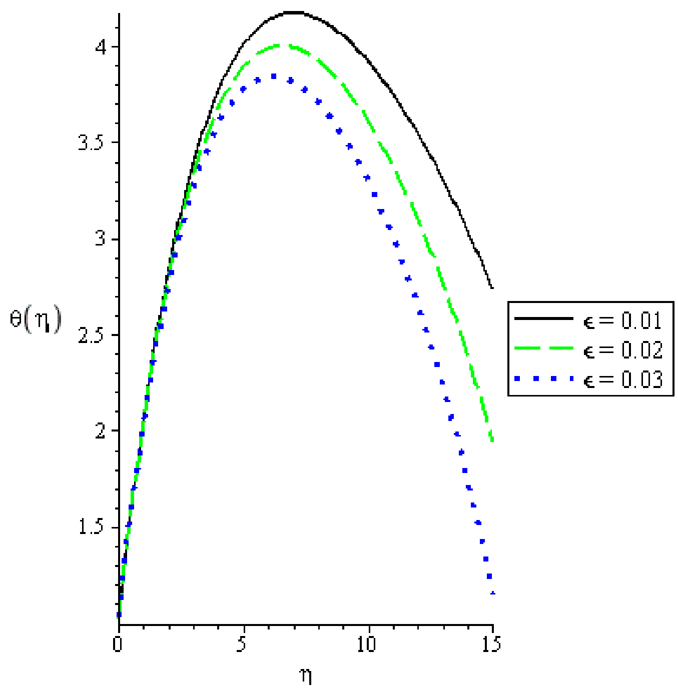


Figure 4.31: Effect of $\in$ on $\theta(\eta)$

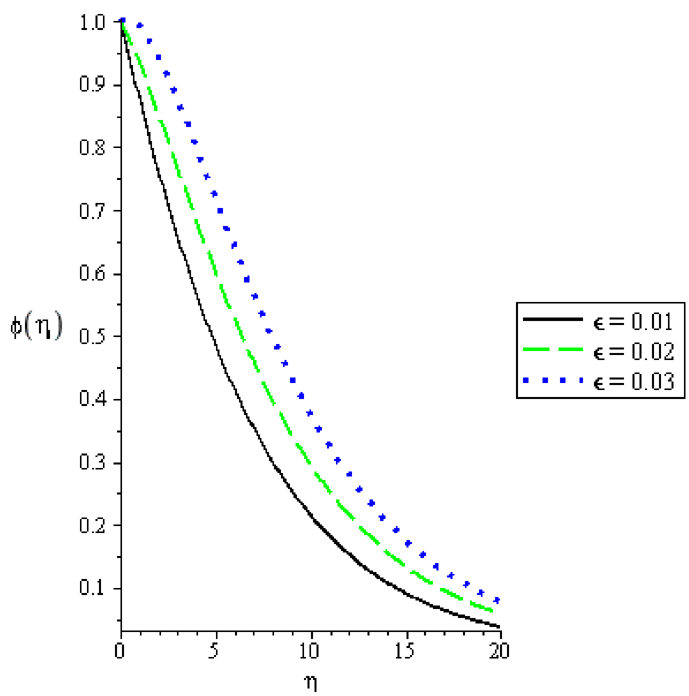


Figure 4.32: Effect of $\in$ on $\phi(\eta)$

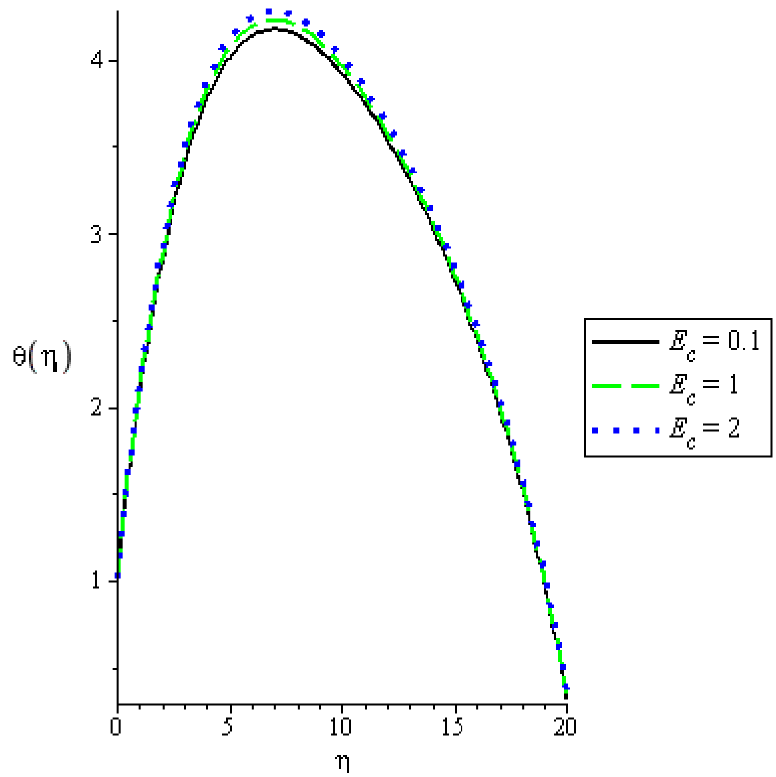


Figure 4.33: Effect of E_c on $\theta(\eta)$

4.1.1.1 Analysis of Results and Discussion for Transient state with Arrhenius chemical reaction

Table 4.1 above demonstrates agreement between the results obtained using iterative perturbation method and purely fourth-order Runge Kutta numerical integration approach coupled with shooting method at small and moderate parameter values. Generally, the difference is of order 10^{-1} and 10^{-2}.

Figures 4.1 to 4.2 display the profiles of dimensionless x and y components of velocity for different values of local Reynolds number R_e.It was observed that as the local renold number increases, the velocity profiles decreases along x and increase in y direction respectively and consequently decreases boundary layer thickness. Figures 4.3 to 4.4

presents the effects of local renolds number on both temperature and concentration profiles. Increase in local renolds number causes the temperature to increase and decrease the concentration with the boundary layer thickness reducing. Figures 4.5 to 4.6 gives the effect of prandtl number $\Pr$ on both temperature and concentration profiles. It was noted that increase in prandtl number enhances both temperature and concentration. The impact of radiation parameter R on temperature profiles is depicted by figures 4.7. An increasing value of radiation parameter decreases temperature . Figures 4.8 to 4.9 illustrate the influence of permeability parameter γ along the primary and the secondary velocity profiles. For increasing values of permeable parameter, all the velocity profiles increases. Figures 4.10 to 4.12 portray the effects of magnetic parameter M on the velocities and temperature profiles. Incremental behavior was observed on the velocities and temperature profile for increasing values of magnetic parameter. Figures 4.13 to 4.16 depict the velocities, temperature and concentration profiles for various value of velocity ratio ψ . An augmentation in ψ indicates an decrease in the velocity $f'(\eta)$ and $g'(\eta)$. The temperature and concentration also increases with increasing values of ψ . Figure 4.17 show the distribution of primary velocity for increasing values of $G_{r\theta}$. An increase is observed on the velocity. Figure 4.18 displays the impact of heat source strength on the temperature profile which increases for increasing values of Q . Figures 4.19 illustrate the concentration distribution for increasing values of S_c . Increasing values of S_c provides reduction in concentration. Figures 4.20 reveal the impact of increasing Brownian diffusion parameter

N_b on temperature. An increase was seen in the temperature. In figures 4.21 to 4.22, Increase is seen on both temperature and concentration profiles with increasing values of N_t. An increasing impact was found on the temperature and decreasing effect on the concentration profiles with increase in the Frank-kamenetskii and chemical reaction parameters δ and σ in figures 4.23 to 4.24. Figures 4.25 to 4.28 show the influence of unsteadiness parameter on the velocity, temperature and concentration profiles. For increasing values of the unsteadiness parameter, the primary velocity reduce and the secondary velocity increase. Also for increasing value of a, the temperature increases and concentration decreases. Figures 4.29 to 4.32 display the effects of activation energy parameter $\in$ on velocities, temperature and concentration profiles. An increasing impact is seen along the velocity in the x axis with opposite impact on the y axis velocity, and tempreture reduces with reverse effect on the concentration for increasing values of activating energy parameter. Figure 4.33 gives the effect of ecket number. the temperature increases with increasing values E_c.

4.1.2. Steady state with Arrhenius chemical reaction

The graphical responses for the steady state with Arrhenius chemical reaction are provided in the figures 4.34 to 4.40. The simulation were carried out with values of

$$R_e = 0.1,\ R = 10,\ \Pr = 0.71,\ S_c = 0.22,\ G_{r\theta} = 0.1,\ G_{r\phi} = 0.1,\ \psi = 1,\ \gamma = 1,$$
$$a = 0, \delta = 0.1,\ \Omega = 1,\ \in = 0.01,\ \mathrm{N}_b = 0.1,\ \mathrm{N}_t = 0.3,\ E_c = 0.1 Q = 0.2,$$
$$\sigma = 0.1,\ \mathrm{M} = 1\ and\ b = 0.2$$

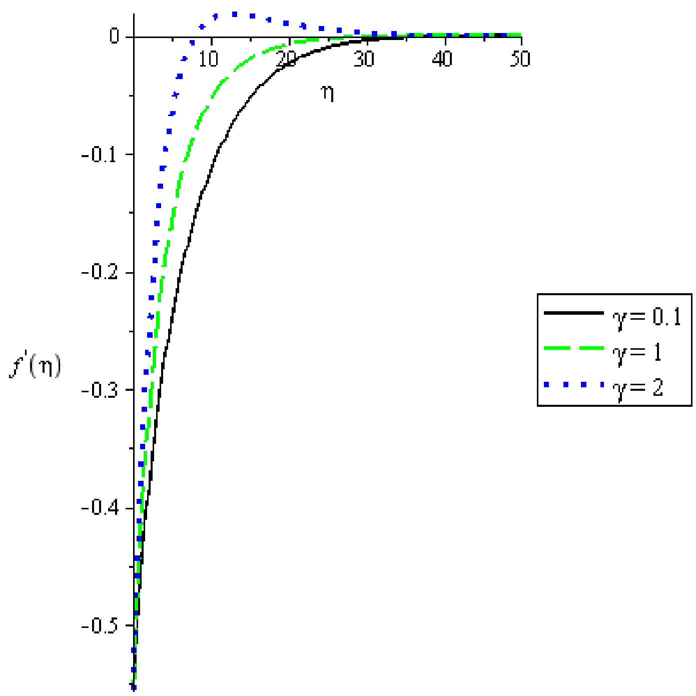


Figure 4.34: Effect of γ **on** $f'(\eta)$

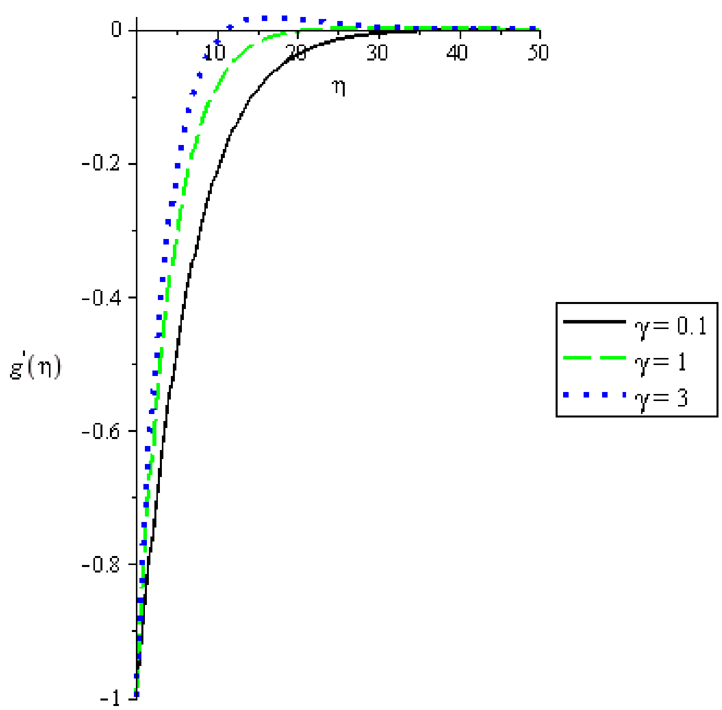


Figure 4.35: Effect of γ **on** $g'(\eta)$

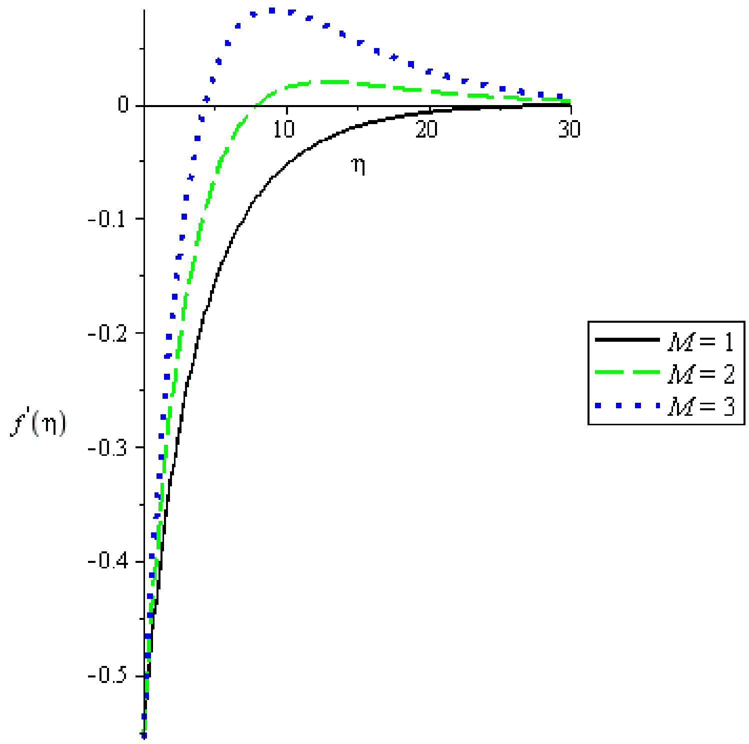


Figure 4.36: Effect of M **on** $f'(\eta)$

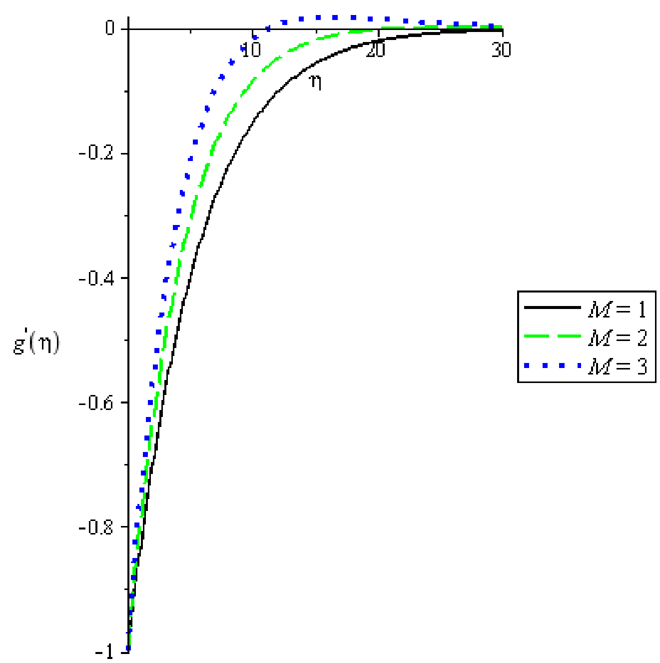


Figure 4.37: Effect of M **on** $g'(\eta)$

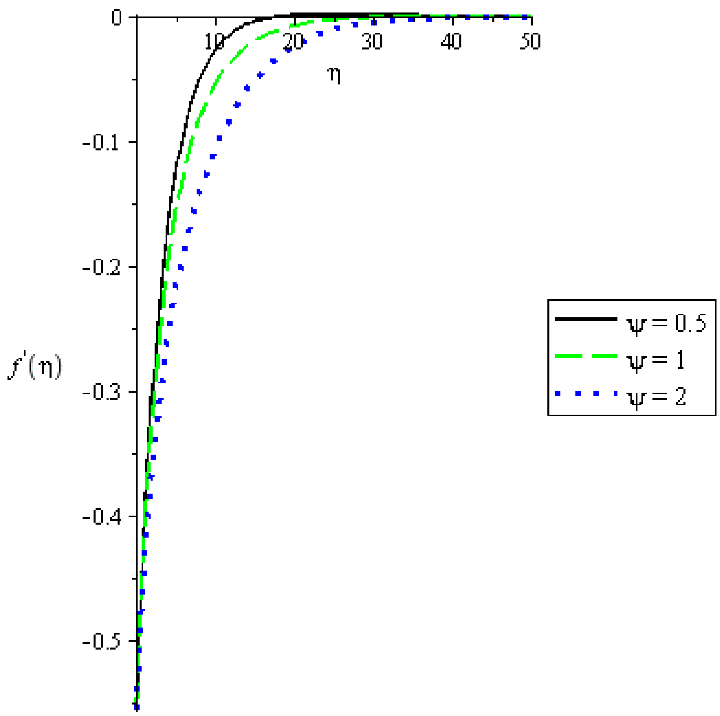


Figure 4.38: Effect of ψ on $f'(\eta)$

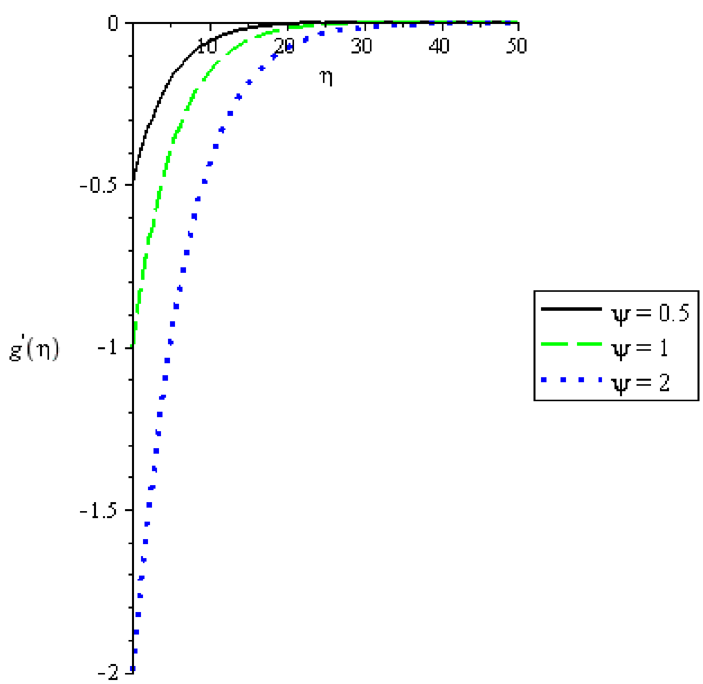


Figure 4.39: Effect of ψ on $g'(\eta)$

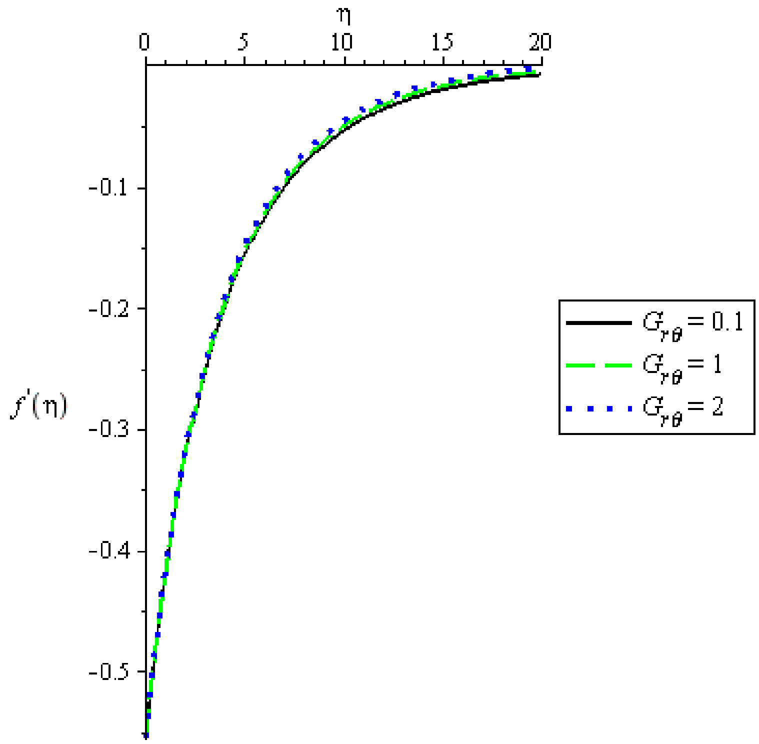


Figure 4.40: Effect of $G_{r\theta}$ on $f'(\eta)$

4.1.2.1 Analysis of Results and Discussion for Steady state with Arrhenius chemical reaction

Figures 4.34 to 4.35 present the influence of permeability γ on velocity profiles. It can be clearly seen that as the permeability increases, the fluid velocity increases from both x and y direction. The impact of magnetic parameter M is shown by figures 4.36 to 4.37. It is observed that velocity profiles increase for increasing values magnetic parameter. Figures 4.38 to 4.39 show the effect of velocity ratio parameter ψ . For increasing value of velocity ratio, the velocity in x and y direction decreases. The velocities increase for increasing values of thermal grashof parameter $G_{r\theta}$ in figures 4.40.

4.1.3. Transient state with chemical reaction of Constant rate

The graphical responses for the steady state with Arrhenius chemical reaction are provided in the figures 4.41 to 4.46. The simulation were carried out with values of

$$R_e = 0.1,\ R = 10,\ \Pr = 0.71,\ S_c = 0.22,\ G_{r\theta} = 0.1,\ G_{r\phi} = 0.1,\ \psi = 1,\ \gamma = 1,$$
$$a = -1, \delta = 0.1,\ \Omega = 1,\ \in = 0.01,\ N_b = 0.1,\ N_t = 0.3,\ E_c = 0.1 Q = 0.2,$$
$$\sigma = 0.1,\ M = 1\ and\ b = 0.2$$

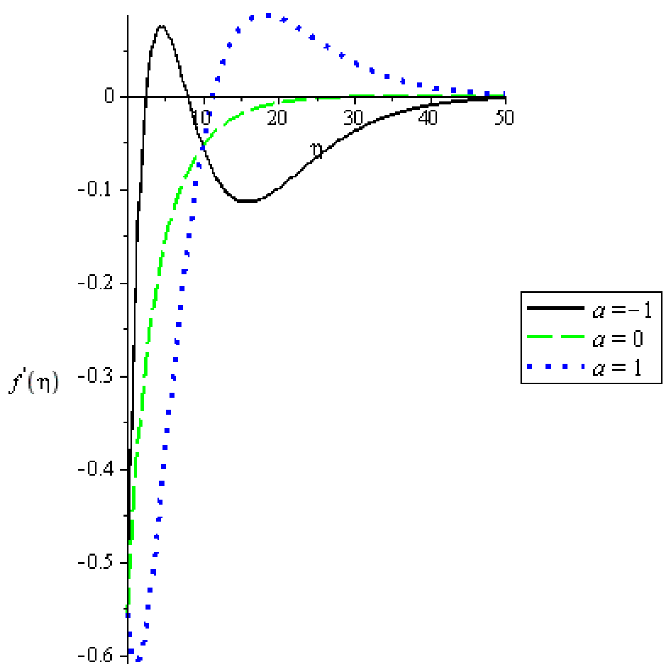


Figure 4.41: Effect of a **on** $f'(\eta)$

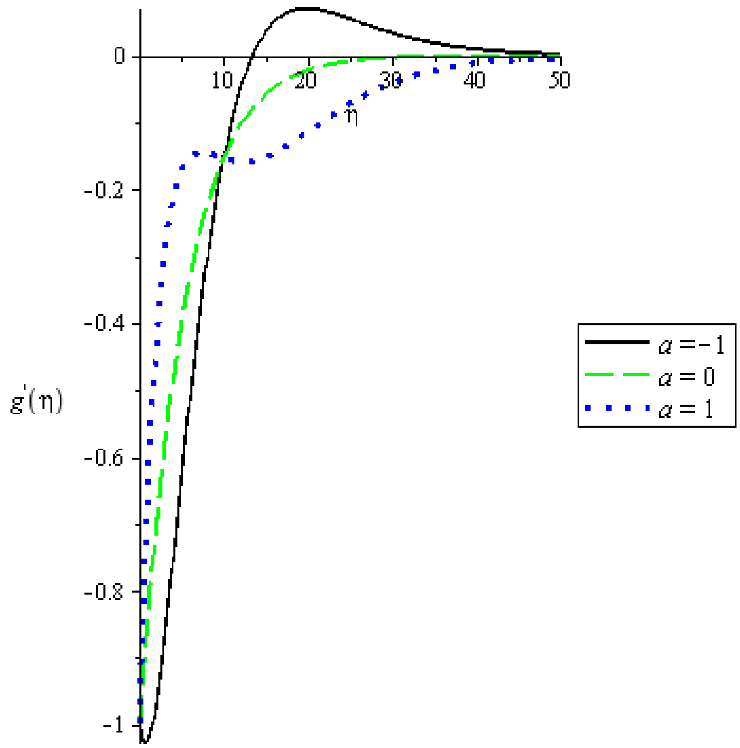


Figure 4.42: Effect of a **on** $g'(\eta)$

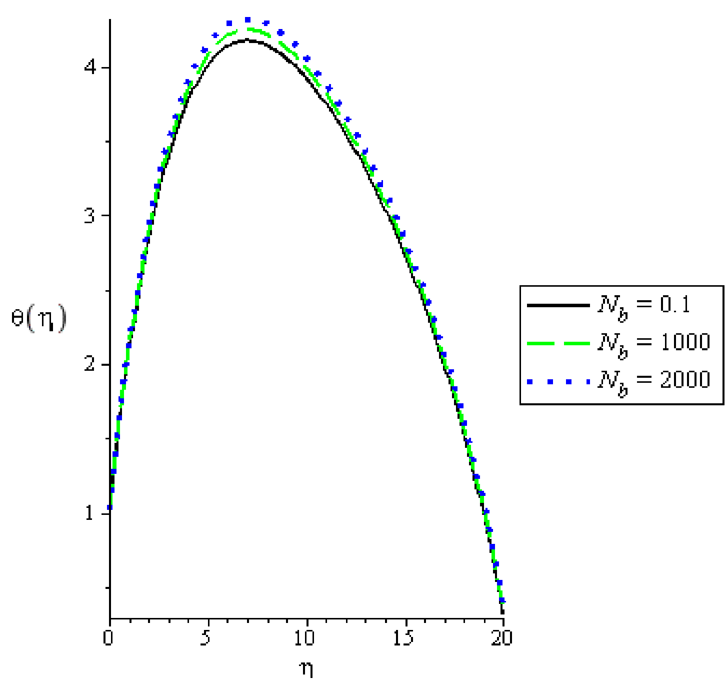


Figure 4.43: Effect of N_b **on** $\theta(\eta)$

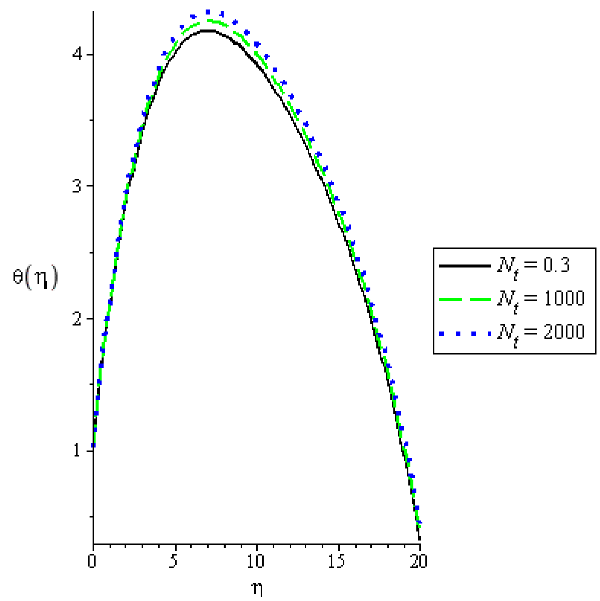


Figure 4.44: Effect of N_t **on** $\theta(\eta)$

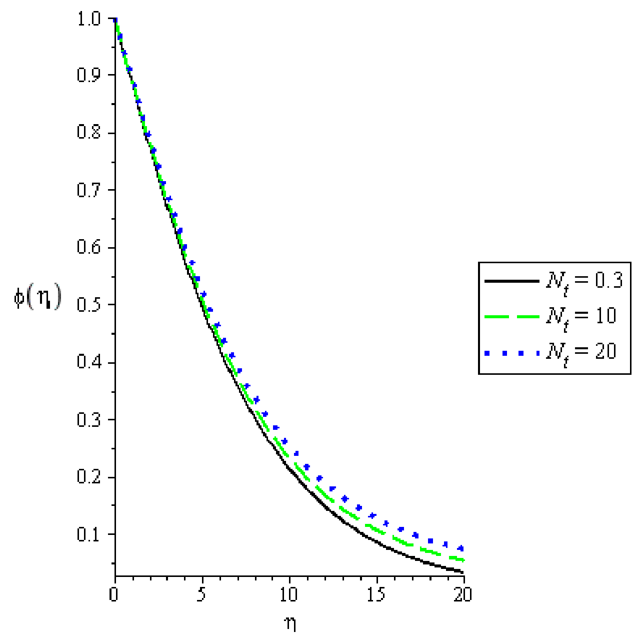


Figure 4.45: Effect of N_t **on** $\phi(\eta)$

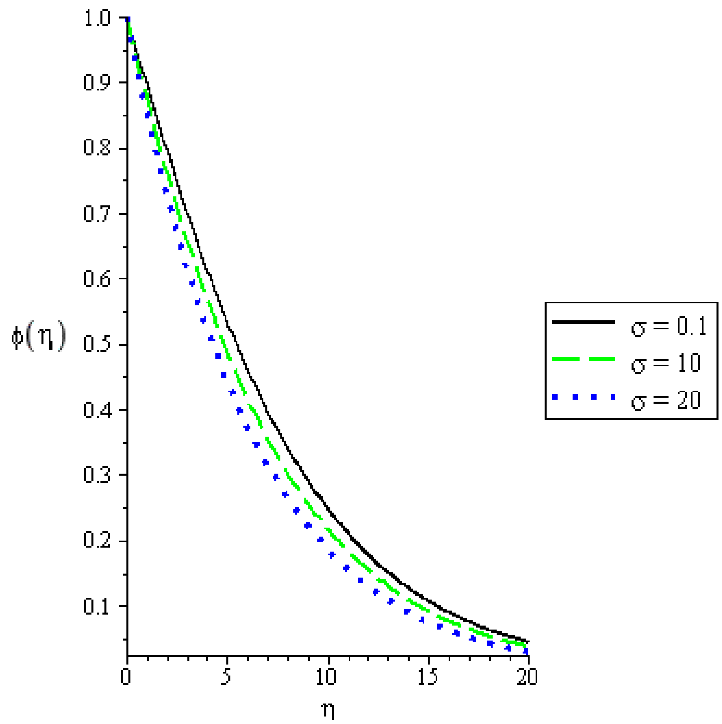


Figure 4.46: Effect of σ on $\phi(\eta)$

4.1.3.1 Analysis of Results and Discussion for Transient state with Chemical Reaction of Constant Reaction Rate

Figures 4.41 to 4.42 have been plotted to demonstrate the effect of velocities along x and y axis for different values a. It is noticed that the velocity profiles decreases along x and increase along y with multiple values of unseadiness parameter a. The temperature profile for selected values of N_b is presented in figures 4.43. The tempreture increases for increasing values Brownian diffusion N_b. The temperature and concentration profiles generated due to increasing Thermophoresis parameter N_t are ploted in figures 4.44 to 4.45. An increase was seen in both the tempreture and concentration. Figure 4.46 explain the impact of σ on concentration . but the concentration shows its usual trend of gradually decaying for increasing values of chemical reaction σ.

4.1.4.1 Steady State with Chemical Reaction of Constant Reaction Rate

The graphical responses for the steady state with Arrhenius chemical reaction are provided in the figures 4.47 to 4.50. The simulation were carried out with values of

$$R_e = 0.1,\ R = 10,\ \Pr = 0.71,\ S_c = 0.22,\ G_{r\theta} = 0.1,\ G_{r\phi} = 0.1,\ \psi = 1,\ \gamma = 1,$$
$$a = 0, \delta = 0.1,\ \Omega = 1,\ \in = 0.01,\ \mathrm{N}_b = 0.1,\ \mathrm{N}_t = 0.3,\ E_c = 0.1 Q = 0.2,$$
$$\sigma = 0.1,\ \mathrm{M} = 1\ and\ b = 0.2$$

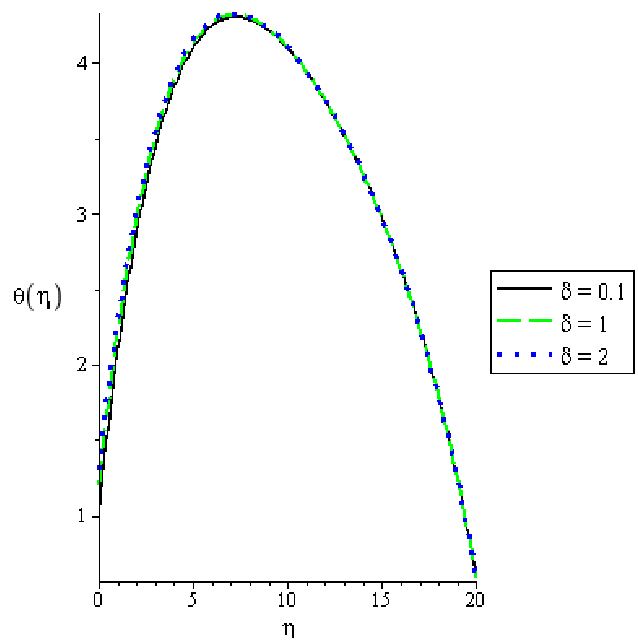


Figure 4.47: Effect of δ on $\theta(\eta)$

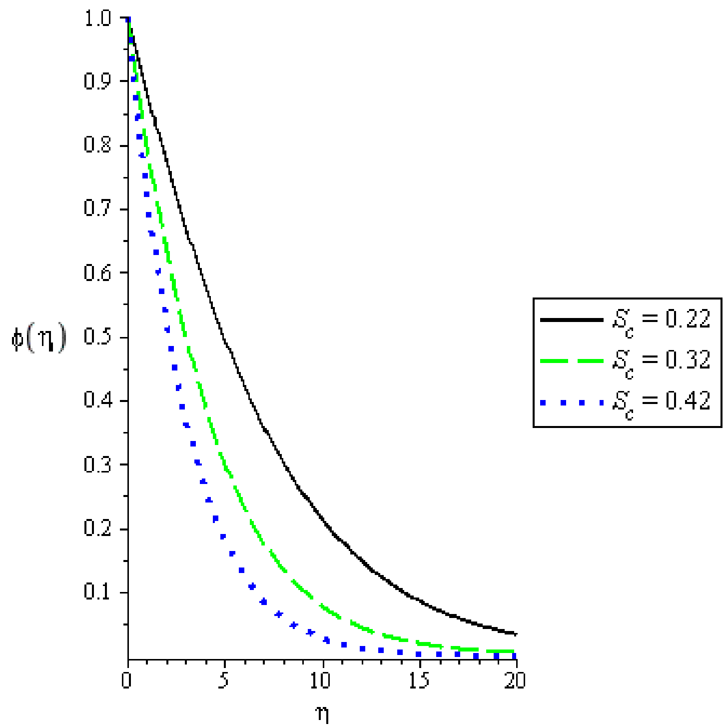


Figure 4.48: Effect of S_c **on** $\phi(\eta)$

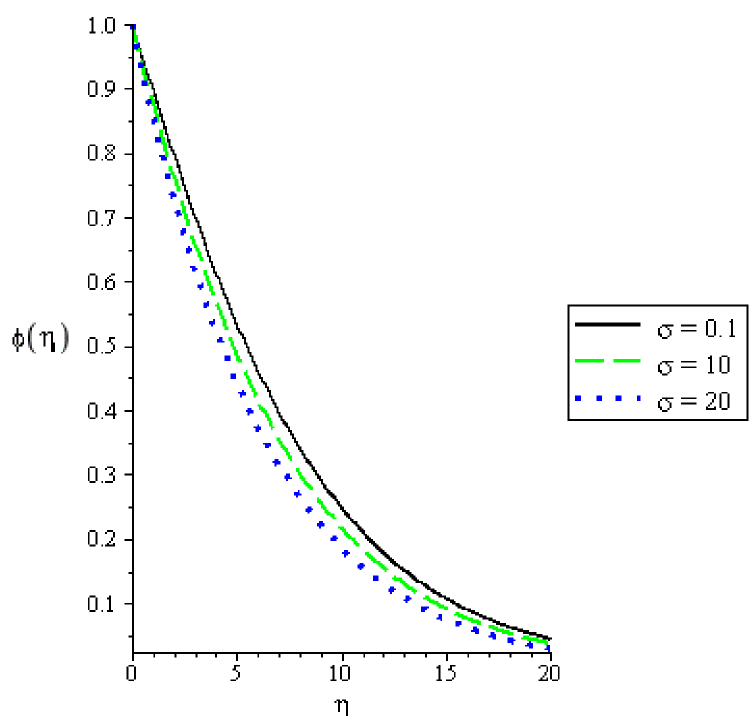


Figure 4.49: Effect of σ **on** $\phi(\eta)$

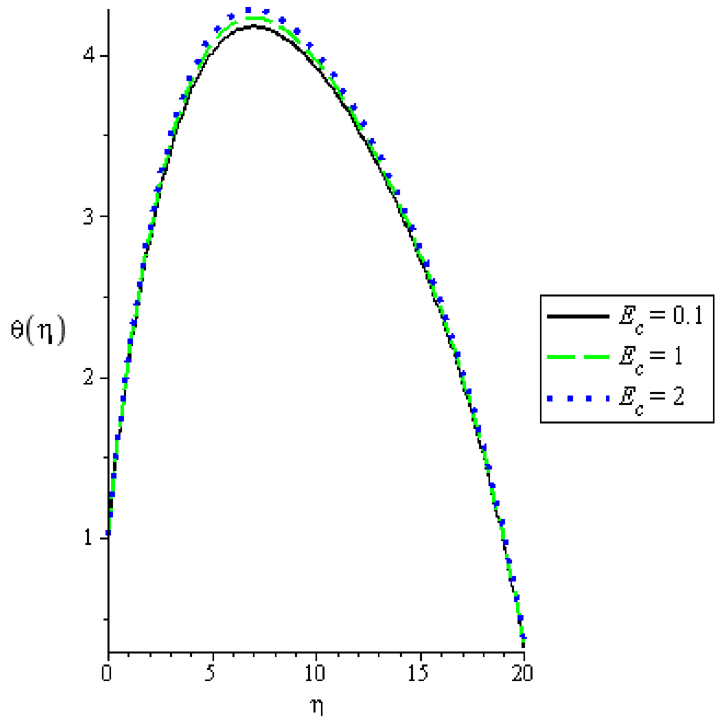


Figure 4.50: Effect of E_c on $\theta(\eta)$

4.1.4.1 Analysis of Results and Discussion for Steady State with Chemical Reaction of Constant Reaction Rate

It is evident in figure 4.47 that temperature increases for increasing number of Frank-kamenetski parameter δ. The effects of S_c and σ on concentration are shown in Figures 4.48 to 4.49. Increasing values in S_c and σ results gradual concentration reduction. The behaviour of Eckert number on temperature profile is depicted by figure 4.50. For increasing values of E_c, the temperature increases.

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

This present work has been able to extend the flow concept of Anuradha and priyadharshini (2016) by incorporating Magnetic field effect, Porosity into momentum equations, Arrhenius chemical reaction and extension of 1 dimensional shrinking sheet problem into three dimensional one shrunk in two direction holding z axis as constant. The problems were looked at in four ways viz unsteady state with Arrhenius chemical reaction, steady state with Arrhenius chemical reaction, unsteady state with chemical reaction of constant reaction rate and steady state with chemical reaction of constant reaction rate. The obtained equations of the formulated problems are converted into Ordinary differential equations (ODEs) using similarity transformation variables. The ODEs were solved using Iteration perturbation method (IPM) and the comparisons of the results were established with the shooting technique. The effects of various dimensionless flow parameters like local Reynolds number, prandtl number, velocity ratio, radiation , Frank-kamenetskii, magnetic, thermal grashof number, activation energy parameter, porosity, unsteadiness, schmidt number, Brownian diffusion, thermopheresis, permeability, Ecket number, heat source and chemical reaction on the velocity, temperature and concentration are analysed. It was discovered that:

I. Results of the IPM are in quite agreement with the shooting technique.
II. The boundary conditions are satisfied by all the presented graphs.

III. Increase in the values of local Reynolds number, velocity ratio and unsteadiness decreases the fluid velocity along x axis while permeability, magnetic effect, thermal grashof and activation energy increase the velocity.

IV. Fluid velocity is increased along y axis for increasing values of local Reynolds number, permeability, magnetic effect and unsteadiness while velocity ratio and activation energy is against the velocity.

V. Temperature strength is enhanced with increase in local Reynolds number, Prandtl, magnetic effect, heat source, velocity ratio, Brownian diffusion, thermophoresis effect, Eckert, Frank-kemenetskii number and unsteadiness parameters and it is decreased by Radiation and activation energy .

VI. Concentration appreciates with increase in Prandtl, velocity ratio, thermophoresis effect and activation energy and decreases with local Reynolds number, Schimidt number, chemical reaction parameter and unsteadiness.

VII. The heat transfer and chemical reaction rates have no significant difference for all the cases.

5.2 Recommendations

We recommend that further research should be on Hybrid nanofluid and not to be restricted on dynamics of incompressible nanofluid, and during the course of this study, it was observed that many industrial and engineering process care less about implementing the result of several research work, thus, the outcome of this research is recommended to be used in industrial processes, technology and engineering since the present work serve as the scientific tool for understanding the dynamics of Arrhenius and constant reaction for different cases considered.

5.3 **Contribution to knowledge**

The frame work has been able to improve upon the existing work by achieving model formulation, and also incorporating magnetic field effect, Porosity ,Arrhenius chemical reaction, extending the work into three dimensional one shrunk in two direction helding z axis as constant, and the condition for the existence of unique solution of the model by actual solution method and Derrick and Grossman approach was established and also the properties of solution of the model was examined by upper and lower solution method.

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