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Full Length Research Article

# MODIFIED SUCCESSIVE OVERRELAXATION (SOR) TYPE METHODS FOR M-MATRICES

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**ABSTRACT**
The SOR is a basic iterative method for solution of the linear system $Ax = b$. Such systems can easily be solved using direct methods such as Gaussian elimination. However, when the coefficient matrix $A$ is large and sparse, iterative methods such as the SOR become indispensable. A new preconditioner for speeding up the convergence of the SOR iterative method for solving the linear system $Ax = b$ is proposed. Arising from the preconditioner, two new preconditioned iterative techniques of the SOR method are developed. The preconditioned iterations are applied to the linear system whose coefficient matrix is an $M-$matrix. Convergence of the preconditioned iterations is established through standard procedures. Numerical examples and results comparison are in conformity with the analytic results. More so, it is established that the spectral radii of the proposed preconditioned SOR $G_1$ and $G_2$ are less than that of the classical SOR, which implies faster convergence.

**Keywords:** SOR method, preconditioner, *M*-matrix, convergence, spectral radius

**INTRODUCTION**
The discretization by finite differences of elliptic partial differential equations that appear in many areas of science and engineering in most cases results into an associated linear system of equations

$$A = b \tag{1}$$

where the coefficient matrix $A$, being an $n \times n$ square matrix , is large and sparse, and usually, has certain particular structures and properties, such as belonging to the class of $M-$matrices (that is a matrix for which $a_{ii} > 0,\ a_{ij} \le 0\ (i \ne j),\ A$ is nonsingular and $A^{-1} \ge 0$); and $x$ and $b$ are $n-$dimensional vectors. Suppose $A = M - N$ is a regular splitting of the matrix $A$ in (1). Then, the general basic iteration method for solving (1) is of the form

$$x^{(k+1)} = Gx^{(k)} + c\ , \qquad k = 0,1,2,\cdots \tag{2}$$

where $G = M^{-1}N$ and $c = M^{-1}b$. The necessary and sufficient condition for convergence of the iterative method (2) entails that the spectral radius of the method be less than 1, and the smaller it is, the faster its convergence. The goal of preconditioning is to speed up the convergence of an iterative method by decreasing the spectral radius of the iteration matrix. A preconditioned linear system is obtained by applying the matrix $P = I + S$, where $I$ is the identity matrix and $S$ is a sparse matrix whose nonzero entries are the negatives of the corresponding entries of $A$, to system (1) thus

$$PAx = Pb \tag{3}$$

and its corresponding general basic iterative method

$$x^{(k+1)} = M_p^{-1}N_px^{(k)} + M_p^{-1}Pb\ , \qquad k = 0,1,2,\cdots \tag{4}$$

results from the splitting $PA = M_p - N_p$. It is assumed, for simplicity, that the diagonal entries of $A$ are unit elements and thus $A$ has the usual splitting $A = I - L - U$, where $-L$ and $-U$ are strictly lower and strictly upper triangular matrices, respectively. Arising from this splitting the iteration matrix of the classical SOR method is described by $G_S = (I - \omega L)^{-1}\{(1 - \omega)I + \omega U\}$. The SOR method was developed independently by Frankel (1950), Young (1950) and Young (1954).

A modified SOR method was first proposed by Devogelaere (1950). Sisler (1972) and Sisler (1973) focused more on the use of more than one parameter for the SOR method. The Accelerated Overrelaxation (AOR) method, introduced by Hadjidimos (1978), was an improvement of Sisler's method. Ever since, a great number of papers have been written on improving the convergence of the SOR method. Bai and Chai (2003) proposed a new class of SOR methods called asymptotically optimal SOR (AOSOR) methods for solving large sparse linear systems by choosing the relaxation parameter in a dynamic fashion according to known information at the current iterate step. Dehghan and Hajarian (2009) applied two new preconditioner techniques, $\bar{P} = I + \bar{S}$ and $\tilde{P} = I + \tilde{S}$, to the successive overrelaxation iterative method for solving $L-$matrix linear systems under mild assumptions on the coefficient matrix $A$. Moussavi (2009) generalized the method of Sisler and provided a range for the second parameter on which the two-parameter method proved to be better than the SOR method. Youssef (2012) introduced the KSOR, a new variant of the SOR method that results from exploitation of the hidden explicit characterization of linear functions. It was proved that the KSOR can converge for all values of the relaxation parameter $\omega^* \in \mathbb{R} - [-2, 0]$ not only for $\omega \in (0, 2)$ as in the SOR method. Ndanusa and Adeboye (2012) proposed a preconditioned SOR method of the type $I + S$ for accelerating the convergence of the classical SOR method. Youssef *et al.* (2016) introduced a line version of the KSOR method for solving systems of linear equations, the LKSOR. Adapted from the KSOR method, the LKSOR employs the same philosophy as obtained in the line SOR method, LSOR. The LKSOR exploits the advantages of the LSOR in addition to those of the KSOR to obtain an efficient iterative algorithm. Zhang *et al.* (2016) proposed some necessary and sufficient conditions for convergence of the SOR iterative methods, including FSOR, BSOR and SSOR, for linear systems with weak $H-$matrices. This present work is a further attempt at accelerating the rate of convergence of the SOR method by introducing a preconditioned version of the method.

## MATERIALS AND METHODS

### Preliminaries

To provide the preconditioned effect on (1) we let $P = I + S$ in (3) where $I$ is the identity matrix and $S$ is a sparse matrix whose nonzero entries are the negatives of corresponding entries of $A$.

$$S = \begin{cases} -a_{ij}, & j = i+1, \text{ for } i < 2 \text{ and } j = i-1 \text{ for } i > j \\ 0, & \text{otherwise } 0 \ \ 0000000000000000000 \end{cases}$$

Equation (3) is now written in the form

$$\tilde{A}x = \tilde{b} \quad (5)$$

where $\tilde{A} = PA$ and $\tilde{b} = Pb$. Equation (5), which has the same solution as equation (1), is referred to as a preconditioned system while the transformation matrix $P$ is called the preconditioning matrix or the preconditioner. Thus the preconditioner $P$ must be nonnegative so as to make $\tilde{A}$ an $L$ −matrix. From (5) results

$$\tilde{A} = PA = (I + S)(I - L - U)$$
$$= I - L - U + S - SL - SU$$
$$= I - L - U - L_S - U_S + D_1 - L_1 - U_1$$

where $S = -L_S - U_S$ and $-SL - SU = D_1 - L_1 - U_1$

Therefore,

$$\tilde{A} = (I + D_1) - (L + L_S + L_1) - (U + U_S + U_1)$$

That is,

$$\tilde{A} = \tilde{D} - \tilde{L} - \tilde{U}$$

Where $\tilde{D} = I + D_1$, $\tilde{L} = L + L_S + L_1$ and $\tilde{U} = U + U_S + U_1$.

The application of overrelaxation parameter $\omega$ to (5) results in the following

$$\omega\tilde{A}x = \omega\tilde{b} \quad (6)$$

The matrix $\omega\tilde{A}$ is subjected to a regular splitting

$$\omega\tilde{A} = \omega(\tilde{D} - \tilde{L} - \tilde{U})$$
$$= \omega(I + D_1 - \tilde{L} - \tilde{U})$$
$$I - I + \omega I + \omega D_1 - \omega\tilde{L} - \omega\tilde{U}$$
$$= I - \omega\tilde{L} + \omega D_1 - I + \omega I - \omega\tilde{U}$$
$$= [I - \omega(\tilde{L} - D_1)] - \{(1-\omega)I + \omega\tilde{U}\}$$
$$= M - N$$

where $M = [I - \omega(\tilde{L} - D_1)]$ and $N = \{(1-\omega)I + \omega\tilde{U}\}$.

Hence, the preconditioned SOR scheme is defined as

$$x^{(k+1)} = [I - \omega(\tilde{L} - D_1)]^{-1}\{(1-\omega)I + \omega\tilde{U}\}x^{(k)} + [I - \omega(\tilde{L} - D_1)]^{-1}\omega b'$$

That is,

$$x^{(k+1)} = G_1 x^{(k)} + c \quad (7)$$

where $G_1 = [I - \omega(\tilde{L} - D_1)]^{-1}\{(1-\omega)I + \omega\tilde{U}\}$ is the preconditioned SOR iteration matrix.

Also, from (6)

$$\omega\tilde{A} = \omega(\tilde{D} - \tilde{L} - \tilde{U})$$
$$= \omega\tilde{D} - \omega\tilde{L} - \omega\tilde{U}$$
$$\tilde{D} - \tilde{D} + \omega\tilde{D} - \omega\tilde{L} - \omega\tilde{U}$$
$$= (\tilde{D} - \omega\tilde{L}) - [(1-\omega)\tilde{D} + \omega\tilde{U}]$$

is another splitting of the preconditioned coefficient matrix $\omega A' = M - N$, where $M = (\tilde{D} - \omega\tilde{L})$ and $N = [(1-\omega)\tilde{D} + \omega\tilde{U}]$, from whence the second preconditioned SOR iterative scheme is defined thus

$$x^{(k+1)} = G_2 x^{(k)} + c \quad (8)$$

where $G_2 = M^{-1}N = (\tilde{D} - \omega\tilde{L})^{-1}[(1-\omega)\tilde{D} + \omega\tilde{U}]$ and $c = M^{-1}\omega b' = (\tilde{D} - \omega\tilde{L})^{-1}\omega b'$.

The resultant entries of the matrix $\tilde{A} = (\tilde{a}_{ij})$ are characterized as follow.

$$\left.\begin{array}{ll}
\tilde{a}_{ii} = 1 - a_{is}a_{si}, & (i,s) = (1,2),(n,n-1) \\
\tilde{a}_{ii} = 1 - a_{i-1,i}a_{i,i-1} - a_{i,i+1}a_{i+1,i}, & i = 2(1)n-1 \\
\tilde{a}_{ij} = 0, & (i,j) = (1,2),(n,n-1) \\
\tilde{a}_{i,i-1} = -a_{i,i+1}a_{i+1,i-1}, & i = 2(1)n-1 \\
\tilde{a}_{i,i+1} = -a_{i,i-1}a_{i-1,i+1}, & i = 2(1)n-1 \\
\tilde{a}_{1,i} = a_{1,i} - a_{12}a_{2,i}, & i = 3(1)n \\
\tilde{a}_{n,i} = a_{n,i} - a_{n-1,i}a_{n,n-1}, & i = 1(1)n-2 \\
\tilde{a}_{i,i-2} = a_{i,i-2} - a_{i-1,i-2}a_{i,i-1} - a_{i,i+1}a_{i+1,i-2}, & i = 3(1)n-1 \\
\tilde{a}_{i,i+2} = a_{i,i+2} - a_{i-1,i+2}a_{i,i-1} - a_{i,i+1}a_{i+1,i+2}, & i = 2(1)n-2 \\
\tilde{a}_{i,1} = a_{i,1} - a_{i-1,1}a_{i,i-1} - a_{i,n}a_{n,1}, & i = 4(1)n-1 \\
\tilde{a}_{i,n} = a_{i,n} - a_{i-1,n}a_{i,i-1} - a_{i,i+1}a_{i+1,n}, & i = 2(1)n-2
\end{array}\right\} \quad (9)$$

The $M$ −matrix structure of $\tilde{A}$ entails that

$$1 - a_{is}a_{si} > 0, \quad (i,s) = (1,2),(n,n-1)$$

and

$$1 - a_{i-1,i}a_{i,i-1} - a_{i,i+1}a_{i+1,i} > 0, \quad i = 2(1)n-1$$

That is, $1 > a_{is}a_{si} \geq 0$ and $1 > a_{i-1,i}a_{i,i-1} + a_{i,i+1}a_{i+1,i} \geq 0$. Hence, we must have that

$0 \leq a_{is}a_{si} < 1, (i,s) = (1,2),(n,n-1)$ and $0 \leq a_{i-1,i}a_{i,i-1} + a_{i,i+1}a_{i+1,i} < 1,\ i = 2(1)n-1$

### Convergence Theorems

The following lemmas are needed in order to prove our main theorems.

**Lemma 1 (Varga (1981))** Let $A \geq 0$ be an irreducible $n \times n$ matrix. Then,

i. $A$ has a positive real eigenvalue equal to its spectral radius.
ii. To $\rho(A)$ there corresponds an eigenvector $x > 0$.
iii. $\rho(A)$ increases when any entry of $A$ increases.
iv. $\rho(A)$ is a simple eigenvalue of $A$.

**Lemma 2 (Varga (1981))**

i. Let $A$ be a nonnegative matrix. Then

If $\alpha x \leq Ax$ for some nonnegative vector $x, x \neq 0$, then $\alpha \leq \rho(A)$.

ii. If $Ax \leq \beta x$ for some positive vector $x$, then $\rho(A) \leq \beta$. Moreover, if $A$ is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector $x$, then $\alpha \leq \rho(A) \leq \beta$ and $x$ is a positive vector.

**Lemma 3 (Li and Sun(2000))** Let $A = M - N$ be an $M$ −splitting of $A$. Then the splitting is convergent, i.e., $\rho(M^{-1}N < 1)$, if and only if $A$ is a nonsingular $M$ −matrix.

**Theorem 1** Let $G_{SOR} = (I - \omega L)^{-1}\{(1-\omega)I + \omega U\}$, $G_1 = [I - \omega(\tilde{L} - D_1)]^{-1}\{(1-\omega)I + \omega\tilde{U}\}$ and $G_2 = (\tilde{D} - \omega\tilde{L})^{-1}\{(1-\omega)\tilde{D} + \omega\tilde{U}\}$ be the SOR, first preconditioned SOR and second preconditioned SOR iteration matrices respectively. If $A$ is an irreducible $M$ −matrix with $0 \leq a_{is}a_{si} < 1, (i,s) = (1,2),(n,n-1)$, $0 \leq a_{i-1,i}a_{i,i-1} + a_{i,i+1}a_{i+1,i} < 1,\ i = 2(1)n-1$ and $0 < \omega < 1$, then $G_{SOR}$, $G_1$ and $G_2$ are nonnegative and irreducible matrices.

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**Proof:** The $G_{SOR}, G_1$ and $G_2$ reduce to $I$ when $\omega = 0$. For $\omega < 0$ and $\omega > 1$, negative entries appear in these matrices. Thus range of values of $\omega$ that ensures nonnegativity of these matrices is, $0 < \omega < 1$.

Given $0 < \omega < 1$, $(1-\omega)I + \omega U \geq 0$, since $U \geq 0$. Also, $(I - \omega L)^{-1} = I + \omega L + \omega^2 L^2 + \cdots + \omega^{n-1}L^{n-1} \geq 0$, since $L \geq 0$. Hence $G_{SOR} = (I-\omega L)^{-1}[(1-\omega)I + \omega U] \geq 0$, that is, a nonnegative matrix. For $0 < \omega < 1$,

$$\begin{aligned} G_{SOR} &= [I + \omega L + \omega^2 L^2 + \cdots + \omega^{n-1}L^{n-1}][(1-\omega)I \\ &\quad + \omega U] \\ &= (1-\omega)I + \omega(1-\omega)L + \omega U + \omega^2 LU + \omega^2(1-\omega)L^2 \\ &\quad + \omega^3 L^2 U + \cdots \\ &= (1-\omega)I + \omega(1-\omega)L + \omega U + \text{nonnegative terms} \end{aligned}$$

Since $A = I - L - U$ is irreducible, so also is the matrix $(1-\omega)I + \omega(1-\omega)L + \omega U$ because the coefficients of $I, L$ and $U$ are not zero and less than 1 in absolute value. Hence, $G_{SOR}$ is an irreducible matrix.

The iteration matrix $G_1$ is defined by

$$G_1 = [I - \omega(\tilde{L} - D_1)]^{-1}\{(1-\omega)I + \omega\tilde{U}\}$$

Since $\tilde{L} \geq 0$, $\tilde{U} \geq 0$, $-D_1 \geq 0$, and for $0 < \omega < 1$, $(1-\omega)I + \omega\tilde{U} \geq 0$ and $[I - \omega(\tilde{L} - D_1]^{-1} = I + \omega(\tilde{L} - D_1) + \omega^2(\tilde{L} - D_1)^2 + \cdots + \omega^{n-1}(\tilde{L} - D_1)^{n-1} \geq 0$. Hence, $G_1 = [I - \omega(\tilde{L} - D_1)]^{-1}\{(1-\omega)I + \omega\tilde{U}\} \geq 0$, and therefore $G_1$ is a nonnegative matrix.

Let the coefficient matrix $A = I - L - U$ be an irreducible matrix; then the preconditioned matrix $\tilde{A}$ is defined by

$$\begin{aligned} \tilde{A} &= PA = (I+S)A = (I - L_s - U_s)A \\ &= (I - L_s - U_s)(I - L - U) \\ &= I - L_s - U_s - L + L_sL + U_sL - U + L_sU + U_sU \\ &= I - L_s - U_s - L + L_sL - (U_sL)_L - (U_sL)_U - U \\ &\quad - (L_sU)_L - (L_sU)_U + U_sU \\ &= I - L - L_s + L_sL - (L_sU)_L - (U_sL)_L - U - U_s + U_sU \\ &\quad - (U_sL)_U - (L_sU)_U \\ &= I - (L + L_s - L_sL + (L_sU)_L + (U_sL)_L) \\ &\quad - (U + U_s - U_sU + (U_sL)_U \\ &\quad + (L_sU)_U) \\ &= I - \tilde{L} - \tilde{U} \end{aligned}$$

where $\tilde{L} = L + L_s - L_sL + (L_sU)_L + (U_sL)_L$, $\tilde{U} = U + U_s - U_sU + (U_sL)_U + (L_sU)_U$ and $-(T)_L$ and $-(T)_U$ denote the strictly lower and strictly upper parts of the matrix $T$ respectively. Since $A$ is irreducible, it is obvious that $\tilde{A} = I - \tilde{L} - \tilde{U}$ is irreducible, since it inherits the nonzero structure of the irreducible matrix $A$. Now,

$$\begin{aligned} G_1 &= [I - \omega(\tilde{L} - D_1)]^{-1}\{(1-\omega)I + \omega\tilde{U}\} \\ &= [I + \omega(\tilde{L} - D_1) + \omega^2(\tilde{L} - D_1)^2 + \cdot\cdot \\ &\quad \cdot + \omega^{n-1}(\tilde{L} - D_1)^{n-1}]\{(1-\omega)I \\ &\quad + \omega\tilde{U}\} \\ &= (1-\omega)I + \omega\tilde{U} + \omega(1-\omega)(\tilde{L} - D_1) + \omega^2(\tilde{L} - D_1)\tilde{U} \\ &\quad + \omega^2(1-\omega)(\tilde{L} - D_1)^2 + \cdots \\ &= (1-\omega)I + \omega(1-\omega)\tilde{L} + \omega\tilde{U} + \omega(1-\omega)(-D_1) \\ &\quad + \omega^2(\tilde{L} - D_1)\tilde{U} \\ &\quad + \omega^2(1-\omega)(\tilde{L} - D_1)^2 + \cdots \\ &= (1-\omega)I + \omega(1-\omega)\tilde{L} + \omega\tilde{U} + \text{nonnegative terms} \end{aligned}$$

Since $\tilde{A} = I - \tilde{L} - \tilde{U}$ is irreducible, it implies, for $0 < \omega < 1$, the matrix $(1-\omega)I + \omega(1-\omega)\tilde{L} + \omega\tilde{U}$ is also irreducible, because the coefficients of $I$, $\tilde{L}$ and $\tilde{U}$ are different from zero and less than one in absolute value. Therefore, the matrix $G_1 = [I - \omega(\tilde{L} - D_1)]^{-1}\{(1-\omega)I + \omega\tilde{U}\}$ is irreducible. Hence $G_1$ is a nonnegative and irreducible matrix.

Similarly,

$$\begin{aligned} G_2 &= (\tilde{D} - \omega\tilde{L})^{-1}[(1-\omega)\tilde{D} + \omega\tilde{U}] \\ &= [\tilde{D}(I - \omega\tilde{D}^{-1}\tilde{L})]^{-1}[(1-\omega)\tilde{D} + \omega\tilde{U}] \\ &= [\tilde{D}(I - \omega\tilde{D}^{-1}\tilde{L})]^{-1}[(1-\omega)\tilde{D} + \omega\tilde{U}] \\ &= (I - \omega\tilde{D}^{-1}\tilde{L})^{-1}\tilde{D}^{-1}[(1-\omega)\tilde{D} + \omega\tilde{U}] \\ &= (I - \omega\tilde{D}^{-1}\tilde{L})^{-1}[(1-\omega)I + \omega\tilde{D}^{-1}\tilde{U}] \\ &= [I + \omega\tilde{D}^{-1}\tilde{L} + \omega^2(\tilde{D}^{-1}\tilde{L})^2 + \cdots \\ &\quad + \omega^{n-1}(\tilde{D}^{-1}\tilde{L})^{n-1}][(1-\omega)I \\ &\quad + \omega\tilde{D}^{-1}\tilde{U}] \\ &= (1-\omega)I + \omega(1-\omega)\tilde{D}^{-1}\tilde{L} + \omega\tilde{D}^{-1}\tilde{U} \\ &\quad + \text{nonnegative terms} \end{aligned}$$

Using similar arguments it is conclusive that $G_2 = (\tilde{D} - \omega\tilde{L})^{-1}[(1-\omega)\tilde{D} + \omega\tilde{U}]$ is a nonnegative and irreducible matrix.

**Theorem 2** Let $G_{SOR} = (I - \omega L)^{-1}\{(1-\omega)I + \omega U\}$ and $G_1 = [I - \omega(\tilde{L} - D_1)]^{-1}\{(1-\omega)I + \omega\tilde{U}\}$ be the SOR and the preconditioned SOR iteration matrices respectively. If $0 < \omega < 1$ and $A$ is an irreducible $M$ −matrix with $0 \leq a_{is}a_{si} < 1, (i,s) = (1,2), (n, n-1)$, $0 \leq a_{i-1,i}a_{i,i-1} + a_{i,i+1}a_{i+1,i} < 1$, $i = 2(1)n-1$, then

a) $\rho(G_1) < \rho(G_{SOR})$, if $\rho(G_{SOR}) < 1$
b) $\rho(G_1) = \rho(G_{SOR})$, if $\rho(G_{SOR}) = 1$
c) $\rho(G_1) > \rho(G_{SOR})$, if $\rho(G_{SOR}) > 1$

**Proof:** It is established in Theorem 1 that the $G_{SOR}$ and $G_1$ are nonnegative and irreducible matrices. Now, suppose that $\rho(G_{SOR}) = \gamma$, then there exists a positive vector $y = (y_1, y_2, \cdots, y_n)$ such that

$$G_{SOR}y = \gamma y$$

which implies

$$\begin{aligned} (I - \omega L)^{-1}\{(1-\omega)I + \omega U\}y &= \gamma y \\ (1-\omega)I + \omega U &= \gamma(I - \omega L) \qquad (10) \end{aligned}$$

And for this $y > 0$

$$\begin{aligned} G_1 y - \gamma y &= [I - \omega(\tilde{L} - D_1)]^{-1}\{(1-\omega)I + \omega\tilde{U}\}y - \gamma y \\ &= [I - \omega(\tilde{L} - D_1)]^{-1}\{(1-\omega)I + \omega\tilde{U}\}y \\ &\quad - \gamma[I - \omega(\tilde{L} - D_1)]^{-1}[I \\ &\quad - \omega(\tilde{L} - D_1)]y \\ &= [I - \omega(\tilde{L} - D_1)]^{-1}\{(1-\omega)I + \omega\tilde{U} - \gamma\omega D_1 - \gamma(I \\ &\quad - \omega\tilde{L})\}y \\ &= [I - \omega(\tilde{L} - D_1)]^{-1}\{(1 - \omega - \gamma)I + \omega\tilde{U} + \gamma\omega\tilde{L} \\ &\quad - \gamma\omega D_1\}y \\ &= [I - \omega(\tilde{L} - D_1)]^{-1}\{(1 - \omega - \gamma)I + \omega(U + U_s + U_1) \\ &\quad + \gamma\omega(L + L_s + L_1) - \gamma\omega D_1\}y \\ &= [I - \omega(\tilde{L} - D_1)]^{-1}\{(1 - \omega - \gamma)I + (\omega U + \gamma\omega L) \\ &\quad + (\omega U_s + \omega L_s) + (\gamma\omega L_s - \omega L_s) \\ &\quad + (-\omega D_1 + \omega L_1 + \omega U_1) \\ &\quad + (\gamma\omega L_1 - \omega L_1) + (-\gamma\omega D_1 \\ &\quad + \omega D_1)\}y \end{aligned}$$

Since from (10),

$$\omega U + \gamma\omega L = -(1-\omega-\gamma)$$
$$= \left[I-\omega(\tilde{L}-D_1)\right]^{-1}\{-\omega S + \omega SL + \omega SU + (\gamma-1)\omega L_s + (\gamma-1)\gamma\omega L_1 + (\gamma-1)(-D_1)\}y$$
$$= \left[I-\omega(\tilde{L}-D_1)\right]^{-1}\{\omega(\gamma-1)(L_1+L_s-D_1) + S - \omega S - S + \omega SL + \omega SU\}y$$
$$= \left[I-\omega(\tilde{L}-D_1)\right]^{-1}\{\omega(\gamma-1)(L_1+L_s-D_1) + (1-\omega)S - S(I-\omega)L + \omega SU\}y$$
$$= \left[I-\omega(\tilde{L}-D_1)\right]^{-1}\{\omega(\gamma-1)(L_1+L_s-D_1) + S[(1-\omega)I + \omega U - (I-\omega)L]\}y$$
$$= \left[I-\omega(\tilde{L}-D_1)\right]^{-1}\{\omega(\gamma-1)(L_1+L_s-D_1) + S[\gamma(I-\omega L) - (I-\omega)L]\}y$$
$$= \left[I-\omega(\tilde{L}-D_1)\right]^{-1}\{\omega(\gamma-1)(L_1+L_s-D_1) + \gamma S(I-\omega L) - S(I-\omega)L\}y$$
$$= (\gamma-1)\left[I-\omega(\tilde{L}-D_1)\right]^{-1}\{\omega(L_1+L_s-D_1) + S(I-\omega L)\}y$$
$$= [(\gamma-1)/\gamma]\left[I-\omega(\tilde{L}-D_1)\right]^{-1}\{\gamma\omega(L_1+L_s-D_1) + (1-\omega)S + \omega SU\}y$$

Suppose $T = Vy$, where $V = \left[I-\omega(\tilde{L}-D_1)\right]^{-1}\{\gamma\omega(L_1+L_s-D_1) + (1-\omega)S + \omega SU\}$. Then $V = \left[I-\omega(\tilde{L}-D_1)\right]^{-1}\{\gamma\omega(L_1+L_s-D_1) + (1-\omega)S + \omega SU\} \geq 0$, since $\gamma\omega(L_1-D_1) \geq 0$, $\gamma\omega L_s + (1-\omega)S \geq 0$ and $\omega SU \geq 0$. Also, $\left[I-\omega(\tilde{L}-D_1)\right]^{-1} = I + \omega(\tilde{L}-D_1) + \omega^2(\tilde{L}-D_1)^2 + \cdots + \omega^{n-1}(\tilde{L}-D_1)^{n-1} \geq 0$, since $\tilde{L} \geq 0$ and $-D_1 \geq 0$. Therefore, $V = \left[I-\omega(\tilde{L}-D_1)\right]^{-1}\{\gamma\omega(L_1+L_s-D_1) + (1-\omega)S + \omega SU\} \geq 0$. Consequently, $T = \left[I-\omega(\tilde{L}-D_1)\right]^{-1}\{\gamma\omega(L_1+L_s-D_1) + (1-\omega)S + \omega SU\}y \geq 0$, since $y > 0$.

a) If $\gamma < 1$, then $G_1y - \gamma y \leq 0$ but not equal to 0. Therefore, $G_1y \leq \gamma y$. Hence,
$$\rho(G_1) < \gamma = \rho(G_{SOR})$$
b) If $\gamma = 1$, then $G_1y - \gamma y = 0$. Therefore, $G_1y = \gamma y$. Hence,
$$\rho(G_1) = \gamma = \rho(G_{SOR})$$
c) If $\gamma > 1$, then $G_1y - \gamma y \geq 0$ but not equal to 0. Therefore, $G_1y \geq \gamma y$. Hence,
$$\rho(G_1) > \gamma = \rho(G_{SOR})$$

**Theorem 3** Let $G_{SOR} = (I-\omega L)^{-1}\{(1-\omega)I + \omega U\}$ and $G_2 = \left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{(1-\omega)\tilde{D} + \omega\tilde{U}\}$ and be the SOR and preconditioned SOR iteration matrices respectively. If $0 < \omega < 1$ is and $A \in \mathbb{R}^{nxn}$ is an irreducible $M$ −matrix with $0 \leq a_{is}a_{si} < 1, (i,s) = (1,2), (n,n-1)$, $0 \leq a_{i-1,i}a_{i,i-1} + a_{i,i+1}a_{i+1,i} < 1$, $i = 2(1)n-1$, then

a) $\rho(G_2) < \rho(G_{SOR})$, if $\rho(G_{SOR}) < 1$
b) $\rho(G_2) = \rho(G_{SOR})$, if $\rho(G_{SOR}) = 1$
c) $\rho(G_2) > \rho(G_{SOR})$, if $\rho(G_{SOR}) > 1$

**Proof:** Theorem 1 established that $G_{SOR}$ and $G_2$ are nonnegative and irreducible matrices. Let $\rho(G_{SOR}) = \gamma$, then there exists a positive vector $y = (y_1, y_2, \cdots, y_n)^T$, such that
$$G_{SOR}y = \gamma y$$
Or,
$$(I-\omega L)^{-1}\{(1-\omega)I + \omega U\}y = \gamma y$$
$$(1-\omega)I + \omega U = \gamma(I-\omega L) \quad (11)$$
Therefore, for this $y > 0$,
$$G_2y - \gamma y = \left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{(1-\omega)\tilde{D} + \omega\tilde{U}\}y - \gamma y$$
$$= \left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{(1-\omega)\tilde{D} + \omega\tilde{U}\}y - \left(\tilde{D}-\omega\tilde{L}\right)^{-1}\left(\tilde{D}-\omega\tilde{L}\right)\gamma y$$
$$= \left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{(1-\omega)\tilde{D} + \omega\tilde{U} - \gamma\left(\tilde{D}-\omega\tilde{L}\right)\}y$$
$$= \left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{(1-\omega-\gamma)\tilde{D} + \gamma\omega\tilde{L} + \omega\tilde{U}\}y$$
$$= \left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{(1-\omega-\gamma)(I+D_1) + \gamma\omega(L+L_S+L_1) + \omega(U+U_S+U_1)\}y$$
$$= \left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{(1-\omega-\gamma)D_1 + \gamma\omega L_1 + \gamma\omega L_S + \gamma\omega L + \omega U_S + \omega U_1 + (1-\omega-\gamma)I + \omega U\}y$$
$$= \left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{(1-\omega-\gamma)D_1 + \gamma\omega L_1 + \gamma\omega L_S + \omega U_S + \omega U_1\}y$$
$$= \left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{(1-\omega-\gamma)D_1 + \gamma\omega L_1 + \gamma\omega L_S + \omega U_S + \omega U_1\}y$$
$$= \left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{(\gamma-1)(-D_1) + (\gamma-1)\omega L_1 - \omega(D_1 - L_1 - U_1) + \gamma\omega L_S + \omega U_S\}y$$
$$= \left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{(\gamma-1)(-D_1+\omega L_1) + \omega SL + \omega SU + (\gamma-1)\omega L_S + \omega(L_S+U_S)\}y$$
$$= \left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{(\gamma-1)(-D_1+\omega L_1+\omega L_s) + (1-\omega)S + \omega SU - S(I-\omega L)\}y$$
$$= \left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{(\gamma-1)(-D_1+\omega L_1+\omega L_s) + S[(1-\omega)I + \omega U] - S(I-\omega L)\}y$$
$$= \left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{(\gamma-1)(-D_1+\omega L_1+\omega L_s) + \gamma S(I-\omega L) - S(I-\omega L)\}y$$
$$= \left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{(\gamma-1)(-D_1+\omega L_1+\omega L_s) + (\gamma-1)S(I-\omega L)\}y$$
$$= (\gamma-1)\left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{(-D_1+\omega L_1+\omega L_s) + [(\gamma-1)S(I-\omega L)]/\gamma\}y$$
$$= [(\gamma-1)/\gamma]\left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{\gamma(-D_1+\omega L_1+\omega L_s) + (1-\omega)S + \omega SU\}y$$

Let $T = Vy$, where $V = \left(\tilde{D}-\omega\tilde{L}\right)^{-1}\{\gamma(-D_1+\omega L_1+\omega L_s) + (1-\omega)S + \omega SU\}$. It is obvious that $\gamma(-D_1+\omega L_1+\omega L_s) + (1-\omega)S + \omega SU \geq 0$, $(1-\omega)S \geq 0$ and $\gamma(-D_1+\omega L_1+\omega L_s) \geq 0$. Since $\tilde{D}$ is a nonsingular matrix, we let $\tilde{D}-\omega\tilde{L}$ be a splitting of some matrix $J$, i.e., $J = \tilde{D}-\omega\tilde{L}$. Also, $\tilde{D}$ is an $M$ −matrix and $\omega\tilde{L} \geq 0$. Thus, $J = \tilde{D}-\omega\tilde{L}$ is an $M$ −splitting. Now, $\omega\tilde{D}^{-1}\tilde{L}$ is a strictly lower triangular matrix, and by implication its eigenvalues lie on its main diagonal; in this case they are all zeros. Therefore, $\rho\left(\omega\tilde{D}^{-1}\tilde{L}\right) = 0$. since $\rho\left(\omega\tilde{D}^{-1}\tilde{L}\right) < 1$, $J = \tilde{D}-\omega\tilde{L}$ is a convergent splitting. By the foregoing, $J = \tilde{D}-\omega\tilde{L}$ is an $M$ −splitting and $\rho\left(\omega\tilde{D}^{-1}\tilde{L}\right) < 1$, we employ Lemma 3 to establish that $J$ is an $M$ −matrix. Since $J$ is an $M$ −matrix, by definition, $J^{-1} = (\tilde{D}-\omega\tilde{L})^{-1} \geq 0$. Thus, $V \geq 0$ and $T \geq 0$.

(i) If $\gamma < 1$, then $G_2y - \gamma y \leq 0$ but not equal to 0. Therefore, $G_2y \leq \gamma y$. From Lemma 2, we have $\rho(G_2) < \gamma = \rho(G_{SOR})$.
(ii) If $\gamma = 1$, then $G_2y - \gamma y = 0$. Therefore, $G_2y = \gamma y$. From Lemma 2, we have $\rho(G_2) = \gamma = \rho(G_{SOR})$.

If $\gamma > 1$, then $G_2y - \gamma y \geq 0$ but not equal to 0. Therefore,

$G_2 y \geq \gamma y$. From Lemma 2, we have $\rho(G_2) > \gamma = \rho(G_{SOR})$.

## NUMERICAL EXAMPLES

Some numerical examples are presented here to demonstrate the convergence results obtained in the preceding section.

***Example* 1** Let the coefficient matrix $A$ of the linear system (1) be given by the following $4 \times 4$ matrix

$$A = \begin{pmatrix} 1 & -12/43 & -10/43 & 0 \\ -15/49 & 1 & 0 & -10/49 \\ -13/49 & 0 & 1 & -12/49 \\ 0 & -13/55 & -3/11 & 1 \end{pmatrix}$$

***Example 2*** Let the coefficient matrix $A$ of the linear system (1) be given by the following $6\times 6$ matrix

$$A = \begin{pmatrix} 1 & -0.5 & -0.1 & -0.1 & -0.1 & -0.1 \\ -0.2 & 1 & 0 & 0 & 0 & -0.5 \\ -0.2 & -0.1 & 1 & -0.3 & -0.1 & -0.2 \\ -0.1 & 0 & -0.2 & 1 & -0.3 & -0.1 \\ -0.3 & -0.2 & -0.1 & -0.1 & 1 & -0.2 \\ -0.2 & -0.3 & -0.2 & -0.1 & -0.1 & 1 \end{pmatrix}$$

In what follows, the computations on the spectral radii $\rho(G_{SOR})$, $\rho(G_1)$, $\rho(G_2)$ and $\rho(G_M)$ of SOR, Equation (7), Equation (8) and Milaszewicz (1987) iterative matrices respectively are performed with Maple 2019 to yield some comparison results which are presented in Tables I and II.

**TABLE I**: Comparison of results for Example 1

| $\omega$ | $\rho(G_{SOR})$ | $\rho(G_1)$ | $\rho(G_2)$ | $\rho(G_M)$ |
|---|---|---|---|---|
| **0.1** | 0.9497367342 | 0.9361539872 | 0.9313266464 | 0.9380715996 |
| **0.2** | 0.8966535886 | 0.8691709495 | 0.8602138051 | 0.8734847743 |
| **0.3** | 0.8403551413 | 0.7986687786 | 0.7863562552 | 0.8059047100 |
| **0.4** | 0.7803331902 | 0.7241616199 | 0.7093593790 | 0.7349022671 |
| **0.5** | 0.7159112281 | 0.6450073375 | 0.6286921041 | 0.6599063782 |
| **0.6** | 0.6461456061 | 0.5603109522 | 0.5435997828 | 0.5801179797 |
| **0.7** | 0.5696315417 | 0.4687251140 | 0.4529231291 | 0.4943364599 |
| **0.8** | 0.4840647588 | 0.3679597310 | 0.3546513357 | 0.4005512726 |
| **0.9** | 0.3850038069 | 0.2531573189 | 0.2444419148 | 0.2947010963 |

**TABLE II**: Comparison of results for Example 2

| $\omega$ | $\rho(G_{SOR})$ | $\rho(G_1)$ | $\rho(G_2)$ | $\rho(G_M)$ |
|---|---|---|---|---|
| **0.1** | 0.9823319839 | 0.9772511691 | 0.9758102796 | 0.9777710180 |
| **0.2** | 0.9631164669 | 0.9523262676 | 0.9495245040 | 0.9535806170 |
| **0.3** | 0.9421143950 | 0.9248532414 | 0.9208064069 | 0.9271176074 |
| **0.4** | 0.9190280945 | 0.8943569161 | 0.8892299441 | 0.8979905550 |
| **0.5** | 0.8934796835 | 0.8602151299 | 0.8542418838 | 0.8656961892 |
| **0.6** | 0.8649781519 | 0.8215871178 | 0.8151013233 | 0.8295693642 |
| **0.7** | 0.8328665903 | 0.7772887176 | 0.7707753260 | 0.7886994501 |
| **0.8** | 0.7962324918 | 0.7255549759 | 0.7197430272 | 0.7417797247 |
| **0.9** | 0.7537428175 | 0.6635289477 | 0.6595824489 | 0.6868076758 |

It is shown in Table I that $\rho(G_1)$ is far less than the $\rho(G_{SOR})$ with $\omega$ increasing from 0.1 to 0.9, an indication of effectiveness of the preconditioned SOR scheme whose iteration matrix is given by $G_1$. More so, $\rho(G_1)$ exhibited better convergence than the $\rho(G_M)$. On the whole, $\rho(G_2)$ showed better performance than the $\rho(G_{SOR})$, $\rho(G_1)$ and $\rho(G_M)$. Similarly, the results of Example 2, presented in Table II, the $\rho(G_2)$ performs better than the $\rho(G_1)$, $\rho(G_M)$ and $\rho(G_M)$ in that order. It follows from Tables I and II that the two preconditioned SOR schemes introduced in this paper are superior to the classical SOR method of Young (1950).

**Conclusion**

In this paper, two preconditioned versions of the SOR iterative method are proposed. Some necessary and sufficient conditions for convergence of the modified SOR methods are imposed on the linear systems with $M$ −matrices. Numerical experiments revealed that the modified methods have smaller spectral radii than the SOR, which indicates their effectiveness in accelerating the convergence of the existing method

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