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On Exponentially Fitted Improved Runge-Kutta Method of Order Five for Solving Oscillatory Problems

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ABSTRACT

An exponentially fitted Improved Runge-Kutta (IRK) method with 5 stages is constructed. The new method integrates exactly differential equations where the solution is oscillatory, that is, whose solutions can be expressed as linear combinations of the set of functions $\{e^{\lambda t}, e^{-\lambda t}\}, \lambda \in \mathbb{C}$, and in particular $\{\sin(\omega t), \cos(\omega t)\}$, where $\lambda = i\omega,\ \omega \in \mathbb{R}$. Analysis of algebraic order of the new method indicates that it is of order five like the classical five stages IRK method. Numerical experiments established the effectiveness and efficiency of the exponentially fitted method over the non-fitted method.

Keywords: Exponential fitting, IRK method, Oscillatory solution, Initial value problem

INTRODUCTION

In response to the need to find approximate solutions to ordinary differential equations whose solutions cannot be found analytically, a great many numerical methods have been constructed. One of such methods is the classical Runge-Kutta (RK) method. While the Improved Runge-Kutta (IRK) methods represent an improvement on the classical RK methods, the need to develop efficient and accurate methods when solutions of initial value problems exhibit exponential or oscillatory behaviour led to exponentially fitted methods. Exponentially fitted methods are thus generally derived to find the numerical integration of initial value problems whose solutions are linear combinations of the functions $\{x^j e^{\lambda t}, x^j e^{-\lambda t}\}$ where $\lambda \in \mathbb{C}$ (λ can be a real or complex number). One procedure that has proven to be very useful toward the construction of numerical methods that approximate the solutions to first order initial value problems (IVPs) is the adaptation technique. The issue of how to choose the frequency in exponentially fitted methods is very difficult. For linear oscillators, the frequency of the method is the same as the frequency of the solution of the differential equation. However, for nonlinear problems, the frequency of the method is, in general, different from the frequency of the exact solution (Ramos and Vigor-Aguiar, 2010). In the case of linear multistep methods, a first good theoretical foundation, which required that the frequency be estimated in advance, was proposed by Gautschi (1961). Following this, Vigor-Aguiar and Ferrandiz (1998) proposed a general procedure for the construction of adapted multistep algorithms. Methods of this type require that the frequency ω that appears in the numerical method be chosen near the frequency of the exact solution, or it is assumed that the frequency is known in advance.

The study of exponentially fitted Runge-Kutta (EFRK) methods is a recent development and of minimal quantity. One of the techniques to construct EFRK methods is to select the coefficients of the method so that it integrates exactly a set of linearly independent functions which are chosen depending on the nature of the solutions of the differential system is solved. In Paternoster (1998) some implicit Runge-Kutta-(Nystrom) methods of low algebraic order were derived by employing the linear stage representation of a Runge-Kutta method introduced in Albrecht's approach. Also, in Simos et al. (1994) and Simos (1998), a fourth order explicit Runge-Kutta-(Nystrom) method which integrates certain particular first-order initial value problems with periodic or exponential solutions was constructed. On the other hand, Berghe et al. (2000) introduced a general technique for the construction of exponentially fitted methods. Of recent, quite a number of authors have studied EFRK methods; they include, Berghe et al (2001), Williams and Simos (2003), Franco (2004), Calvo et al. (2009), Rabiei et al. (2014) and Monovasilis et al. (2015).

This research work is aimed at developing an exponentially fitted fifth order Improved Runge-Kutta method (EFIRK5), applying the developed method to solving initial value problems whose solutions are oscillatory or exponential and comparing the results with those the classical IRK method

MATERIALS AND METHODS

Order conditions for EFIRK5

The general explicit IRK method for solving the initial value problem

Similarly, if we let $z = \omega h$, then

$$y'(x) = f(x, y), \qquad y(x) = y_0 \tag{1}$$

takes the form

$$y_{n+1} = y_n + hb_1 f(x_n, y_n) - hb_{-1} f(x_{n-1}, y_{n-1}) + h\sum_{i=2}^{s} b_i\big(f(x_n + c_i h, Y_i) - f(x_{n-1} + c_i h, Y_{-i})\big) \tag{2}$$

where,

$$\left.\begin{aligned} Y_i &= \gamma_i y_n + h\sum_{j=1}^{i-1} a_{ij} f(x_n + c_j h, Y_j) \\ Y_{-i} &= \gamma_i y_{n-1} + h\sum_{j=1}^{i-1} a_{ij} f(x_{n-1} + c_j h, Y_{-j}) \end{aligned}\right\} \tag{3}$$

(Rabiei *et al.*, 2013).

The IVP (1) is integrated exactly by an EFIRK if its solution $y(x)$ takes the form

$$y_n = y(x_n) = e^{i\omega x_n} \tag{4}$$

Consequently,

$$y_{n+1} = y(x_n + h) = e^{i\omega(x_n+h)} \tag{5}$$

$$y_{n-1} = y(x_n - h) = e^{i\omega(x_n-h)} \tag{6}$$

$$y_n' = i\omega e^{i\omega x_n} = f(x_n, y_n) \tag{7}$$

$$y_{n-1}' = i\omega e^{i\omega(x_n-h)} = f(x_{n-1}, y_{n-1}) \tag{8}$$

And from (3) results

$$e^{i\omega(x_n+c_i h)} = \gamma_i e^{i\omega x_n} + h\sum_{j=1}^{i-1} a_{ij}\,(i\omega) e^{i\omega(x_n+c_j h)} \tag{9}$$

$$e^{i\omega(c_i h)} = \gamma_i + (i\omega)h\sum_{j=1}^{i-1} a_{ij}\, e^{i\omega(c_j h)} \tag{10}$$

Let $\omega h = z$, where ω is the frequency and h the step size, then (10) becomes

$$e^{ic_i z} = \gamma_i + iz\sum_{j=1}^{i-1} a_{ij}\, e^{ic_j z} \tag{11}$$

$$cos(c_i z) + i\,sin(c_i z) = \gamma_i + iz\sum_{j=1}^{i-1} a_{ij}\big[\cos(c_j z) + i\sin(c_j z)\big] \tag{12}$$

$$\boldsymbol{cos(c_i z) = \gamma_i - z\sum_{j=1}^{i-1} a_{ij}\sin(c_j z), \qquad i = 2, 3, \ldots, s} \tag{13}$$

$$sin(c_i z) = z\sum_{j=1}^{i-1} a_{ij}\,cos(c_j z), \qquad i = 2,3,\ldots,s \tag{14}$$

Furthermore, manipulation of (2) results into

$$\begin{aligned} e^{i\omega(x_n+h)} &= e^{i\omega(x_n)} + hb_1(i\omega)e^{i\omega(x_n)} - hb_{-1}(i\omega)z \\ &\quad + h\sum_{i=2}^{s} b_i\,(i\omega)\big[e^{i\omega(x_n+c_i h)} - e^{i\omega(x_n-h+c_i h)}\big] \end{aligned} \tag{15}$$

$$e^{i\omega(h)} = 1 + hb_1(i\omega) - hb_{-1}(i\omega)e^{-i\omega(h)} + h(i\omega)\sum_{i=2}^{s} b_i\left[e^{i\omega(c_i h)}\big(1 - e^{-i\omega(h)}\big)\right] \tag{16}$$

$$\boldsymbol{e^{iz} = 1 + izb_1 - izb_{-1}e^{-iz} + iz\sum_{i=2}^{s} b_i\left[e^{ic_iz} - e^{iz(c_i-1)}\right]} \quad (17)$$

$$\begin{aligned}cos(z) + i\,sin(z) &= 1 + izb_1 - izb_{-1}[cos(z) - i\,sin(z)]\\ &+iz\sum_{i=2}^{s} b_i\left[cos(c_iz) + i\,sin(c_iz) - \left(\cos\left(z(c_i-1)\right) + i\sin\left(z(c_i-1)\right)\right)\right] \quad (18)\end{aligned}$$

$$\boldsymbol{cos(z) = 1 - zb_{-1}\,sin(z) - z\sum_{i=2}^{s} b_i\,sin(c_iz) + z\sum_{i=2}^{s} b_i\,sin\left(z(c_i-1)\right)} \quad (19)$$

$$sin(z) = zb_1 - zb_{-1}\,cos(z) + z\sum_{i=2}^{s} b_i\,cos(c_iz) - z\sum_{i=2}^{s} b_i\,cos\left(z(c_i-1)\right) \quad (20)$$

Equations (13), (14), (19) and (20) are now the relations of order conditions of the proposed exponentially-fitted methods. These equations are solved in order to determine the coefficients $c_{i,}\gamma_i\,, a_{ij}\ and\ b_i$ of an EFRK5 method by choosing values for a free parameter from existing coefficients of an IRK5 method. They replace the equations of order conditions of two-step IRK5 methods derived by Rabiei et al. (2013) as represented by (21

$$\left.\begin{aligned}&\text{order 1:}\quad b_1 - b_{-1} = 1\\ &\text{order 2:}\ b_{-1} + \sum_{i=2}^{s} b_i = \frac{1}{2}\\ &\text{order 3:}\quad \sum_{i=2}^{s} b_i\,c_i = \frac{5}{12}\\ &\text{order 4:}\quad \sum_{i=2}^{s} b_i\,c_i^2 = \frac{1}{3}\\ &\qquad \sum_{i=2,j=1}^{s} b_i\,a_{ij}c_j = \frac{1}{6}\\ &order\ 5\text{:}\ \sum_{i=2}^{s} b_i\,c_i^3 = \frac{31}{120}\\ &\qquad \sum_{i=2,j=1}^{s} b_ic_i\,a_{ij}c_j = \frac{31}{240}\\ &\qquad \sum_{i=2,j=1}^{s} b_i\,a_{ij}c_j^2 = \frac{31}{360}\\ &\qquad \sum_{i=2,j=1}^{s} b_i\,a_{ij}a_{jk}c_k = \frac{31}{720}\end{aligned}\right\} \quad (21)$$

Derivation of EFIRK5 with $\boldsymbol{s = p = 5}$

The Butcher table of coefficients for the Improved Runge-Kutta method of order five (IRK5-5) of Rabiei et al. (2013) is presented in Table 1

Table 1: Coefficients of IRK5-5 Method

0					
$\frac{1}{4}$	$\frac{1}{4}$				
$\frac{1}{4}$	$\frac{-727}{1000}$	$\frac{732}{1000}$			
$\frac{1}{2}$	$\frac{573}{1000}$	$\frac{-225}{1000}$	$\frac{334}{1000}$		
$\frac{3}{4}$	$\frac{175}{1000}$	$\frac{12}{1000}$	$\frac{56}{1000}$	$\frac{552}{1000}$	
$\frac{1}{45}$	$\frac{46}{45}$	$\frac{-96}{1000}$	$\frac{29}{1000}$	$\frac{-1}{10}$	$\frac{29}{45}$

To derive the fifth order five stages exponentially fitted IRK (EFIRK5-5) method, we make the substitution $s = 5, c_1 = 0, \gamma_1 = 1$ in the relations (13) and (14)

for $i = 2$

$$\cos(c_2 z) - \gamma_2 = 0 \tag{22}$$
$$\sin(c_2 z) - z a_{2,1} = 0 \tag{23}$$

for $i = 3$

$$\cos(c_3 z) - \gamma_3 + z a_{32} \sin(c_2 z) = 0 \tag{24}$$
$$\sin(c_3 z) - z[a_{31} + a_{32}\cos(c_2 z)] = 0 \tag{25}$$

for $i = 4$

$$\cos(c_4 z) - \gamma_4 + z a_{42} \sin(c_2 z) + z a_{43} \sin(c_3 z) = 0 \tag{26}$$
$$\sin(c_4 z) - z[a_{41} + a_{42}\cos(c_2 z) + a_{43}\cos(c_3 z)] = 0 \tag{27}$$

for $i = 5$

$$\cos(c_5 z) - \gamma_5 + z[a_{52}\sin(c_2 z) + a_{53}\sin(c_3 z) + a_{54}\sin(c_4 z)] = 0 \tag{28}$$
$$\sin(c_5 z) - z[a_{51} + a_{52}\cos(c_2 z) + a_{53}\cos(c_3 z) + a_{54}\cos(c_4 z)] = 0 \tag{29}$$

And substituting $s = 5, c_1 = 0$ in (19) and (20) results in

$$\begin{aligned}\cos(z) - 1 + z b_{-1}\sin(z) + z[b_2\sin(c_2 z) + b_3\sin(c_3 z) + b_4\sin(c_4 z) + b_5\sin(c_5 z)]\\ - z[b_2\sin((c_2 - 1)z) + b_3\sin((c_3 - 1)z) + b_4\sin((c_4 - 1)z)\\ + b_5\sin((c_5 - 1)z)\end{aligned} \tag{30}$$

$$\begin{aligned}\sin(z) - z b_1 + z b_{-1}\cos(z) - z[b_2\cos(c_2 z) + b_3\cos c_3 z + b_4\cos c_4 z + b_5\cos c_5 z]\\ + z[b_2\cos((c_2 - 1)z) + b_3\cos((c_3 - 1)z) + b_4\cos((c_4 - 1)z)\\ + b_5\cos((c_5 - 1)z)] = 0\end{aligned} \tag{31}$$

Equations (24) – (31) thus become the equations of order conditions; they now replace order conditions (21). We then substitute the values of the free parameters $c_2, c_3, c_4, c_5, a_{32}, a_{42}, a_{43}, a_{52}, a_{53}, a_{54}$ from Table I into equations (22), (24), (26),(28) which we again solve together with the four additional equations obtained from (21):

$$\left.\begin{aligned} b_1 - b_{-1} - 1 = 0\\ b_{-1} + b_1 + b_2 + b_3 + b_5 - \frac{1}{2} = 0\\ b_2 c_2 + b_3 c_3 + b_4 c_4 + b_5 c_5 - \frac{5}{12} = 0\\ b_3 a_{32} c_2 + b_4 a_{42} c_2 + b_4 a_{43} c_3 + b_5 a_{52} c_2 + b_5 a_{53} c_3 + b_5 a_{54} c_4 - \frac{1}{6} = 0 \end{aligned}\right\} \tag{32}$$

The values of the remaining parameters in terms of the free parameters are computed thus:

$$\left.\begin{aligned} b_{-1} &= -\frac{1}{3}\frac{M_1}{M_2} \\ b_1 &= \frac{1}{3}\frac{M_3}{M_4} \\ b_2 &= \frac{1}{549}\frac{M_5}{M_6} \\ b_3 &= -\frac{1}{1089}\frac{M_7}{M_8} \\ b_4 &= \frac{1}{6}\frac{M_9}{M_{10}} \\ b_5 &= -\frac{1}{6}\frac{M_{11}}{M_{12}} \end{aligned}\right\} \tag{33}$$

$$M_1 = -2z\cos\left(\tfrac{1}{4}z\right) + 2z\cos\left(\tfrac{3}{4}z\right) - 3\sin(z) + 3z \tag{34}$$

$$M_2 = \left[1 - 2\cos\left(\tfrac{1}{4}z\right) + 2\cos\left(\tfrac{3}{4}z\right) - \cos(z)\right]z \tag{35}$$

$$M_3 = -3z\cos(z) - 4z\cos\left(\tfrac{1}{4}z\right) + 4z\cos\left(\frac{3}{4}z\right) + 3\sin(z) \tag{36}$$

$$M_4 = \left[1 - 2\cos\left(\frac{1}{4}z\right) + 2\cos\left(\frac{3}{4}z\right) - \cos(z)\right]z \tag{37}$$

$$\begin{aligned} M_5 = {}& -108 + 380\sin\left(\tfrac{1}{2}z\right)z\cos\left(\tfrac{1}{4}z\right) - 380\sin\left(\tfrac{1}{2}z\right)z\cos\left(\tfrac{3}{4}z\right) - 66\sin\left(\tfrac{1}{2}z\right)z\cos(z) \\ & + 216\cos(z) + \cdots - 108z\sin(z) + 513z\sin\left(\tfrac{1}{4}z\right) \end{aligned} \tag{38}$$

$$\begin{aligned} M_6 = z\Big[& -\cos(z)\sin\left(\tfrac{3}{4}z\right) + 2\cos(z)\sin\left(\tfrac{1}{2}z\right) - \cos(z)\sin\left(\tfrac{1}{4}z\right) + 2\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{3}{4}z\right) \\ & - 4\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{1}{2}z\right) + \cdots + \sin\left(\tfrac{1}{4}z\right)\Big] \end{aligned} \tag{39}$$

$$\begin{aligned} M_7 = {}& -765 - 1070\sin\left(\tfrac{1}{2}z\right)z\cos\left(\tfrac{1}{4}z\right) + 1070\sin\left(\tfrac{1}{2}z\right)z\cos\left(\tfrac{3}{4}z\right) + 51\sin\left(\tfrac{1}{2}z\right)z\cos(z) \\ & + 1530\cos(z) + \cdots + 294z\sin\left(\tfrac{1}{4}z\right) \end{aligned} \tag{40}$$

$$\begin{aligned} M_8 = z\Big[& -\cos(z)\sin\left(\tfrac{3}{4}z\right) + 2\cos(z)\sin\left(\tfrac{1}{2}z\right) - \cos(z)\sin\left(\tfrac{1}{4}z\right) + 2\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{3}{4}z\right) \\ & - 4\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{1}{2}z\right) + \cdots + \sin\left(\tfrac{1}{4}z\right)\Big] \end{aligned} \tag{41}$$

$$\begin{aligned} M_9 = {}& -6 + 4z\cos\left(\tfrac{1}{4}z\right)\sin(z) - 10z\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{3}{4}z\right) - 10z\cos\left(\tfrac{1}{4}z\right)\sin\left(\tfrac{1}{4}z\right) \\ & - 4z\cos\left(\tfrac{3}{4}z\right)\sin(z) + \cdots + 12\cos(z) + 12\cos\left(\tfrac{1}{4}z\right) \end{aligned} \tag{42}$$

$$\begin{aligned} M_{10} = z\Big[& -\cos(z)\sin\left(\tfrac{3}{4}z\right) + 2\cos(z)\sin\left(\tfrac{1}{2}z\right) - \cos(z)\sin\left(\tfrac{1}{4}z\right) + 2\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{3}{4}z\right) \\ & - 4\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{1}{2}z\right) + \cdots + \sin\left(\tfrac{1}{4}z\right)\Big] \end{aligned} \tag{43}$$

$$\begin{aligned} M_{11} = {}& 2z\cos(z)\sin\left(\tfrac{1}{4}z\right) - 6z\sin(z)\sin\left(\tfrac{1}{4}z\right) + 4z\sin\left(\tfrac{1}{4}z\right) - 3 + 2z\cos\left(\tfrac{1}{4}z\right)\sin(z) \\ & - 2z\cos\left(\tfrac{3}{4}z\right)\sin(z) + \cdots + 6\cos(z) + 6\cos\left(\tfrac{1}{4}z\right) \end{aligned} \tag{44}$$

$$\begin{aligned} M_{12} = z\Big[& -\cos(z)\sin\left(\tfrac{3}{4}z\right) + 2\cos(z)\sin\left(\tfrac{1}{2}z\right) - \cos(z)\sin\left(\tfrac{1}{4}z\right) + 2\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{3}{4}z\right) \\ & - 4\cos\left(\tfrac{3}{4}z\right)\sin\left(\tfrac{1}{2}z\right) + \cdots + \sin\left(\tfrac{1}{4}z\right)\Big] \end{aligned} \tag{45}$$

$$\left.\begin{aligned}
\gamma_1 &= 1\\
\gamma_2 &= \cos\left(\tfrac{1}{4}\,z\right)\\
\gamma_3 &= \cos\left(\tfrac{1}{4}\,z\right) + \frac{183}{250} z \sin\left(\tfrac{1}{4}\,z\right)\\
\gamma_4 &= \cos\left(\tfrac{1}{2}\,z\right) + \frac{137}{125} z \sin\left(\tfrac{1}{4}\,z\right)\\
\gamma_5 &= \cos\left(\tfrac{3}{4}\,z\right) + \frac{17}{250} z \sin\left(\tfrac{1}{4}\,z\right) + \frac{69}{125} z \sin\left(\tfrac{1}{2}\,z\right)
\end{aligned}\right\} \qquad (46)$$

Next, the Taylor expansions of (33) and (46) are obtained as (47) and (48) respectively:

$$\left.\begin{aligned}
b_{-1} &= \frac{1}{45} + \frac{13}{3780} z^2 + \frac{17}{172800} z^4 + O(z^6)\\
b_1 &= \frac{46}{45} + \frac{13}{3780} z^2 + \frac{17}{172800} z^4 + O(z^6)\\
b_2 &= \frac{-157}{1647} - \frac{85}{276696} z^2 - \frac{4867}{88542720} z^4 + O(z^6)\\
b_3 &= \frac{236}{8235} - \frac{136807}{11067840} z^2 + \frac{452591}{3541708800} z^4 + O(z^6)\\
b_4 &= -\frac{1}{10} + \frac{139}{10080} z^2 + \frac{227}{3225600} z^4 + O(z^6)\\
b_5 &= \frac{29}{45} - \frac{313}{60480} z^2 + \frac{271}{19353600} z^4 + O(z^6)
\end{aligned}\right\} \qquad (47)$$

$$\left.\begin{aligned}
\gamma_1 &= 1\\
\gamma_2 &= 1 - \frac{1}{32} z^2 + \frac{1}{6144} z^4 + O(z^6)\\
\gamma_3 &= 1 + \frac{67}{2000} z^2 - \frac{131}{256000} z^4 + O(z^6)\\
\gamma_4 &= 1 - \frac{387}{4000} z^2 + \frac{887}{384000} z^4 + O(z^6)\\
\gamma_5 &= 1 - \frac{29}{4000} z^2 + \frac{1309}{768000} z^4 + O(z^6)
\end{aligned}\right\} \qquad (48)$$

The coefficients are substituted into the order conditions (21) and the Taylor expansion obtained as

$$\left.\begin{aligned}
&order\ 1:\quad b_1 - b_{-1} = 1 + O(z^6)\\
&order\ 2:\ b_{-1} + \textstyle\sum_{i=2}^{s} b_i = \frac{1}{2} + O(z^6)\\
&order\ 3:\quad \textstyle\sum_{i=2}^{s} b_i\, c_i = \frac{5}{12} + O(z^6)\\
&order\ 4:\ \textstyle\sum_{i=2}^{s} b_i\, c_i^2 = \frac{1}{3} - \frac{1}{4608} z^2 + \frac{29}{2064384} z^4 + O(z^6)\\
&\textstyle\sum_{i=2,j=1}^{s} b_i\, a_{ij} c_j = \frac{1}{6} + O(z^6)\\
&order\ 5:\ \textstyle\sum_{i=2}^{s} b_i\, c_i^3 = \frac{31}{120} - \frac{209}{322560} z^2 + \frac{1223}{103219200} z^4 + O(z^6)\\
&\textstyle\sum_{i=2,j=1}^{s} b_i c_i\, a_{ij} c_j = \frac{23261}{180000} + \frac{29249}{120960000} z^2 + \frac{265997}{38707200000} z^4 + O(z^6)\\
&\textstyle\sum_{i=2,j=1}^{s} b_i\, a_{ij} c_j^2 = \frac{323}{3750} - \frac{7199}{20160000} z^2 + \frac{6233}{6451200000} z^4 + O(z^6)\\
&\textstyle\sum_{i=2,j=1}^{s} b_i\, a_{ij} a_{jk} c_k = \frac{428617}{10000000} + \frac{25554019}{3360000000} z^2 + \frac{48749627}{1075200000000} z^4 + O(z^6)
\end{aligned}\right\} \qquad (49)$$

It is observed that (49) reduces to the order conditions of the IRK5-5, represented by (21), as z approaches zero; which confirms that the EFIRK5-5 method is of order five.

Choice of Frequency

According to Berghe et al. (2000), a mathematical theory to determine the frequency ω in an exact way is non-existent. However, a study of the local truncation error (LTE) was made, with the goal of making the LTE as small as possible, out of which follows a simple heuristically chosen algorithm to estimate the ω in each integration interval $[x_n, x_{n+1}]$ in the following way

$$\left.\begin{array}{ll} \omega = \sqrt{-\dfrac{y^{(2)}(x_n)}{y(x_n)}}\,, & n = 0, \cdots \ \text{if}\ y(x_n) \neq 0 \ \text{and} \\ \omega = 0 & \text{otherwise} \end{array}\right\} \qquad (50)$$

In the case where $y(x)$ is a linear combination of $sin(\alpha x)$ and $cos(\alpha x)$ and $y(x_n) \neq 0$, then it is obtained that $\omega = \alpha$ and $y(x)$ will be integrated exactly. The expressions for the occurring second derivatives can be obtained analytically from the given ODEs or calculated numerically using previously derived $y(x_{n-j})$ values. The ω values used are then in each integration interval taken as the positive square root of the numerically obtained ω^2. Also, if negative ω^2-values are obtained, ω is replaced in the corresponding formulae by $(i^2 = -1)$. In fact, in this case, the exponential functions instead of the trigonometric ones are integrated.Stability Analysis

The general exponentially-fitted improved Runge-Kutta (EFIRK) method (2) can be rewritten as

$$y_{n+1} = y_n + hb_1k_1 - hb_{-1}k_{-1} + h\sum_{i=2}^{s} b_i(k_i - k_{-i}) \qquad (51)$$

where $k_1 = f(x_n, y_n)$, $k_{-1} = f(x_{n-1}, y_{n-1})$, $k_i = f(x_n + c_ih, \gamma_iy_n + h\sum_{j=1}^{i-1} a_{ij}\,k_j)$ and $k_{-i} = f(x_n + c_ih, \gamma_{-1}y_{n-1} + h\sum_{j=1}^{i-1} a_{ij}\,k_{-j})$.

Apply (51) with $s = 5$ to the scalar test problem

$$y' = \lambda y, \quad \text{where } \lambda \in \mathbb{C} \qquad (52)$$

For finding the region of absolute stability, we set $\lambda h = \bar{h}$ to obtain the following expression for the stability polynomial

$$\begin{aligned} y_{n+1} = \Bigg[&\left(\sum_{i=2,j,k=1}^{5} b_i a_{ij} a_{jk} a_{km} c_m\right)\bar{h}^5 + \left(\sum_{i=2,j=1}^{5} b_i a_{ij} a_{jk} a_k\right)\bar{h}^4 + \left(\sum_{i=2,j=1}^{5} b_i a_{ij} c_j\right)\bar{h}^3 \\ &+ \left(\sum_{i=2}^{5} b_i c_i\right)\bar{h}^2 + \left(\sum_{i=2}^{5} b_i\right)\bar{h}\Bigg] y_n \\ &- \Bigg[\left(\sum_{i=2,j,k,m=1}^{5} b_i a_{ij} a_{jk} a_{km} c_m\right)\bar{h}^5 + \left(\sum_{i=2,j,k=1}^{5} b_i a_{ij} a_{jk} a_k\right)\bar{h}^4 \\ &+ \left(\sum_{i=2,j=1}^{5} b_i a_{ij} c_j\right)\bar{h}^3 + \left(\sum_{i=2}^{5} b_i c_i\right)\bar{h}^2 + \left(b_{-1} + \sum_{i=2}^{5} b_i\right)\bar{h}\Bigg] \end{aligned} \qquad (53)$$

We note that the expressions for the coefficients of $\bar{h}$it's in (53) are the order conditions up to order five given in (21). By setting $y_{n+1} = \xi^2$ and $y_n = \xi$ and substituting for these order conditions using values from the Taylor series expansions presented in (47), as z tends to zero, we obtain the stability polynomials for the derived EFIRK5 as

$$\begin{aligned} \rho(\xi, \bar{h}) = \xi^2 - \left(\frac{1}{120}\bar{h}^5 + \frac{428617}{10000000}\bar{h}^4 + \frac{1}{6}\bar{h}^3 + \frac{5}{12}\bar{h}^2 + \frac{3}{2}\bar{h}\right)\xi + \frac{1}{120}\bar{h}^5 + \frac{428617}{10000000}\bar{h}^4 \\ + \frac{1}{6}\bar{h}^3 + \frac{5}{12}\bar{h}^2 + \frac{1}{2}\bar{h} \end{aligned} \qquad (54)$$

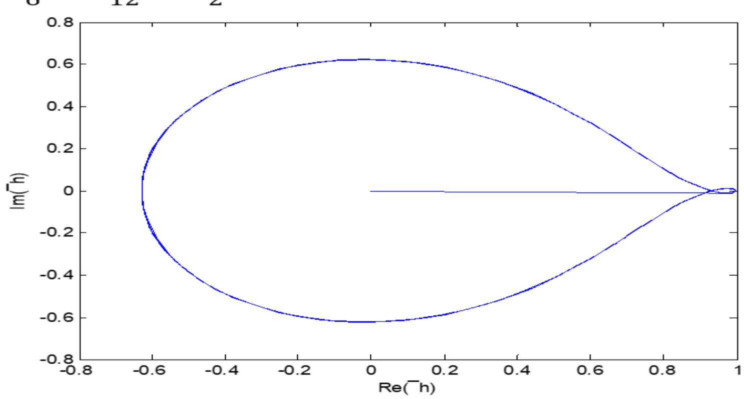

Figure 1 The region of absolute stability for the EFIRK5-5

Figure 1: shows the region of absolute stability of EFIRK5-5 and its interval of absolute stability is (-0.68, 1)

RESULTS AND DISCUSSION

We apply the new method to two problems. We consider the following problems:

Problem 1:

$$y'(x) = 10\cos(10x), \qquad y(0) = 0 \qquad (55)$$

Which has a solution of the form $y(x) = \sin(10x)$. Equation (55) is solved numerically on the interval $0 \le x \le 1$ and with $\omega = 10$ using the aforementioned method.

Problem 2:

$$y'(x) = x\,\mathrm{e}^{-3x} + 2x, \qquad y(0) = -\frac{1}{9} \qquad (56)$$

With a solution of the form$y(x) = x^2 - \frac{1}{3}x\mathrm{e}^{-3x} - \frac{1}{9}\mathrm{e}^{-3x}$, where the frequency ω is computed as $\omega = 5$.

In all our numerical illustrations, we compare the following methods.

i. The original improved Runge-Kutta method of order five given Table I, that is method of table 1 without exponential fitting, indicated as method IRK5-5
ii. The new exponentially fitted improved Runge-Kutta method of order five indicated as method EFIRK5-5
iii. The analytic method which gives the exact solution, indicated as y(x).

Table 2: Comparison of results for petroleum 1 in [0, 1], h = 0.05, w = 10

x	$y(x)$	EFIRK5-5	Error	IRK5-5	Error
0.05	0.4794255380	0.4794255380	1.3934367E-24	0.4794294216	3.88300269500E-06
0.10	0.8414709848	0.8414709848	1.1943620E-24	0.8414117419	5.92429509959E-05
0.15	0.9974949866	0.9974949866	3.4378570E-24	0.9973107823	1.84204278485E-04
0.20	0.9092974268	0.9092974268	6.1811980E-24	0.9089531377	3.44289091319E-04
0.25	0.5984721441	0.5984721441	8.7527210E-24	0.5979718411	5.00303044164E-04
0.30	0.1411200081	0.1411200081	1.0522827E-23	0.1405059596	6.14048480185E-04
0.35	-0.3507832277	-0.3507832277	1.1058133E-23	-0.3514409042	6.57676549635E-04
0.40	-0.7568024953	-0.7568024953	1.0227575E-23	-0.7574230009	6.20505579529E-04
0.45	-0.9775301177	-0.9775301177	8.2345060E-24	-0.9780417540	5.11636319734E-04
0.50	-0.9589242747	-0.9689242747	5.5668960E-24	-0.9592819984	3.57723761994E-04
0.55	-0.7055403256	-0.7055403256	2.8778710E-24	-0.7057367766	1.96451068332E-04
0.60	-0.2794154999	-0.2794154999	8.2579700E-25	-0.2794828016	6.73034187396E-05
0.65	0.2151199881	0.2151199881	8.6907000E-26	0.2151180874	1.90066201770E-06
0.70	0.6569865987	0.6569865987	3.6322100E-25	0.6569703430	1.62556740129E-05
0.75	0.9379999768	0.9379999768	2.0659740E-25	0.9378931229	1.06853847140E-04
0.80	0.9893582466	0.9893582466	4.6044590E-24	0.9891067330	2.51513588897E-04
0.85	0.7984871126	0.7984871126	7.3571660E-24	0.7980722955	4.14817149315E-04
0.90	0.4121184852	0.4121184852	9.6501370E-24	0.4115617031	5.56782121394E-04
0.95	-0.0751511205	-0.0751511205	1.0921971E-23	-0.0757937709	6.42650528768E-04
1.00	-0.5440211109	-0.5440211109	1.0861282E-23	-0.5446725097	6.51398790545E-04

Table 3: Comparison of Results for Problem 2 in $[0,1]$, $h = 0.05$, $\omega = 5$

x	$y(x)$	EFIRK5-5	Error	IRK5-5	Error
0.05	-0.1074793525	-0.1074793526	2.47189770E-11	-0.1074781472	1.205316372E-06
0.10	-0.0970070763	-0.0970070776	1.35991889E-09	-0.0970024177	4.658623556E-06
0.15	-0.0802289799	-0.0802289825	-0.0802289E-09	-0.0802200398	8.940196992E-06
0.20	-0.0575665131	-0.0575665165	3.39311803E-09	-0.0575535890	1.292404682E-05
0.25	-0.0293490519	-0.0293490561	4.14203276E-09	-0.0293323772	1.667471608E-05
0.30	0.00416862738	0.0041686226	4.75271549E-09	0.0041888719	2.024454367E-05
0.35	0.04279195715	0.0427919519	5.24914482E-09	0.0428156329	2.367581192E-05
0.40	0.08637474819	0.0863747425	5.65130128E-09	0.0864017507	2.700253273E-05
0.45	0.13480948749	0.1348094815	5.97581595E-09	0.1348397394	3.025193119E-05
0.50	0.18801939996	0.1880193937	6.23651583E-09	0.1880528456	3.344567637E-05
0.55	0.24595197135	0.2459519649	6.44488239E-09	0.2459885723	3.660090092E-05
0.60	0.30857367922	0.3085736726	6.61043669E-09	0.3086134103	3.973104439E-05
0.65	0.37586572098	0.3758657142	6.74106276E-09	0.3759085675	4.284654969E-05
0.70	0.44782056360	0.4478205567	6.84327891E-09	0.4478665190	4.595543712E-05
0.75	0.52443916891	0.5244391619	6.92246526E-09	0.5244882327	4.906377667E-05
0.80	0.60572877320	0.6057287662	6.98305431E-09	0.6057809493	5.217607574E-05
0.85	0.69170112063	0.6917011136	7.02869059E-09	0.6917564162	5.529559665E-05
0.90	0.78237106698	0.7823710599	7.06236412E-09	0.7824294916	5.842461608E-05
0.95	0.87775548496	0.8777554779	7.08652209E-09	0.8778170496	6.156463634E-05
1.00	0.97787241406	0.9778724069	7.10316205E-09	0.9779371306	6.471655692E-05

Table 4: Comparison of Results for Problem 2 in $[0,1]$, $h = 0.05$, $\omega = 5$

h	EFIRK5-5	IRK5-5	NFEs
$\mathbf{0.05}$	1.10666180000E-23	6.577447507946E-04	10000
$\mathbf{0.025}$	2.29628809000E-22	3.327995692713E-04	20000
$\mathbf{0.0125}$	1.07980091732E-19	1.666193812173E-04	40000
$\mathbf{0.00625}$	3.77879924219E-17	8.332803981257E-05	80000
$\mathbf{0.003125}$	1.00321288098E-15	4.166602431409E-05	160000
$\mathbf{0.0015625}$	3.33811029343E-13	2.0833253635942-05	320000

h	EFIRK5-5	IRK5-5	NFEs
$\mathbf{0.05}$	7.12276483049E-09	6.66292897824E-03	10000

0.025	2.48840446673E-10	3.33239766414E-03	20000
0.0125	8.20344713044E-12	1.66643253125E-03	40000
0.00625	2.62930101153E-13	8.33274769920E-04	80000
0.003125	6.91208257424E-12	4.16652022036E-04	160000
0.0015625	9.53459932039E-10	2.08329671700E-04	320000

The error in each step of integration within the interval [0, 1] as given in Table II reveals that the EFIRK5-5 method is more accurate than the original IRK5-5 with the same number of function evaluations. From Table III, it is evident that the new method also provides more accurate integration to the problem than the existing IRK5-5 with the same computational efficiency. It is clear from Table IV that the EFIRK5-5 gives more accurate results than the IRK5-5 method. On the other hand, the accuracy of the new method decreases as the step size grows smaller which also indicates that the original method is recovered from our new method as h approaches zero. In Table V It is observed that EFIRK5-5 approaches the original IRK5-5 method as h tends to zero.

CONCLUSION

A new five-stage exponentially fitted improved Runge-Kutta method has been constructed. The method was applied to solve oscillatory and exponential problems. The results of the numerical examples revealed that for all the problems considered, the new exponentially fitted method is much more efficient than the original method without exponential fitting.

All computations were carried out using MAPLE 2019 S oftware Package.

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