Homotopy Perturbation Method (HPM) for Solving Mathematical Modeling of Monkey Pox Virus

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Abstract

Mathematical modeling of real life problems such as transmission dynamics of infectious diseases resulted into non-linear differential equations which make it difficult to solve and have exact solution. Consequently, semi-analytical and numerical methods are used to solve these model equations. In this paper we used Homotopy Perturbation Method (HPM) to solve the mathematical modeling of Monkeypox virus. The solutions of HPM were validated numerically with the Runge-Kutta-Fehlberg 4-5th order built-in in Maple software. It was observed that the two solutions were in agreement with each other.

Keywords: Mathematical modeling, monkeypox, homotopy perturbation method.

1 Introduction

Non-linear differential equations have been solved by scientists and engineers using Homotopy Perturbation Method (HPM) in recent times. HPM reduce the difficult problem into a simple problem which is easier to solve. This method was first proposed by He [1] and improved upon by other researchers [2,3,4,5]. Homotopy, is an important part of differential topology. The Homotopy Perturbation Method (HPM) gives an approximate analytical solution in a series form. Several researchers have used HPM successfully to solve different real life application problems such as, bifurcation, asymptotology, nonlinear wave equations, oscillators with discontinuities [6,7,8,9], reaction-diffusion equation and heat radiation equation [10,11], and Approximate Solution of Susceptible Infected Recovered (SIR) Model of Infectious Diseases [12].

Monkeypox is a virus from rodent, which occurs mostly in West and Central Africa. The monkeypox virus is identified based on biological characteristics and endonuclease patterns of viral DNA. The virus can infect rabbit skin and transmitted consecutively by intracerebral vaccination of mice [13]. The rain forests of central and western Africa are the hosts of Monkeypox until 2003, when the first cases were reported in the Western Hemisphere [14]. The virus can be transmitted between humans through the respiratory (airborne) contact and contact with the bodily fluids of an infected person. Sharing the same bed and room, or using the same utensils with infected person is another way of the virus spreads [15].

The World Health Organization (WHO) was notified in 2017of suspected cases of human monkeypox in Nigeria and from September to December, 172 suspected and 61 confirmed cases have been reported in different states of the country.

In this paper the semi-analytical solution, Homotopy Perturbation Method (HPM) was used to solved the mathematical modeling of monkeypox virus. The details of the dynamics of the disease and other analysis can be found in [16]. It is difficult to solve and have the exact solution of the model equations, therefore Runge-Kutta-Fehlberg 4-5th order built-in in Maple was use to validate the solution.
This paper has four sections; section I is the introduction and related work, section II is the materials and method, section III is the result and discussion and section IV is the conclusion.

2 Materials and Methods

Formulation of Homotopy Perturbation Method (HPM)
Considered the non-linear differential equation, [4].

$$A(u) - f(r) = 0, \qquad r \in \Omega, \tag{2.1}$$

Subject to the boundary condition of:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma \tag{2.2}$$

Where A is a differential operator, B is a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω. The differential operator A can be divided into two parts of L and N, where L is the linear part, while N is a non-linear one.
Equation (2.1) can therefore, be rewritten as follows:

$$L(u)+N(u)-f(r)=0, \qquad r\in\Omega,, \tag{2.3}$$

Using the Homotopy technique, we construct a Homotopy $v(r,p):\Omega\times[0,1]\to R$, which satisfies

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[L(v)+N(v)-f(r)]=0 \tag{2.4}$$

$$p\in[0,1], \quad r\in\Omega \tag{2.5}$$

Or

$$H(v,p)=L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)]=0 \tag{2.6}$$

where :

$L(u)$ is the linear part

$L(u)=L(v)-L(u_0)+pL(u_0)$

And $N(u)$ is the non-linear part

$$N(u)=pN(v)$$

In equation (2.4) $p\in[0,1]$ is an embedding parameter and u_0 is an initial approximation of (2.1) that satisfied the boundary condition. Considering equations (2.4) and (2.6), gives:

$$H(v,0)=L(v)-L(u_0)=0 \tag{2.7}$$

and

$$H(v,1)=L(v)+N(v)-f(r)=0 \tag{2.8}$$

The changing process of p from zero to unity is just of $H(v,r)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v)-L(u_0)$ and $A(v)-f(r)$ are called homotopy.

According to HPM, we can first use the embedding parameter p as a small parameter and assume that the solution of equations (2.4) and (2.6) can be written as power series in

$$p: v = v_0 + pv_1 + p^2 v_2 + \ldots \tag{2.9}$$

Setting $p = 1$ result in the approximate solution of (2.1) is:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots \tag{2.10}$$

The series (2.10) is convergent for most cases. However, the convergence rate depends on the non-linear operator $A(v)$. The following observations are made by He (2000, 2006):

The derivatives of $N(v)$ with respect to v must be small because the parameter p may be large, i.e. $p \to 1$.

The norm of $L^{-1}\frac{\partial N}{\partial V}$ must be smaller than one so that the series converges.

Model Equations

$$\frac{dS_h}{dt} = \Lambda_h - \left(\frac{\alpha_1 I_r}{N_r} + \frac{\alpha_2 I_h}{N_h}\right)S_h - (\mu_h + \varepsilon)S_h \tag{2.11}$$

$$\frac{dI_h}{dt} = \left(\frac{\alpha_1 I_r}{N_r} + \frac{\alpha_2 I_h}{N_h}\right)S_h - (\mu_h + \delta_h + \tau)I_h \tag{2.12}$$

$$\frac{dQ_h}{dt} = \tau I_h - (\mu_h + \delta_h + \gamma_h)Q_h \tag{2.13}$$

$$\frac{dR_h}{dt} = \varepsilon S_h + \gamma_h Q_h - \mu_h R_h \tag{2.14}$$

$$\frac{dS_r}{dt} = \Lambda_r - \frac{\alpha_1 I_r S_r}{N_r} - \mu_r S_r \tag{2.15}$$

$$\frac{dI_r}{dt} = \frac{\alpha_1 I_r S_r}{N_r} - (\mu_r + \delta_r + \gamma_r)I_r \tag{2.16}$$

$$\frac{dR_r}{dt} = \gamma_r I_r - \mu_r R_r \tag{2.17}$$

$$N_h = S_h + I_h + Q_h + R_h \tag{2.18}$$

$$N_r = S_r + I_r + R_r \tag{2.19}$$

Equation (2.11) to (2.17) can be written as

$$\left.\begin{aligned}
&\frac{dS_h}{dt} - \Lambda_h + \left(\frac{\alpha_1 I_r}{N_r} + \frac{\alpha_2 I_h}{N_h}\right)S_h + A_1 S_h = 0\\
&\frac{dI_h}{dt} - \left(\frac{\alpha_1 I_r}{N_r} + \frac{\alpha_2 I_h}{N_h}\right)S_h + A_2 I_h = 0\\
&\frac{dQ_h}{dt} - \tau I_h + A_3 Q_h = 0\\
&\frac{dR_h}{dt} - \varepsilon S_h - \gamma_h Q_h + \mu_h R_h = 0\\
&\frac{dS_r}{dt} - \Lambda_r + \frac{\alpha_1 I_r S_r}{N_r} + \mu_h S_r = 0\\
&\frac{dI_r}{dt} - \frac{\alpha_1 I_r S_r}{N_r} + A_4 I_r = 0\\
&\frac{dR_r}{dt} - \gamma_r I_r + \mu_r R_r = 0
\end{aligned}\right\} \tag{2.20}$$

Where,

$A_1 = (\mu_h + \varepsilon)$, $A_2 = (\mu_h + \delta_h + \tau)$, $A_3 = (\mu_h + \delta_h + \gamma_h)$, $A_4 = (\mu_r + \delta_r + \gamma_r)$,

$S_h(0)=S_{h0}$, $I_h(0)=I_{h0}$ $Q_h(0)=Q_{h0}$ $R_h(0)=R_{h0}$, $S_r(0)=S_{r0}$, $I_r(0)=I_{r0}$, $R_r(0)=R_{r0}$
(2.21)

Let,

$$\left.\begin{aligned}
S_h &= a_0 + pa_1 + p^2a_2 + p^3a_3 + p^4a_4 + \cdots \\
I_h &= b_0 + pb_1 + p^2b_2 + p^3b_3 + p^4b_4 + \cdots \\
Q_h &= c_0 + pc_1 + p^2c_2 + p^3c_3 + p^4c_4 + \cdots \\
R_h &= d_0 + pd_1 + p^2d_2 + p^3d_3 + p^4d_4 + \cdots \\
S_r &= x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + \cdots \\
I_r &= y_0 + py_1 + p^2y_2 + p^3y_3 + p^4y_4 + \cdots \\
R_r &= z_0 + pz_1 + p^2z_2 + p^3z_3 + p^4z_4 + \cdots
\end{aligned}\right\} \qquad (2.22)$$

Applying HPM to the first equation of (2.20) gives

$$(1-p)\frac{dS_h}{dt} + p\left[\frac{dS_h}{dt} - \Lambda_h + \left(\frac{\alpha_1 I_r}{N_r} + \frac{\alpha_2 I_h}{N_h}\right)S_h + A_1 S_h\right] = 0 \qquad (2.23)$$

Substituting the first equation of (2.22) into (2.23) gives

$$\begin{aligned}
&a_0' + pa_1' + p^2a_2' + p^3a_3' + p^4a_4' + \cdots \\
&\quad + p\frac{\alpha_1}{N_r}\left(a_0 + pa_1 + p^2a_2 + p^3a_3 + p^4a_4 + \cdots\right)\left(y_0 + py_1 + p^2y_2 + p^3y_3 + p^4y_4 + \cdots\right) \\
&+ p\frac{\alpha_2}{N_h}\left(a_0 + pa_1 + p^2a_2 + p^3a_3 + p^4a_4 + \cdots\right)\left(b_0 + pb_1 + p^2b_2 + p^3b_3 + p^4b_4 + \cdots\right) \\
&+ pA_1\left(a_0 + pa_1 + p^2a_2 + p^3a_3 + p^4a_4 + \cdots\right) - p\Lambda_h = 0
\end{aligned}$$

(2.24)

Simplifying and collecting the coefficient of powers of p in (2.24) gives

$$\left.\begin{aligned}
&p^0: a_0' = 0\\
&p^1: a_1' + \frac{\alpha_1}{N_r} a_0 y_0 + \frac{\alpha_2}{N_h} a_0 b_0 + A_1 a_0 - \Lambda_h = 0\\
&p^2: a_2' + \frac{\alpha_1}{N_r}(a_0 y_1 + a_1 y_0) + \frac{\alpha_2}{N_h}(a_0 b_1 + a_1 b_0) + A_1 a_1 = 0\\
&p^3: a_3' + \frac{\alpha_1}{N_r}(a_0 y_2 + a_1 y_1 + a_2 y_0) + \frac{\alpha_2}{N_h}(a_0 b_2 + a_1 b_1 + a_2 b_0) + A_1 a_2 = 0\\
&p^4: a_4' + \frac{\alpha_1}{N_r}(a_0 y_3 + a_1 y_2 + a_2 y_1 + a_3 y_0) + \frac{\alpha_2}{N_h}(a_0 y_3 + a_1 y_2 + a_2 y_1 + a_3 y_0) + A_1 a_3 = 0
\end{aligned}\right\} \quad (2.25)$$

Applying HPM to the second equation of (2.20) gives

$$(1-p)\frac{dI_h}{dt} + p\left[\frac{dI_h}{dt} - \left(\frac{\alpha_1 I_r}{N_r} + \frac{\alpha_2 I_h}{N_h}\right)S_h + A_2 I_h\right] = 0 \quad (2.26)$$

Substituting the second equation of (2.22) into (2.26) gives

$$\begin{aligned}
&b_0' + pb_1' + p^2 b_2' + p^3 b_3' + p^4 b_4' + \cdots\\
&\quad - p\frac{\alpha_1}{N_r}(a_0 + pa_1 + p^2 a_2 + p^3 a_3 + p^4 a_4 + \cdots)(y_0 + py_1 + p^2 y_2 + p^3 y_3 + p^4 y_4 + \cdots)\\
&\quad - p\frac{\alpha_2}{N_h}(a_0 + pa_1 + p^2 a_2 + p^3 a_3 + p^4 a_4 + \cdots)(b_0 + pb_1 + p^2 b_2 + p^3 b_3 + p^4 b_4 + \cdots)\\
&\quad + pA_2(b_0 + pb_1 + p^2 b_2 + p^3 b_3 + p^4 b_4 + \cdots) = 0
\end{aligned} \quad (2.27)$$

Simplifying and collecting the coefficient of powers of p in (2.27) gives

$$\left.\begin{aligned}
&p^0 : b_0' = 0\\
&p^1 : b_1' - \frac{\alpha_1}{N_r} a_0 y_0 - \frac{\alpha_2}{N_h} a_0 b_0 + A_2 b_0 = 0\\
&p^2 : b_2' - \frac{\alpha_1}{N_r}\left(a_0 y_1 + a_1 y_0\right) - \frac{\alpha_2}{N_h}\left(a_0 b_1 + a_1 b_0\right) + A_2 b_1 = 0\\
&p^3 : b_3' - \frac{\alpha_1}{N_r}\left(a_0 y_2 + a_1 y_1 + a_2 y_0\right) - \frac{\alpha_2}{N_h}\left(a_0 b_2 + a_1 b_1 + a_2 b_0\right) + A_2 b_2 = 0\\
&p^4 : b_4' - \frac{\alpha_1}{N_r}\left(a_0 y_3 + a_1 y_2 + a_2 y_1 + a_3 y_0\right) - \frac{\alpha_2}{N_h}\left(a_0 y_3 + a_1 y_2 + a_2 y_1 + a_3 y_0\right) + A_2 b_3 = 0
\end{aligned}\right\} \tag{2.28}$$

Applying HPM to the third equation of (2.20) gives

$$(1-p)\frac{dQ_h}{dt} + p\left[\frac{dQ_h}{dt} - \tau I_h + A_3 Q_h\right] = 0 \tag{2.29}$$

Substituting the third equation of (2.22) into (2.29) gives

$$\begin{aligned}
&c_0' + pc_1' + p^2 c_2' + p^3 c_3' + p^4 c_4' + \cdots - p\tau\left(b_0 + pb_1 + p^2 b_2 + p^3 b_3 + p^4 b_4 + \cdots\right)\\
&+ pA_3\left(c_0 + pc_1 + p^2 c_2 + p^3 c_3 + p^4 c_4 + \cdots\right) = 0
\end{aligned} \tag{2.30}$$

Simplifying and collecting the coefficient of powers of p in (2.30) gives

$$\left.\begin{aligned}
&p^0 : c_0' = 0\\
&p^1 : c_1' - \tau b_0 + A_3 c_0 = 0\\
&p^2 : c_2' - \tau b_1 + A_3 c_1 = 0\\
&p^3 : c_3' - \tau b_2 + A_3 c_2 = 0\\
&p^4 : c_4' - \tau b_3 + A_3 c_3 = 0
\end{aligned}\right\} \tag{2.31}$$

Applying HPM to the fourth equation of (2.20) gives

$$(1-p)\frac{dR_h}{dt} + p\left[\frac{dR_h}{dt} - \varepsilon S_h - \gamma_h Q_h + \mu_h R_h\right] = 0 \tag{2.32}$$

Substituting the fourth equation of (2.22) into (2.32) gives

$$
\begin{aligned}
&d_0' + pd_1' + p^2 d_2' + p^3 d_3' + p^4 d_4' + \cdots - p\varepsilon\left(a_0 + pa_1 + p^2 a_2 + p^3 a_3 + p^4 a_4 + \cdots\right) \\
&- p\gamma_h\left(b_0 + pb_1 + p^2 b_2 + p^3 b_3 + p^4 b_4 + \cdots\right) + p\mu_h\left(d_0 + pd_1 + p^2 d_2 + p^3 d_3 + p^4 d_4 + \cdots\right) = 0
\end{aligned}
\tag{2.33}
$$

Simplifying and collecting the coefficient of powers of p in (2.33) gives

$$
\left.
\begin{aligned}
&p^0 : d_0' = 0 \\
&p^1 : d_1' - \varepsilon a_0 - \gamma_h b_0 + \mu_h d_0 = 0 \\
&p^2 : d_2' - \varepsilon a_1 - \gamma_h b_1 + \mu_h d_1 = 0 \\
&p^3 : d_3' - \varepsilon a_2 - \gamma_h b_2 + \mu_h d_2 = 0 \\
&p^4 : d_4' - \varepsilon a_3 - \gamma_h b_3 + \mu_h d_3 = 0
\end{aligned}
\right\}
\tag{2.34}
$$

Applying HPM to the fifth equation of (2.20) gives

$$
(1 - p)\frac{dS_r}{dt} + p\left[\frac{dS_r}{dt} - \Lambda_r + \frac{\alpha_1 I_r S_r}{N_r} + \mu_h S_r\right] = 0
\tag{2.35}
$$

Substituting the fifth equation of (2.22) into (2.35) gives

$$
\begin{aligned}
&x_0' + px_1' + p^2 x_2' + p^3 x_3' + p^4 x_4' + \cdots \\
&+ p\frac{\alpha_1}{N_r}\left(x_0 + px_1 + p^2 x_2 + p^3 x_3 + p^4 x_4 + \cdots\right)\left(y_0 + py_1 + p^2 y_2 + p^3 y_3 + p^4 y_4 + \cdots\right) \\
&+ p\mu_r\left(x_0 + px_1 + p^2 x_2 + p^3 x_3 + p^4 x_4 + \cdots\right) - p\Lambda_r = 0
\end{aligned}
\tag{2.36}
$$

Simplifying and collecting the coefficient of powers of p in (2.36) gives

$$\left.\begin{aligned}
&p^0 : x_0' = 0\\
&p^1 : x_1' + \frac{\alpha_1}{N_r} x_0 y_0 + \mu_r x_0 - \Lambda_r = 0\\
&p^2 : x_2' + \frac{\alpha_1}{N_r}(x_0 y_1 + x_1 y_0) + \mu_r x_1 = 0\\
&p^3 : x_3' + \frac{\alpha_1}{N_r}(x_0 y_2 + x_1 y_1 + x_2 y_0) + \mu_r x_2 = 0\\
&p^4 : x_4' + \frac{\alpha_1}{N_r}(x_0 y_3 + x_1 y_2 + x_2 y_1 + x_3 y_0) + \mu_r x_3 = 0
\end{aligned}\right\} \tag{2.37}$$

Applying HPM to the sixth equation of (2.20) gives

$$(1-p)\frac{dI_r}{dt} + p\left[\frac{dI_r}{dt} - \frac{\alpha_1 I_r S_r}{N_r} + A_4 I_r\right] = 0 \tag{2.38}$$

Substituting the sixth equation of (2.22) into (2.38) gives

$$\begin{aligned}
&y_0' + py_1' + p^2 y_2' + p^3 y_3' + p^4 y_4' + \cdots\\
&- p\frac{\alpha_1}{N_r}(x_0 + px_1 + p^2 x_2 + p^3 x_3 + p^4 x_4 + \cdots)(y_0 + py_1 + p^2 y_2 + p^3 y_3 + p^4 y_4 + \cdots)\\
&+ pA_4(y_0 + py_1 + p^2 y_2 + p^3 y_3 + p^4 y_4 + \cdots) = 0
\end{aligned} \tag{2.39}$$

Simplifying and collecting the coefficient of powers of p in (2.39) gives

$$\left.\begin{aligned}
p^0 &: y_0' = 0\\
p^1 &: y_1' - \frac{\alpha_1}{N_r} x_0 y_0 + A_4 y_0 = 0\\
p^2 &: y_2' - \frac{\alpha_1}{N_r}(x_0 y_1 + x_1 y_0) + A_4 y_1 = 0\\
p^3 &: y_3' - \frac{\alpha_1}{N_r}(x_0 y_2 + x_1 y_1 + x_2 y_0) + A_4 y_2 = 0\\
p^4 &: y_4' - \frac{\alpha_1}{N_r}(x_0 y_3 + x_1 y_2 + x_2 y_1 + x_3 y_0) + A_4 y_3 = 0
\end{aligned}\right\} \tag{2.40}$$

Applying HPM to the seventh equation of (2.20) gives

$$(1-p)\frac{dR_r}{dt} + p\left[\frac{dR_r}{dt} - \gamma_r I_r + \mu_r R_r\right] = 0 \tag{2.41}$$

Substituting the seventh equation of (2.22) into (2.41) gives

$$\begin{aligned}
&z_0' + pz_1' + p^2 z_2' + p^3 z_3' + p^4 z_4' + \cdots - p\gamma_r\left(y_0 + py_1 + p^2 y_2 + p^3 y_3 + p^4 y_4 + \cdots\right)\\
&+ p\mu_r\left(z_0 + pz_1 + p^2 z_2 + p^3 z_3 + p^4 z_4 + \cdots\right) = 0
\end{aligned} \tag{2.42}$$

Simplifying and collecting the coefficient of powers of p in (2.42) gives

$$\left.\begin{aligned}
p^0 &: z_0' = 0\\
p^1 &: z_1' - \gamma_r y_0 + \mu_r z_0 = 0\\
p^2 &: z_2' - \gamma_r y_1 + \mu_r z_1 = 0\\
p^3 &: z_3' - \gamma_r y_2 + \mu_r z_2 = 0\\
p^4 &: z_4' - \gamma_r y_3 + \mu_r z_3 = 0
\end{aligned}\right\} \tag{2.43}$$

Integrating the first equation of (2.25), (2.28), (2.31), (2.34), (2.37), (2.40) and (2.43) with initial conditions gives

$$\left.\begin{aligned} a_0 &= S_{h0} \\ b_0 &= I_{h0} \\ c_0 &= Q_{h0} \\ d_0 &= R_{h0} \\ x_0 &= S_{r0} \\ y_0 &= I_{r0} \\ z_0 &= R_{r0} \end{aligned}\right\} \tag{2.44}$$

Integrating the second equation of (2.25), (2.28), (2.31), (2.34), (2.37), (2.40) and (2.43) and substituting into (2.44) gives

$$\left.\begin{aligned} a_1 &= B_1 t \\ b_1 &= B_2 t \\ c_1 &= B_3 t \\ d_1 &= B_4 t \\ x_1 &= B_5 t \\ y_1 &= B_6 t \\ z_1 &= B_7 t \end{aligned}\right\} \tag{2.45}$$

where,

$$B_1 = \left(\Lambda_h - \frac{\alpha_1}{N_r} S_{h0} I_{r0} - \frac{\alpha_2}{N_h} S_{h0} I_{h0} - A_1 S_{h0}\right),\ B_2 = \left(\frac{\alpha_1}{N_r} S_{h0} I_{r0} + \frac{\alpha_2}{N_h} S_{h0} I_{h0} - A_2 I_{h0}\right),$$

$$B_3 = \left(\tau I_{h0} - A_3 Q_{h0}\right),\ B_4 = \left(\varepsilon S_{h0} + \gamma_h Q_{h0} - \mu_h R_{h0}\right),\ B_5 = \left(\Lambda_r - \frac{\alpha_1}{N_r} S_{r0} I_{r0} - \mu_r S_{r0}\right),$$

$$B_6 = \left(\frac{\alpha_1}{N_r} S_{r0} I_{r0} - A_4 I_{r0}\right),\ B_7 = \left(\gamma_r I_{r0} - \mu_r R_{r0}\right)$$

Integrating the third equation of (2.25), (2.28), (2.31), (2.34), (2.37), (2.40) and (2.43) substituting (2.44) and (2.45) gives

$$\left.\begin{aligned} a_2 &= -C_1\frac{t^2}{2} \\ b_2 &= C_2\frac{t^2}{2} \\ c_2 &= C_3\frac{t^2}{2} \\ d_2 &= C_4\frac{t^2}{2} \\ x_2 &= -C_5\frac{t^2}{2} \\ y_2 &= C_6\frac{t^2}{2} \\ z_2 &= C_7\frac{t^2}{2} \end{aligned}\right\} \quad (2.46)$$

where,

$$C_1 = \left(\frac{\alpha_1}{N_r}(S_{h0}B_7 + I_{r0}B_1) + \frac{\alpha_2}{N_h}(S_{h0}B_2 + I_{h0}B_1) + A_1B_1\right),$$

$$C_2 = \left(\frac{\alpha_1}{N_r}(S_{h0}B_6 + I_{r0}B_1) + \frac{\alpha_2}{N_h}(S_{h0}B_2 + I_{h0}B_1) - A_2B_2\right),$$

$$C_3 = (\tau B_2 - A_3B_3),\ C_4 = (\varepsilon B_1 + \gamma_h B_3 - \mu_h B_4),$$

$$C_5 = \left(\frac{\alpha_1}{N_r}(S_{r0}B_6 + I_{r0}B_5) + \mu_r B_5\right),$$

$$C_6 = \left(\frac{\alpha_1}{N_r}(S_{r0}B_6 + I_{r0}B_5) - A_4B_6\right),\ C_7 = (\gamma_r B_6 - \mu_r B_7)$$

Integrating the fourth equation of (2.25), (2.28), (2.31), (2.34), (2.37), (2.40) and (2.43) substituting (2.44), (2.45) and (2.46) gives

$$\left.\begin{aligned} a_3 &= -D_1\frac{t^3}{6} \\ b_3 &= D_2\frac{t^3}{6} \\ c_3 &= D_3\frac{t^3}{6} \\ d_3 &= D_4\frac{t^3}{6} \\ x_3 &= -D_5\frac{t^3}{6} \\ y_3 &= D_6\frac{t^3}{6} \\ z_3 &= D_7\frac{t^3}{6} \end{aligned}\right\} \tag{2.47}$$

where,

$$D_1 = \left(\frac{\alpha_1}{N_r}\left(S_{h0}C_6 + 2B_1B_6 - I_{r0}C_1\right) + \frac{\alpha_2}{N_h}\left(S_{h0}B_2 + 2B_1B_2 - I_{h0}C_1\right) - A_1C_1\right)$$

$$D_2 = \left(\frac{\alpha_1}{N_r}\left(S_{h0}C_6 + 2B_1B_6 - I_{r0}C_1\right) + \frac{\alpha_2}{N_h}\left(S_{h0}B_2 + 2B_1B_2 - I_{h0}C_1\right) - A_2C_2\right)$$

$$D_3 = \left(\tau C_2 - A_3C_3\right)$$

$$D_4 = \left(\gamma_h C_3 - \varepsilon C_1 - \mu_h C_4\right)$$

$$D_5 = \left(\frac{\alpha_1}{N_r}\left(S_{r0}C_6 + 2B_5B_6 - I_{r0}C_5\right) - \mu_r C_5\right)$$

$$D_6 = \left(\frac{\alpha_1}{N_r}\left(S_{r0}C_6 + 2B_5B_6 - I_{r0}C_5\right) - A_4C_6\right)$$

$$D_7 = \left(\gamma_r C_6 - \mu_r C_7\right)$$

Integrating the fifth equation of (2.25), (2.28), (2.31), (2.34), (2.37), (2.40) and (2.43) substituting (2.44), (2.45), (2.46) and (2.47) gives

$$\left.\begin{aligned}
a_4 &= -E_1\frac{t^4}{24}\\
b_4 &= E_2\frac{t^4}{24}\\
c_4 &= E_3\frac{t^4}{24}\\
d_4 &= E_4\frac{t^4}{24}\\
x_4 &= -E_5\frac{t^4}{24}\\
y_4 &= E_6\frac{t^4}{24}\\
z_4 &= E_7\frac{t^4}{24}
\end{aligned}\right\} \quad (2.48)$$

Where,

$$E_1 = \left(\frac{\alpha_1}{N_r}(S_{h0}D_6 + 3B_1C_6 - 3C_1B_6 - I_{r0}D_1) + \frac{\alpha_2}{N_h}(S_{h0}D_2 + 3B_1C_2 - 3C_1B_2 - I_{h0}D_1) - A_1D_1\right)$$
$$E_2 = \left(\frac{\alpha_1}{N_r}(S_{h0}D_6 + 3B_1C_6 - 3C_1B_6 - I_{r0}D_1) + \frac{\alpha_2}{N_h}(S_{h0}D_2 + 3B_1C_2 - 3C_1B_2 - I_{h0}D_1) - A_2D_2\right)$$
$$E_3 = (\tau D_2 - A_3D_3)$$
$$E_4 = (\gamma_h D_3 - \varepsilon D_1 - \mu_h D_4)$$
$$E_5 = \left(\frac{\alpha_1}{N_r}(S_{r0}D_6 + 3B_5C_6 + 3C_5B_6 - I_{r0}D_5) - \mu_r D_5\right)$$
$$E_6 = \left(\frac{\alpha_1}{N_r}(S_{r0}D_6 + 3B_5C_6 + 3C_5B_6 - I_{r0}D_5) - A_4D_6\right)$$
$$E_7 = (\gamma_r D_6 - \mu_r D_7)$$

Substituting (2.44) to (2.48) into (2.22) gives

$$\left.\begin{aligned}
S_h &= S_{h0} + pB_1 t - p^2 C_1 \frac{t^2}{2} - p^3 D_1 \frac{t^3}{6} - p^4 E_1 \frac{t^4}{24} + \cdots \\
I_h &= I_{h0} + pB_2 t + p^2 C_2 \frac{t^2}{2} + p^3 D_2 \frac{t^3}{6} + p^4 E_2 \frac{t^4}{24} + \cdots \\
Q_h &= Q_{h0} + pB_3 t + p^2 C_3 \frac{t^2}{2} + p^3 D_3 \frac{t^3}{6} + p^4 E_3 \frac{t^4}{24} + \cdots \\
R_h &= R_{h0} + pB_4 t + p^2 C_4 \frac{t^2}{2} + p^3 D_4 \frac{t^3}{6} + p^4 E_4 \frac{t^4}{24} + \cdots \\
S_r &= S_{r0} + pB_5 t - p^2 C_5 \frac{t^2}{2} - p^3 D_5 \frac{t^3}{6} - p^4 E_5 \frac{t^4}{24} + \cdots \\
I_r &= I_{r0} + pB_6 t + p^2 C_6 \frac{t^2}{2} + p^3 D_6 \frac{t^3}{6} + p^4 E_6 \frac{t^4}{24} + \cdots \\
R_r &= R_{r0} + pB_7 t + p^2 C_7 \frac{t^2}{2} + p^3 D_7 \frac{t^3}{6} + p^4 E_7 \frac{t^4}{24} + \cdots
\end{aligned}\right\} \quad (2.49)$$

$$X_i = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + v_4 + \cdots \quad (2.50)$$

Where,

$$X_i = \{S_h, I_h, Q_h, R_h, S_r, I_r, R_r\} \quad (2.51)$$

Hence, (2.49) becomes

$$\left.\begin{aligned}
S_h &= S_{h0} + B_1 t - C_1 \frac{t^2}{2} - D_1 \frac{t^3}{6} - E_1 \frac{t^4}{24} + \cdots \\
I_h &= I_{h0} + B_2 t + C_2 \frac{t^2}{2} + D_2 \frac{t^3}{6} + E_2 \frac{t^4}{24} + \cdots \\
Q_h &= Q_{h0} + B_3 t + C_3 \frac{t^2}{2} + D_3 \frac{t^3}{6} + E_3 \frac{t^4}{24} + \cdots \\
R_h &= R_{h0} + B_4 t + C_4 \frac{t^2}{2} + D_4 \frac{t^3}{6} + E_4 \frac{t^4}{24} + \cdots \\
S_r &= S_{r0} + B_5 t - C_5 \frac{t^2}{2} - D_5 \frac{t^3}{6} - E_5 \frac{t^4}{24} + \cdots \\
I_r &= I_{r0} + B_6 t + C_6 \frac{t^2}{2} + D_6 \frac{t^3}{6} + E_6 \frac{t^4}{24} + \cdots \\
R_r &= R_{r0} + B_7 t + C_7 \frac{t^2}{2} + D_7 \frac{t^3}{6} + E_7 \frac{t^4}{24} + \cdots
\end{aligned}\right| \quad (2.52)$$

3 Result and Discussions

Numerical Solution

In Table 3.1 are the variables and parameters of the model equation and their definitions. The values of table 3.1 were estimated for the purpose of numerical solutions. The tables 3.2 to 3.8 are the numerical solutions of each of the equation (2.52). The HPM solution was validated with Runge-Kutta to see the agreement between the two solutions. The errors are minimal.

Table 3.1: Definition of Variables and Parameters

Variables/Parameters	Definition	Values
S_h	susceptible Humans	10000
I_h	Infected Humans	500
Q_h	Quarantine Infected Humans	1000
R_h	Recovered Humans	300
S_r	Susceptible Rodents	500
I_r	Infected Rodents	200
R_r	Recovered Rodents	50
Λ_h	Recruitment Rate of Humans	65000
Λ_r	Recruitment Rate of Rodents	5000
α_1	Contact Rate of Rodents to Humans	0.001
α_2	Contact Rate of Humans to Humans	0.1
α_3	Contact Rate of Rodents to Rodents	0.01
μ_h	Natural Death Rate of Humans	0.015
δ_h	Disease Induced Death Rate of Humans	0.0001
γ_h	Recovery Rate of Humans	0.25
γ_r	Recovery Rate of Rodents	0.2
τ	Progression Rate from Infected to Quarantine	0.50
ε	Effectiveness Public Enlightenment Campaign	0.25
μ_r	Natural Death Rate of Rodents	0.01
δ_r	Disease Induced Death Rate of Rodents	0.001
N_h	Total Population of Humans	100000000
N_r	Total Population of Rodents	10000

Table 3.2: Numerical Solution for Susceptible Humans

T	Runge-Kutta	HPM	Error
0.0	10000.0000	10000.0000	0
0.1	16153.0850	16153.0850	4.93074E-05
0.2	22145.2435	22145.2426	0.000827317
0.3	27980.6848	27980.6786	0.006220119
0.4	33663.5081	33663.4822	0.02587213
0.5	39197.7056	39197.6268	0.07867993
0.6	44587.1643	44586.9697	0.194616374
0.7	49835.6706	49835.2519	0.418711643
0.8	54946.9112	54946.0984	0.812878598
0.9	59924.4760	59923.0180	1.457988925
1.0	64771.8618	64769.4036	2.458272091

Table 3.3: Numerical Solution for Infected Humans

t	Runge-Kutta	HPM	Error
0.0	500.0000	500.0000	0
0.1	474.9230	474.9230	-3.35343E-06
0.2	451.1161	451.1162	-5.34592E-05
0.3	428.5149	428.5153	-0.000391696
0.4	407.0581	407.0597	-0.001606786
0.5	386.6874	386.6923	-0.004841169
0.6	367.3478	367.3597	-0.011888652
0.7	348.9867	349.0121	-0.025420482
0.8	331.5544	331.6036	-0.049079107
0.9	315.0040	315.0915	-0.087586605
1.0	299.2903	299.4373	-0.146971

Table 3.4: Numerical Solution for Quarantine Humans

t	Runge-Kutta	HPM	Error
0.0	1000.0000	1000.0000	0
0.1	997.8831	997.8831	5.54961E-06
0.2	994.6159	994.6158	9.13095E-05
0.3	990.2896	990.2889	0.000686259
0.4	984.9899	984.9870	0.002842679
0.5	978.7972	978.7886	0.008610256
0.6	971.7872	971.7660	0.021209245

0.7	964.0308	963.9853	0.045438487
0.8	955.5945	955.5066	0.087842946
0.9	946.5408	946.3838	0.15690524
1.0	936.9282	936.6647	0.263453118

Table 3.5: Numerical Solution for Recovered Humans

t	Runge-Kutta	HPM	Error
0.0	300.0000	300.0000	0
0.1	651.5348	651.5348	-5.062E-05
0.2	1154.1676	1154.1684	-0.0008617
0.3	1803.6770	1803.6835	-0.006512
0.4	2595.9540	2595.9811	-0.0271016
0.5	3526.9986	3527.0811	-0.0824508
0.6	4592.9177	4593.1217	-0.2039295
0.7	5789.9211	5790.3599	-0.4387205
0.8	7114.3195	7115.1712	-0.8516212
0.9	8562.5224	8564.0497	-1.5272787
1.0	10131.0334	10133.6081	-2.5747044

Table 3.6: Numerical Solution for Susceptible Rodents

t	Runge-Kutta	HPM	Error
0.0	500.0000	500.0000	0
0.1	999.2489	999.2488	-2.72905E-09
0.2	1497.9978	1497.9977	1.4191E-07
0.3	1996.2472	1996.2473	-2.5584E-07
0.4	2493.9980	2493.9980	2.4893E-07
0.5	2991.2503	2991.2503	1.28403E-06
0.6	3488.0049	3488.0049	2.68664E-06
0.7	3984.2622	3984.2621	5.55868E-06
0.8	4480.0228	4480.0228	1.12909E-05
0.9	4975.2872	4975.2871	2.05958E-05
1.0	5470.0559	5470.0559	3.45643E-05

Table 3.7: Numerical Solution for Infected Rodents

t	Runge-Kutta	HPM	Error
0.0	200.0000	200.0000	0.00000
0.1	195.8257	195.8316	-0.005872859
0.2	191.7394	191.7624	-0.022993593

0.3	187.7394	187.7900	-0.050641296
0.4	183.8237	183.9118	-0.088129751
0.5	179.9906	180.1254	-0.134808191
0.6	176.2383	176.4283	-0.190062371
0.7	172.5650	172.8184	-0.253315299
0.8	168.9692	169.2932	-0.324028283
0.9	165.4492	165.8509	-0.401701136
1.0	162.0032	162.4891	-0.485873876

Table 3.8: Numerical Solution for Recovered Rodents

t	Runge-Kuta	HPM	Error
0.0	50.0000	50.0000	0
0.1	53.9061	53.9062	-3.9336E-05
0.2	57.7258	57.7261	-0.00030932
0.3	61.4609	61.4619	-0.00102533
0.4	65.1131	65.1155	-0.00238539
0.5	68.6842	68.6887	-0.0045692
0.6	72.1759	72.1836	-0.00773755
0.7	75.5899	75.6019	-0.01203117
0.8	78.9279	78.9454	-0.01757019
0.9	82.1914	82.2158	-0.02445349
1.0	85.3820	85.4147	-0.03275749

Tables 3.2 to 3.5 are the solutions of Susceptible human, Infected human, Quarantine human and Recovered human respectively while tables 3.6 to 3.8 are the solutions of Susceptible rodent, Infected rodent and Recovered rodent.

4 Conclusion

The mathematical model of monkeypox virus was solved semi-analytically using Homotopy Perturbation Method. The solutions of the model equations were validated numerically with the Runge-Kutta in Maple software. Numerical solutions showed that the HPM is in agreement with Runge – Kutta. Semi –analytical solutions help to validated mathematical modeling of infectious disease and others it give better understanding of the model the mathematical epidemiologists and the lay man.

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