INT. J. NUM. MATHS. (2011) Vol. 6 No. 2 pp-345-359

ON A CONTINUOUS BLOCK METHOD FOR THE SOLUTION OF INITIAL VALUE PROBLEMS

U MOHAMMED AND A NDANUSA
Department of Mathematics/Computer Science, Federal University of Technology, Minna, Nigeria

Abstract

In this paper, we convert a conventional linear multistep method for solving initial value problems into the continuous form. The approach of collocation approximation is adopted in the derivation of the schemes. The continuous schemes so derived are then applied as simultaneous integrators to the solution of the initial value problem (ivp) for first order ordinary differential equations (odes). This implementation strategy is expected to produce results that are more accurate and efficient than those given when applied over overlapping intervals in predictor-corrector mode. In addition to eliminating the rigours associated with predictor-corrector methods, the new block method possesses the self-starting feature of Runge-Kutta methods. Numerical experiments confirm the theoretical expectations.

Introduction

Lie and Norsett (1989), Onumanyi, et al (1994), Sirisena (1999), Onumanyi, et al (1999), Yusuph and Onumanyi (2002) and yahaya (2004) have all converted conventional linear multistep methods including hybrid ones into continuous forms through the idea of Multistep Collocation (MC). The Conitnuous Multistep (CM) method, associated

Keywords: Block Method, Linear Multistep Method, Multistep Collocation, Continuous Multistep (CM), Zero-stability

digitalumar@yahoo.com asndanusa@yahoo.com

with conventional linear multistep methods produces piece-wise polynomial solutions over - - steps .: : .. for the first order differential system

$$y' = f(x, y), \qquad y(a) = y_0, \qquad a \le x \le b \tag{1}$$

This research work aims at reformulating a six-step linear multistep method derived by Ndanusa and Adeboye (2008), in its present discrete form, into the continuous form, in order to attain greater efficiency and accuracy.

The method of multistep collocation (MC) and its link to the continuous multistep (CM) method for the solution of equation (1) is described thus. From equation (1), f is given and y is sought as

$$y = a_1\phi_1 + a_2\phi_2 + \ldots + a_p\phi_p \tag{2}$$

Where,

$$a = (a_1, a_2, \ldots, a_p)^T, \qquad \phi = (\phi_1, \phi_2, \ldots, \phi_p)^T \tag{3}$$

Where, T denotes the transpose.

Equation (2) can be re-written as

$$y = (a_1 . a \ldots a_p)^T \; (\varphi_1 . \varphi \ldots \varphi_p)^T \tag{4}$$

The unknown coefficients $a_1 . a \ldots a_p$ are determined using respectively the $r(0 < r \le k)$ interpolation conditions and the *s > 0* distinct collocation conditions, $p = r + s$, as follows

$$\sum_{j=1}^{p} a_j \varphi_j(x_i) = y_i \qquad (i = 1, \ldots, r) \tag{5}$$

$$\sum_{j=1}^{p} a_j \varphi_j'(x_i) = f_i \qquad (i = 1,\dots,r) \tag{6}$$

This is a system of p linear equations from which we can compute values for the unknown coefficients provided equations (5) and (6) are non-singular. For the distinct points x_i and c_i the non-singular system is guaranteed (Yusuph et al (2002)). We can write equations (5) and (6) as a single set of linear equations of the form

$$\left.\begin{aligned} \underline{D}\,\underline{a} &= \underline{F} \\ \underline{a} &= D^{-1}\underline{F} \end{aligned}\right\} \tag{7}$$

where

$$F = (y_1, y_2, \dots, y_r, f_1, f_2, \dots, f_3 \tag{8}$$

Substituting the vector $\underline{\propto}$, given by equation (7) and F given by equation (8) into equation (4) gives

$$y = (y_1, y_2, \dots, y_r, f_1, f_2, \dots, f_3)C^{-1}(\varphi_1, \varphi_2, \dots, \varphi_p)^{T} \tag{9}$$

Equation (9) therefore, is the continuous MC Interplant., where

$$\underline{C^{-1}\phi} = \begin{pmatrix} \sum_{j=1}^{p} C_{n\phi_j} \\ \\ \sum_{j=1}^{p} C_{n\phi} \\ \sum_{j=1}^{p} C_{n-1\phi_j} \\ \\ \sum_{j=1}^{p} C_{n\phi} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \\ \alpha_r \\ \beta_1 \\ \\ \beta_r \end{pmatrix} \tag{10}$$

$$F^{t}C^{t}\phi = \sum_{j=1}^{r} \alpha_j y_j + h_j \left(\sum_{j=1}^{s} \beta_j / h_j \, f_j \right) \quad (11)$$

$$\left.\begin{aligned} \alpha_j &= \sum_{q=1}^{p} C_{qj}\phi_j, \qquad j = 1,\ldots\ldots r \\ \beta_j / h_j &= \sum_{q=1}^{p} \left[\frac{C_{qj+r}}{h_j} \right] \phi_j \qquad j = 1,\ldots\ldots s \end{aligned}\right\} \quad (12)$$

Therefore,

$$y = \sum_{j=1}^{r} \alpha_j y_j + h_j \left(\sum_{j=1}^{s} \beta_j / h_j \right) f_j \qquad (13)$$

is the CM interpolant with variable step-size.

Derivation of the Continuous Formula

We seek to derive a continuous form of the following 6-step linear multistep method (lmm) derived by Ndanusa and Adeboye (2008).

$$y_{n+6} = y_n + h\left[\frac{41}{140} f_{n+6} + \frac{54}{35} f_{n+5} + \frac{27}{140} f_{n+4} + \frac{68}{35} f_{n+3} + \frac{27}{140} f_{n+2} + \frac{54}{35} f_{n+1} + \frac{41}{140} f_n \right] \quad (14)$$

We propose a continuous solution method to equation (1) in the form

$$y_p(x) = \sum_{j=0}^{m+t-1} \alpha_j x^j, \; i = 0,1,\ldots,(m+t-1) \quad (15)$$

Where $m=6,\ t=1,\ p=m+t-1$. Also, $\propto_j,\ j=0,1,\ldots,(m+t-1)$ are the parameters to be determined. *p* is the degree of the polynomial interpolant of our choice. Specifically, we interpolate equation (15) at $\{x_n\}$ and collocate at $\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}, x_{n+6}\}$ using the method described above.

The general form of the new method is thus

$$y(x)=\alpha_0 y_n+\beta_0 f_n+\beta_1 f_{n+1}+\beta_2 f_{n+2}+\beta_3 f_{n+3}+\beta_4 f_{n+4}+\beta_5 f_{n+5}+\beta_6 f_{n+6} \qquad (16)$$

The matrix D of the new method expressed as:

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & 7x_{n+4}^6 \\ 0 & 1 & 2x_{n+5} & 3x_{n+5}^2 & 4x_{n+5}^3 & 5x_{n+5}^4 & 6x_{n+5}^5 & 7x_{n+5}^6 \\ 0 & 1 & 2x_{n+6} & 3x_{n+6}^2 & 4x_{n+6}^3 & 5x_{n+6}^4 & 6x_{n+6}^5 & 7x_{n+6}^6 \end{pmatrix} \qquad (17)$$

The above matrix is solved using either matrix inversion technique or Gaussian elimination method to obtain the values of the parameters $\alpha_j,\ j=0,1,\ldots,m+t-1$ and then substituting them into equation (15) to obtain the scheme expressed in the form.

$$y_k(x)=\sum_{j=0}^{k-1}\alpha_j(x)y_{n+j}+h\sum_{j=0}^{k-2}\beta_j(x)f_{n+j} \qquad (18)$$

If we let $k = 6$, we obtain the continuous form

$y(x) = y_n +$

$$\left[12(x-x_n)^7 - 294(x-x_n)^6 h + 2940(x-x_n)^5 h^2 - 15435(x-x_n)^4 h^3 + 45475(x-x_n)^3 h^4 - 74088(x-x_n)^2 h^5 + 60480(x-x_n) h^6\right] f_n$$

$$+\left[-3(x-x_n)^7 + 70(x-x_n)^6 h - 651(x-x_n)^5 h^2 + 3045(x-x_n)^4 h^3 - 7308(x-x_n)^3 h^4 + 7560(x-x_n)^2 h^5\right] f_{n+1}$$

$$+\left[60(x-x_n)^7 - 1330(x-x_n)^6 h + 11508(x-x_n)^5 h^2 - 48405(x-x_n)^4 h^3 + 98280(x-x_n)^3 h^4 - 75600(x-x_n)^2 h^5\right] f_{n+2}$$

$$+\left[-15(x-x_n)^7 + 315(x-x_n)^6 h - 441(x-x_n)^5 h^2 + 9765(x-x_n)^4 h^3 - 17780(x-x_n)^3 h^4 + 12600(x-x_n)^2 h^5\right] f_{n+3}$$

$$+\left[60(x-x_n)^7 - 1190(x-x_n)^6 h + 8988(x-x_n)^5 h^2 - 32235(x-x_n)^4 h^3 + 55440(x-x_n)^3 h^4 - 37800(x-x_n)^2 h^5\right] f_{n+4}$$

$$+\left[-3(x-x_n)^7 + 56(x-x_n)^6 h - 399(x-x_n)^5 h^2 + 1365(x-x_n)^4 h^3 - 2268(x-x_n)^3 h^4 + 1512(x-x_n)^2 h^5\right] f_{n+5}$$

$$+\left[12(x-x_n)^7 - 210(x-x_n)^6 h + 1428(x-x_n)^5 h^2 - 4725(x-x_n)^4 h^3 + 7672(x-x_n)^3 h^4 - 5040(x-x_n)^2 h^5\right] f_{n+6} \quad (19)$$

Evaluating equation (19) at $x = x_{n+6}$ we obtain equation (14). In the same vein, Evaluating equation (19) at the points $x = x_{n+5}, x = x_{n+4}, x = x_{n+3}, x = x_{n+2}, \ and \ x = x_{n+1}$ we obtain respectively

$$y_{n+5} = y_n + h\left[\frac{3715}{12096} f_n + \frac{725}{504} f_{n+1} + \frac{2125}{4032} f_{n+2} + \frac{250}{189} f_{n+3} + \frac{3875}{4032} f_{n+4} + \frac{235}{504} f_{n+5} - \frac{275}{12096} f_{n+6}\right] \quad (20)$$

$$y_{n+4} = y_n + h\left[\frac{286}{945} f_n + \frac{464}{315} f_{n+1} + \frac{128}{315} f_{n+2} + \frac{1504}{945} f_{n+3} + \frac{58}{315} f_{n+4} + \frac{16}{315} f_{n+5} - \frac{8}{945} f_{n+6}\right] \quad (21)$$

$$y_{n+3} = y_n + h\left[\frac{137}{448} f_n + \frac{81}{56} f_{n+1} + \frac{1161}{2240} f_{n+2} + \frac{34}{35} f_{n+3} + \frac{729}{2240} f_{n+4} + \frac{27}{280} f_{n+5} - \frac{29}{2240} f_{n+6}\right] \quad (22)$$

$$y_{n+2} = y_n + h\left[\frac{1139}{3780} f_n + \frac{94}{63} f_{n+1} + \frac{11}{1260} f_{n+2} + \frac{332}{945} f_{n+3} - \frac{269}{1260} f_{n+4} + \frac{22}{315} f_{n+5} - \frac{37}{3780} f_{n+6}\right] \quad (23)$$

$$y_{n+1} = y_n + h\left[\frac{19087}{60480} f_n + \frac{2713}{2520} f_{n+1} - \frac{15487}{20160} f_{n+2} + \frac{586}{945} f_{n+3} - \frac{6737}{20160} f_{n+4} + \frac{263}{2520} f_{n+5} - \frac{863}{60480} f_{n+6}\right] \quad (24)$$

Equations (14), (20) to (24) constitute the members of a zero-stable block integrator of order $(8,7,7,7,7,7)^T$, with

$$C_{9,8} = \left(-\frac{9}{1460}, \frac{275}{24192}, \frac{8}{945}, \frac{9}{896}, \frac{8}{945}, \frac{275}{24192} \right).$$

In order to apply the block method to the solution of the IVP (1) on the subinterval $[x_0, x_6]$, equations (14), (20) to (24) are solved simultaneously, when *n = 0*, to form the 1-block 6 point method as given by equation (25). This method enables us to obtain the starting values y_1, y_2, y_3, y_4 *and* y_5 without recourse to any predictor for the production of these values, as evident in the scheme developed by Ndanusa and Adeboye (2008).

Stability Analysis

We can put the six integrators represented by equations (14), (20) to (24) into the matrix-equation form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_{n} \end{pmatrix}$$

$$
+h\begin{pmatrix}
\frac{2713}{2520} & \frac{15487}{20160} & \frac{586}{945} & -\frac{6737}{20160} & \frac{263}{2520} & -\frac{863}{60480} \\
\frac{94}{63} & \frac{11}{1260} & \frac{332}{945} & -\frac{269}{1260} & \frac{22}{315} & \frac{37}{3780} \\
\frac{81}{56} & \frac{1161}{2240} & \frac{34}{35} & \frac{729}{2240} & \frac{27}{280} & \frac{29}{2240} \\
\frac{464}{315} & \frac{128}{315} & \frac{1504}{945} & \frac{58}{315} & \frac{16}{315} & \frac{8}{945} \\
\frac{725}{504} & \frac{2125}{4032} & \frac{250}{189} & \frac{3875}{4032} & \frac{235}{504} & \frac{275}{12096} \\
\frac{54}{32} & \frac{27}{140} & \frac{68}{35} & \frac{27}{140} & \frac{54}{35} & \frac{41}{140}
\end{pmatrix}
\begin{pmatrix} f_{n+1} \\ f \\ f \\ f \\ f \\ f_n \end{pmatrix}
$$

$$
+h\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \frac{19087}{60480} \\
0 & 0 & 0 & 0 & 0 & \frac{1139}{3780} \\
0 & 0 & 0 & 0 & 0 & \frac{137}{448} \\
0 & 0 & 0 & 0 & 0 & \frac{286}{9450} \\
0 & 0 & 0 & 0 & 0 & \frac{3715}{12096} \\
0 & 0 & 0 & 0 & 0 & \frac{41}{140}
\end{pmatrix}
\begin{pmatrix} f \\ f \\ f \\ f \\ f \\ f \end{pmatrix}
\qquad (25)
$$

The first characteristic polynomial of the proposed 1-block 6 step method is

$$\rho(R) = \det[RA^{(0)} - A^{(1)}]$$

$$\rho(R) = \det\left[R\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix} - \begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{pmatrix}\right] \quad (26)$$

$$= \det\begin{pmatrix}R&0&0&0&0&-1\\0&R&0&0&0&-1\\0&0&R&0&0&-1\\0&0&0&R&0&-1\\0&0&0&0&R&1\\0&0&0&0&0&R-1\end{pmatrix} = \left[R^{5}(R-1)\right] \quad (27)$$

$P(R) = R^5(R-1)$ *implies*

$R_1 = R_2 = R_3 = R_4 = R_5 = 0$ *or* $R_6 = 1$.

From above, it is evident that the 1-block 6-point method is zero stable as well as consistent, as its order; $(8,7,7,7,7,7)^T > 1$. Thus, it is convergence is ascertained, following Henrici (1962).

We employ the Matlab software package to plot the absolute stability region of the proposed block method. This is done by reformulating a block method as general linear method to obtain the values of the matrices A.B.U.V which are then substituted into stability matrix and stability function, to yield the stability polynomial of the block method. And finally, we plot the absolute stability region of the 6-step block method as shown in figure 1 below. The 6-point block method (3.5) is *A(α)* stable

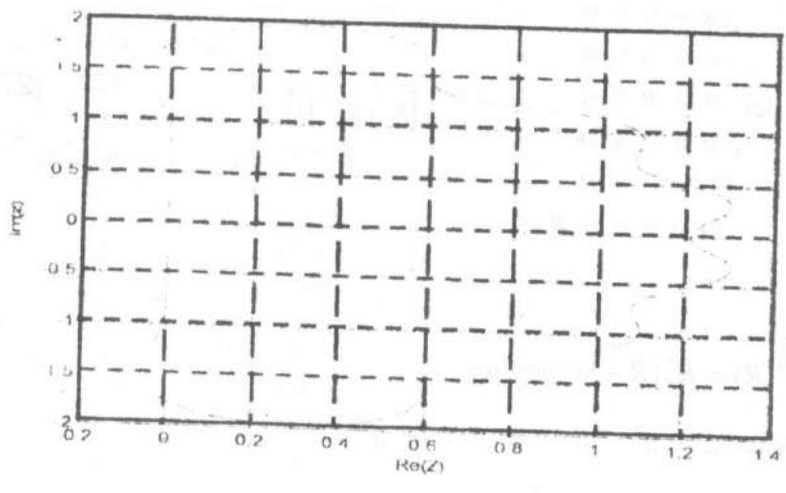

Figure 1: Region of Absolute Stability

From figure 1 above, it is clear that the 6-point block method (24) is *A(α)* stable.

Numerical Experiment

We solve the following differential equation using the 6-point block method we derived. The results are obtained and compared with the exact solutions. Also, the errors of the 6-point block method are compared with those of the predictor-corrector method, represented by equation (14) when applied to solve the same problem.

Consider the initial value problem

$$y' = x + y, \qquad y(0) = 1, \qquad h = 0.1$$

Exact solution: $y(x) = 2e^{x} - x - 1$

Table 5.1: Accuracy Comparison of 6-Step Block Method and 6 step (predictor-corrector) Method

	$y(x)$		Error	
	Exact solution	Block Method	Block method	Predictor-corrector
0	1.0000000000	1.0000000000	0	0
0.1	1.1103418362	1.1103418370	8.0000006619E-10	1.6948462878E-07
0.2	1.2428055163	1.2428055160	3.0000002482E-10	3.7461895075E-07
0.3	1.3997176152	1.3997176160	8.0000006619E-10	6.2102693121E-07
0.4	1.5836493953	1.5836493940	1.2999998855E-09	9.1512116951E-07
0.5	1.7974425414	1.7974425410	4.0000003310E-10	1.2642065803E-06
0.6	2.0442376008	2.0442376020	1.2000000993E-09	7.1145714031E-07
0.7	2.3275054149	2.3275054160	1.1000000910E-09	2.6179097690E-06
0.8	2.6510818570	2.6510818600	3.0000002482E-09	3.3079643345E-06
0.9	3.0192062223	3.0192062250	2.7000002234E-09	6.2842580402E-06
1.0	3.4365636569	3.4365636570	1.0000000827E-10	7.3885805181E-06
1.1	3.9083320479	3.9083320480	1.0000000827E-10	1.1011780565E-05
1.2	4.4402338455	4.4402338490	3.5000002896E-09	1.1849689056E-05
1.3	5.0385933352	5.0385933380	2.8000002317E-09	1.6494681973E-05
1.4	5.7103999337	5.7103999380	4.3000003558E-09	1.8789605289E-05
1.5	6.4633781407	6.4633781460	5.3000004385E-09	2.5346198219E-05

Conclusion

We have converted a six--step implicit linear multistep method into the continuous form. The continuous formulae are immediately employed as block method for direct solution of six point IVPs. The direct solutions are in discrete form which can be substituted into the continuous formula for dense output. The proposed method is self starting, convergent and $A(\alpha)$ stable as shown by the plotted region of absolute stability (Figure 4.1). The method demonstrated satisfactory performance when applied to solve a simple ODE, without recourse to any other method to provide the starting values.

REFERENCES

[1] **Fatunla, S O (1992),** *Parallel Methods for Second Order ODEs*, Computational Ordinary Differential Equations Proceeding of Computer Conference (eds), pp 87-99

[2] **Henrici, P (1962),** *Discrete Variable Methods for ODEs*. John Wiley, New York. *Initial Value Problems*. JOSET, Vol. 4, No. 1, pp 93-99

[3] **Lambert, J D (1973),** *Computational Methods in Ordinary Differential Equations*, John Wiley and Sons, New York, p 278

[4] **Lie, I and Norset, S P (1989),** *Supper Convergence for Multistep Collocation*, Math. Comp. 52, pp 65-79

[5] **Ndanusa, A and Adeboye, K R (2008),** *An Optimal 6-Step Implicit Linear Multistep Method for initial value problems. JOSET. vol.4. No. 1. pp 93-99.*

[6] **Onumanyi, P, et al (1994)**. *New Linear Multistep Method with Continuous Coefficients for First Order Initial Value Problems*. J. Nig. Math. Soc. 13, pp37-51

[7] **Onumanyi, P, et al (1999),** *Continuous Finite difference Approximate for Solving Differential Equations*. Inter J. Comp Maths. 72 No. 1. pp 15-27

[8] **Yahaya, Y A (2004),** *Some Theory and Application of Continuous Linear Multi-step Methods for Ordinary Differential Equations*, PhD thesis (unpublished), University of Jos, Nigeria

[9] **Yusuph, Y A and Onumanyi, P (2002),** *New Multiple FDMs through Multistep Collocation for* $y' = f(x, y)$. Abacus, 29(2) pp 92-100

REFERENCES

[1] **Fatunla, S O (1992),** *Parallel Methods for Second Order ODEs*, Computational Ordinary Differential Equations Proceeding of Computer Conference (eds), pp 87-99

[2] **Henrici, P (1962),** *Discrete Variable Methods for ODEs*. John Wiley, New York. *Initial Value Problems*. JOSET, Vol. 4, No. 1, pp 93-99

[3] **Lambert, J D (1973),** *Computational Methods in Ordinary Differential Equations*, John Wiley and Sons, New York, p 278

[4] **Lie, I and Norset, S P (1989),** *Supper Convergence for Multistep Collocation*, Math. Comp. 52, pp 65-79

[5] **Ndanusa, A and Adeboye, K R (2008),** *An Optimal 6-Step Implicit Linear Multistep Method for initial value problems. JOSET. vol.4. No. 1, pp 93-99.*

[6] **Onumanyi, P, et al (1994).** *New Linear Multistep Method with Continuous Coefficients for First Order Initial Value Problems*. J. Nig. Math. Soc. 13, pp37-51

[7] **Onumanyi, P, et al (1999),** *Continuous Finite difference Approximate for Solving Differential Equations*. Inter J. Comp Maths. 72 No. 1, pp 15-27

[8] **Yahaya, Y A (2004),** *Some Theory and Application of Continuous Linear Multi-step Methods for Ordinary Differential Equations*, PhD thesis (unpublished), University of Jos, Nigeria

[9] **Yusuph, Y A and Onumanyi, P (2002),** *New Multiple FDMs through Multistep Collocation for* $y' = f(x, y)$. Abacus, 29(2) pp 92-100