Convergence of Preconditioned Gauss-Seidel Iterative Method For L −Matrices

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Abstract: *A great many real-life situations are often modeled as linear system of equations, $Ax = b$. Direct methods of solution of such systems are not always realistic, especially where the coefficient matrix A is very large and sparse, hence the recourse to iterative solution methods. The Gauss-Seidel, a basic iterative method for linear systems, is one such method. Although convergence is rarely guaranteed for all cases, it is established that the method converges for some situations depending on properties of the entries of the coefficient matrix and, by implication, on the algebraic structure of the method. However, as with all basic iterative methods, when it does converge, convergence could be slow. In this research, a preconditioned version of the Gauss-Seidel method is proposed in order to improve upon its convergence and robustness. For this purpose, convergence theorems are advanced and established. Numerical experiments are undertaken to validate results of the proved theorems.*

Key Words: *Gauss-Seidel iterative method, Preconditioning, L −-matrix, Splitting, Nonnegative matrix*

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1.0 Introduction

In order to employ iterative solution method for the linear system of algebraic equations $Ax = b$, where the coefficient matrix $A \in \mathbb{R}^{n,n}$ is an irreducible $L -$ matrix, $b \in \mathbb{R}^n$, and x being the vector of unknowns, the generic linear iteration formula takes the form

$$x^{(n)} = Gx^{(n-1)} + c, \qquad n = 0,1,2,\cdots \qquad (1)$$

where $G(= M^{-1}N)$, referred to as the iteration matrix, is a matrix depending upon A and x, and $c(= M^{-1}b)$ is a column vector. Both G_n and c_k are obtained from a regular splitting of the matrix A thus: $A = M - N$. We assume for simplicity, without loss of generality, that the coefficient matrix A has the usual triangular splitting of the form $A = I - L - U$, where I is the identity matrix, $-L$ and $-U$ are the strictly lower and strictly upper triangular parts of A, respectively. By the foregoing, the Gauss-Seidel method is easily described by the relation

$$x^{(n)} = Gx^{(n-1)} + c \qquad n = 0,1,2,\cdots \qquad (2)$$

where $G = (I - L)^{-1}U$ is the Gauss-Seidel iteration matrix, and $c = (I - L)^{-1}b$. The Gauss-Seidel method is known to converge for linear systems with strictly or irreducibly diagonally dominant matrices, invertible $H -$ matrices (generalized strictly diagonally dominant matrices) and Hermitian positive definite matrices. However, as with all basic iterative methods, convergence could be slow; hence the idea of preconditioning.

Preconditioning is the application of a transformation (preconditioner) to a linear system that transforms the system into a form that is more suitable for numerical computation. When preconditioners are applied to linear systems, the associated iterative methods tend to converge asymptotically faster than the unpreconditioned ones. Preconditioning, in relation to classical iterative methods, aims to reduce the spectral radius of the iteration matrix so as to improve convergence. However, when applied to Conjugate Gradient or other Krylov subspace methods, the goal of preconditioning is to increase the condition number of the coefficient matrix A in order to improve convergence.

A diversity of preconditioned Gauss-Seidel iterative techniques has been advanced by various researchers and authors. Among these include the preconditioners of Allahviranloo *et al.* (2012), Gunawardena *et al.* (1991), Hadjidimos *et al.* (2003), Kohno *et al.* (1997), Li (2005), Li and Sun (2000), Milaszewicz (1987), Nazari and Borujeni (2012), Ndanusa and Adeboye (2012), Noutsos and Tzoumas (2006), Zhang *et al.* (2015) and Zheng and Miao (2009). This present research aims to investigate the applicability of the preconditioner of [9] to the classical Gauss-Seidel method in order to improve on its convergence.

2.0 Materials and Methods

2.1 Preliminaries

In order to use the successive overrelaxation (SOR) method to solve the preconditioned linear system, equation 3 is significant

$$PAx = Pb \tag{3}$$

where $P \in \mathbb{R}^{n\times n}$, called the preconditioner, is nonsingular. Ndanusa and Adeboye (2012) proposed the preconditioner $P = I + S$, where I is the $n \times n$ identity matrix and S is a sparse matrix defined by

$$S = \begin{cases} -a_{i1}, & i = 2,\cdots,n \\ -a_{i,i+1}, & i = 1,\cdots,n-1 \\ 0, & \text{otherwise} \end{cases}$$

The nonzero entries of S are the negatives of the corresponding entries of the coefficient matrix A. If $PA = \bar{A}$ and $Pb = \bar{b}$, system (3) is simplified to

$$\bar{A}x = \bar{b} \tag{4}$$

From (4) we obtain

$$\bar{A} = PA = (I + S)(I - L - U)$$
$$= I - L - U + S - SL - SU$$

where,

$$S = -L_S - U_S - SL - SU = D_1 - L_1 - U_1$$

Therefore,

$$\bar{A} = I - L - U - L_S - U_S + D_1 - L_1 - U_1$$
$$= (I + D_1) - (L + L_S + L_1) - (U + U_S + U_1)$$

It implies,

$$\bar{A} = \bar{D} - \bar{L} - \bar{U} \tag{5}$$

where $\bar{D} = I + D_1$, $\bar{L} = L + L_S + L_1$ and $\bar{U} = U + U_S + U_1$ constitute the diagonal, strictly lower and strictly upper components of $\bar{A}$ respectively.

The classical (unpreconditioned) Gauss-Seidel iteration scheme (2) is rewritten as

$$x^{(n)} = (I - L)^{-1}Ux^{(n-1)} + (I - L)^{-1}b \qquad n = 0,1,2,\cdots \tag{6}$$

To construct a preconditioned version of the iteration (6), ~~consider~~ a regular splitting of the preconditioned coefficient matrix $\bar{A}$ is considered and the resulting models are as follow,

$$\bar{A} = (\bar{D} - \bar{L} - \bar{U}) = (I + D_1 - \bar{L} - \bar{U}) = (I - \bar{L}) - (\bar{U} - D_1)$$

Therefore,

$$\bar{A} = M - N = (I - \bar{L}) - (\bar{U} - D_1)$$

is a regular splitting of $\bar{A}$, where $M = (I - \bar{L})$ and $N = (\bar{U} - D_1)$. Therefore, the first preconditioned Gauss-Seidel iterative scheme is defined as

$$x^{(n)} = (I - \bar{L})^{-1}(\bar{U} - D_1)x^{(n-1)} + (I - \bar{L})^{-1}b \tag{7}$$

or equivalently,

$$x^{(n)} = G_1x^{(n-1)} + c \qquad n = 0,1,2,\cdots \tag{8}$$

where the iterative matrix of the preconditioned Gauss-Seidel scheme, G_1, is represented as

$$G_1 = (I - \bar{L})^{-1}(\bar{U} - D_1) \tag{9}$$

Also, from (5), a second preconditioned Gauss-Seidel iteration scheme can be defined as

$$x^{(n)} = (\bar{D} - \bar{L})^{-1}\bar{U}x^{(n-1)} + (\bar{D} - \bar{L})^{-1}b \tag{10}$$

Or more compactly,

$$x^{(n)} = G_2x^{(n-1)} + c \qquad n = 0,1,2,\cdots \tag{11}$$

where

$$G_2 = (\bar{D} - \bar{L})^{-1}\bar{U} \tag{12}$$

is the Gauss-Seidel iteration matrix.

Convergence Analysis

The following lemmas and theorems are advanced in order to establish convergence of the derived preconditioned iterative processes.

Lemma 1 (Varga (1981)) Let $A \geq 0$ be an irreducible matrix. Then,

i. A has a positive real eigenvalue equal to its spectral radius.
ii. For $\rho(A)$ there corresponds an eigenvector $x > 0$.
iii. $\rho(A)$ increases when any entry of A increases.
iv. $\rho(A)$ is a simple eigenvalue of A.

Lemma 2 (Varga (1981)) Let A be a nonnegative matrix. Then

i. If $\alpha x \leq Ax$ for some nonnegative vector $x, x \neq 0$, then $\alpha \leq \rho(A)$.
ii. If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector x, then $\alpha \leq \rho(A) \leq \beta$ and x is a positive vector.

Lemma 3 (Li and Sun (2000)) Let $A = M - N$ be an $M-$splitting of A. Then the splitting is convergent, i.e., $\rho(M^{-1}N < 1)$, if and only if A is a nonsingular $M-$matrix.

Theorem 1 Let $G = (I - L)^{-1}U$, $G_1 = (I - \bar{L})^{-1}(\bar{U} - D_1)$ and $G_2 = (\bar{D} - \bar{L})^{-1}\bar{U}$ be the Gauss-Seidel, the first preconditioned Gauss-Seidel and the second preconditioned Gauss-Seidel iteration matrices respectively. If A is an irreducible $L-$matrix with $0 \leq a_{1i}a_{i1} + a_{i,i+1}a_{i+1,i} < 1$, $i = 2(1)n$, then G, G_1 and G_2 are nonnegative and irreducible matrices.

Proof For A being an L −matrix, it implies that $L \geq 0$ and $U \geq 0$. Then $(I-L)^{-1} = I + L + L^2 + \cdots + L^{n-1} \geq 0$. Thus $G = (I-L)^{-1}U \geq 0$. Hence, G is a nonnegative matrix.

It can also be shown that

$$G = [I + L + L^2 + \cdots + L^{n-1}]U = U + LU + L^2U + \cdots = U + LU + L^2U + \text{nonnegative terms}$$

It can also be shown that $U + LU + L^2U$ is irreducible for irreducible A. Hence, G is an irreducible matrix.

The first preconditioned iteration matrix G_1 is examined as follows.

$$G_1 = (I - \bar{L})^{-1}(\bar{U} - D_1)$$

Since $\bar{L} \geq 0$, $\bar{U} \geq 0, -D_1 \geq 0$, then $(\bar{U} - D_1) \geq 0$ and $(I-L)^{-1} = I + L + L^2 + \cdots + L^{n-1} \geq 0$. Consequently, we must have that

$$G_1 = (I + L + L^2 + \cdots + L^{n-1})(\bar{U} - D_1) = (\bar{U} - D_1) + L(\bar{U} - D_1) + L^2(\bar{U} - D_1) + \cdots + L^{n-1}(\bar{U} - D_1) \geq 0$$

So G_1 is a nonnegative. We can also get that $(\bar{U} - D_1) + L(\bar{U} - D_1) + L^2(\bar{U} - D_1) + \cdots + L^{n-1}(\bar{U} - D_1)$ is irreducible since A is irreducible, hence G_1 is irreducible.

Similarly, we consider

$$G_2 = (\bar{D} - \bar{L})^{-1}\bar{U} = [\bar{D}(I - \bar{D}^{-1}\bar{L})]^{-1}\bar{U} = (I - \bar{D}^{-1}\bar{L})^{-1}\bar{D}^{-1}\bar{U} = [I + \bar{D}^{-1}\bar{L} + (\bar{D}^{-1}\bar{L})^2 + \cdots + (\bar{D}^{-1}\bar{L})^{n-1}]\bar{D}^{-1}\bar{U}$$
$$= \bar{D}^{-1}\bar{U} + (\bar{D}^{-1})^2\bar{L}\bar{U} + (\bar{D}^{-1})^3\bar{L}^2\bar{U} + \text{nonnegative terms}$$

Using similar arguments it is conclusive that $G_2 = (\bar{D} - \bar{L})^{-1}\bar{U}$ is a nonnegative and irreducible matrix.

Theorem 2 Let $G = (I-L)^{-1}U$ and $G_1 = (I - \bar{L})^{-1}(\bar{U} - D_1)$ be the Gauss-Seidel and preconditioned Gauss-Seidel iteration matrices respectively. If A is an irreducible L −matrix with $0 \leq a_{1i}a_{i1} + a_{i,i+1}a_{i+1,i} < 1,\ i = 2(1)n$. Then,

(i) $\rho(G_1) < \rho(G)$, if $\rho(G) < 1$;
(ii) $\rho(G_1) = \rho(G)$, if $\rho(G) = 1$;
(iii) $\rho(G_1) > \rho(G)$, if $\rho(G) > 1$.

Proof Theorem 1 established G and G_1 as nonnegative and irreducible matrices. Suppose $\rho(G) = \lambda$, then there exists a positive vector $x = (x_1, x_2, \cdots, x_n)^T$, such that

$$Gx = \lambda x$$

That is,

$$(I-L)^{-1}Ux = \lambda x$$
$$U = \lambda(I - L) \quad (13)$$

And for this $x > 0$,

$$G_1x - \lambda x = (I - \bar{L})^{-1}(\bar{U} - D_1)x - \lambda(I - \bar{L})^{-1}(I - \bar{L})x$$
$$= (I - \bar{L})^{-1}\{\bar{U} - D_1 - \lambda I + \lambda\bar{L}\}x$$
$$= (I - \omega\bar{L})^{-1}\{(1-\omega)I + \omega\bar{U} - \omega D_1 - \lambda(I - \omega\bar{L})\}x$$
$$= (I - \bar{L})^{-1}\{-\lambda I - D_1 + \lambda(L + L_S + L_1) + (U + U_S + U_1)\}x$$
$$= (I - \bar{L})^{-1}\{-\lambda I + U + \lambda L + \lambda L_S + \lambda L_1 + U_S - D_1 + U_1\}x$$

From (13), $\lambda I = U + \lambda L$

$$G_1x - \lambda x = (I - \bar{L})^{-1}\{\lambda L_S + \lambda L_1 + U_S - D_1 + U_1\}x$$
$$= (I - \bar{L})^{-1}\{(\lambda - 1)L_1 + \lambda L_S + U_S - (D_1 - L_1 - U_1)\}x$$
$$= (I - \bar{L})^{-1}\{(\lambda - 1)L_1 + \lambda L_S + U_S - [-(SL + SU)]\}x$$
$$= (I - \bar{L})^{-1}\{(\lambda - 1)L_1 + \lambda L_S + U_S + SL + SU\}x$$
$$= (I - \bar{L})^{-1}\{(\lambda - 1)L_1 + (\lambda - 1)L_S + (L_S + U_S) + SL + SU\}x$$
$$= (I - \bar{L})^{-1}\{(\lambda - 1)(L_1 + L_S) - S + SL + SU)\}x$$
$$= (I - \bar{L})^{-1}\{(\lambda - 1)(L_1 + L_S) - S + SL + SU)\}x$$
$$= (I - \bar{L})^{-1}\{(\lambda - 1)(L_1 + L_S) - S(I - L) + SU)\}x$$
$$= (I - \bar{L})^{-1}\{(\lambda - 1)(L_1 + L_S) + S[U - (I - L)]\}x$$

From (13), $U = \lambda(I - L)$

$$G_1x - \lambda x = (I - \bar{L})^{-1}\{(\lambda - 1)(L_1 + L_S) + S[\lambda(I - L) - (I - L)]\}x$$
$$= (I - \bar{L})^{-1}\{(\lambda - 1)(L_1 + L_S) + (\lambda - 1)S(I - L)\}x$$
$$= (\lambda - 1)(I - \bar{L})^{-1}\{(L_1 + L_S) + S(I - L)\}x$$

From (13), $(I - L) = U/\lambda$

$$G_1x - \lambda x = (\lambda - 1)(I - \bar{L})^{-1}\{(L_1 + L_S) + SU/\lambda\}x$$
$$= [(\lambda - 1)/\lambda](I - \bar{L})^{-1}\{\lambda\,(L_1 + L_S) + SU\}x$$

Assume $H = Jx$, where $J = (I - \bar{L})^{-1}\{\lambda(L_1 + L_S) + SU\}$. Then $J = (I - \bar{L})^{-1}\{\lambda\,(L_1 + L_S) + SU\} \geq 0$, since $\lambda(L_1 + L_S) \geq 0$, and $SU \geq 0$. Also, $(I - \bar{L})^{-1} = I + \bar{L} + \bar{L}^2 + \cdots + \bar{L}^{n-1} \geq 0$, since $\bar{L} \geq 0$. Therefore, $J = (I - \bar{L})^{-1}\{\lambda\,(L_1 + L_S) + SU\} \geq 0$. Consequently, $H = (I - \bar{L})^{-1}\{\lambda\,(L_1 + L_S) + SU\}x \geq 0$, since $x > 0$.

(i) If $\lambda < 1$, then $G_1x - \lambda x \leq 0$ but not equal to 0. Therefore, $G_1x \leq \lambda x$. From Lemma 2, we have $\rho(G_1) < \lambda = \rho(G)$.

(ii) If $\lambda = 1$, then $G_1x - \lambda x = 0$. Therefore, $G_1x = \lambda x$. From Lemma 2, we have $\rho(G_1) = \lambda = \rho(G)$.

(iii) If $\lambda > 1$, then $G_1x - \lambda x \geq 0$ but not equal to 0. Therefore, $G_1x \geq \lambda x$. From Lemma 2, we have $\rho(G_1) > \lambda = \rho(G)$.

Theorem 3 Let $G = (I - L)^{-1}U$ and $G_2 = (\bar{D} - \bar{L})^{-1}U$ be the Gauss-Seidel and the preconditioned Gauss-Seidel iteration matrices respectively. If A is an irreducible L −matrix with $0 \leq a_{1i}a_{i1} + a_{i,i+1}a_{i+1,i} < 1,\ i = 2(1)n$.
Then,

(i) $\rho(G_2) < \rho(G)$, if $\rho(G) < 1$;
(ii) $\rho(G_2) = \rho(G)$, if $\rho(G) = 1$;
(iii) $\rho(G_2) > \rho(G)$, if $\rho(G) > 1$.

Proof From Theorem 1, G and G_2 are nonnegative and irreducible matrices. Suppose $\rho(G) = \lambda$, then there exists a positive vector $x = (x_1, x_2, \cdots, x_n)^T$, such that (13) holds.
Therefore, for this $x > 0$,

$$G_2x - \lambda x = (\bar{D} - \bar{L})^{-1}\bar{U}x - \lambda x$$
$$= (\bar{D} - \bar{L})^{-1}\bar{U}x - (\bar{D} - \bar{L})^{-1}(\bar{D} - \bar{L})\lambda x$$
$$= (\bar{D} - \bar{L})^{-1}\{\bar{U} - \lambda(\bar{D} - \bar{L})\}x$$
$$= (\bar{D} - \bar{L})^{-1}\{\bar{U} - \lambda\bar{D} + \bar{L}\}x$$
$$= (\bar{D} - \bar{L})^{-1}\{(U + U_S + U_1) - \lambda(I + D_1) + \lambda(L + L_S + L_1)\}x$$
$$= (\bar{D} - \bar{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_S + \lambda L + U_S + U_1 + -\lambda I + U)\}x$$

But, from (13), $-\lambda I + U = -\lambda L$

$$G_2x - \lambda x = (\bar{D} - \bar{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_S + U_S + U_1 - \lambda L)\}x$$
$$= (\bar{D} - \bar{L})^{-1}\{(\lambda - 1)(-D_1) + (\lambda - 1)L_1 - (D_1 - L_1 - U_1) + \lambda L_S - L_S + L_S + U_S)\}x$$
$$= (\bar{D} - \bar{L})^{-1}\{(\lambda - 1)(-D_1 + L_1) - (-(SL + SU)) + (\lambda - 1)L_S + (L_S + U_S)\}x$$
$$= (\bar{D} - \bar{L})^{-1}\{(\lambda - 1)(-D_1 + L_1 + L_S) + SL + SU + (L_S + U_S)\}x$$
$$= (\bar{D} - \bar{L})^{-1}\{(\lambda - 1)(-D_1 + L_1 + L_S) + SU - S + SL\}x$$
$$= (\bar{D} - \bar{L})^{-1}\{(\lambda - 1)(-D_1 + L_1 + L_S) + SU - S(I - L)\}x$$
$$= (\bar{D} - \bar{L})^{-1}\{(\lambda - 1)(-D_1 + L_1 + L_S) + S[U - (I - L)]\}x$$

From equation (13), $U = \lambda(I - L)$

$$G_2x - \lambda x = (\bar{D} - \bar{L})^{-1}\{(\lambda - 1)(-D_1 + L_1 + L_S) + S[\lambda(I - L) - (I - L)]\}x$$
$$= (\bar{D} - \bar{L})^{-1}\{(\lambda - 1)(-D_1 + L_1 + L_S) + (\lambda - 1)S(I - L)\}x$$

From equation (13), $(I - L) = U/\lambda$

$$= (\lambda - 1)(\bar{D} - \bar{L})^{-1}\{(-D_1 + L_1 + L_S) + SU/\lambda\}x$$
$$= [(\lambda - 1)/\lambda](\bar{D} - \bar{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_S + SU\}x$$

Suppose $R = Qx$, with $Q = (\bar{D} - \bar{L})^{-1}\{-\lambda D_1 + \lambda L_1 + \lambda L_S + SU\}$. Obviously, $-\lambda D_1 + \lambda L_1 + \lambda L_S + SU \geq 0$, since $SU \geq 0$, $-\lambda D_1 \geq 0$, $\lambda L_1 \geq 0$ and $\lambda L_S \geq 0$. Since $\bar{D}$ is a nonsingular matrix, we let $\bar{D} - \bar{L}$ be a splitting of some matrix K, i.e., $K = \bar{D} - \bar{L}$. Also, $\bar{D}$ is an M −matrix and $\bar{L} \geq 0$. Thus, $K = \bar{D} - \bar{L}$ is an M −splitting. Now, $\bar{D}^{-1}\bar{L}$ is a strictly lower triangular matrix, and by implication its eigenvalues lie on its main diagonal; in this case they are all zeros. Therefore, $\rho(\bar{D}^{-1}\bar{L}) = 0$. Since $\rho(\bar{D}^{-1}\bar{L}) < 1$, $K = \bar{D} - \bar{L}$ is a convergent splitting. By the foregoing, $K = \bar{D} - \bar{L}$ is an M −splitting and $\rho(\bar{D}^{-1}\bar{L}) < 1$, we employ Lemma 3 to establish that K is an M −matrix. Since K is an M −matrix, by definition, $K^{-1} = (\bar{D} - \bar{L})^{-1} \geq 0$. Thus, $Q \geq 0$ and $R \geq 0$.

(i) If $\lambda < 1$, then $G_2x - \lambda x \leq 0$ but not equal to 0. Therefore, $G_2x \leq \lambda x$. From Lemma 2, we have $\rho(G_2) < \lambda = \rho(G_{SOR})$.

(ii) If $\lambda = 1$, then $G_2x - \lambda x = 0$. Therefore, $G_2x = \lambda x$. From Lemma 2, we have $\rho(G_2) = \lambda = \rho(G_{SOR})$

(iii) If $\lambda > 1$, then $G_2x - \lambda x \geq 0$ but not equal to 0. Therefore, $G_2x \geq \lambda x$. From Lemma 2, we have $\rho(G_2) > \lambda = \rho(G_{SOR})$.

Numerical Experiments

In order to validate the results of the preceding section, the preconditioned Gauss-Seidel methods introduced in this work are applied to Problems 1 and 2. The spectral radii of iteration matrices of the two methods are obtained and compared to those of some other methods.

Problem 1 Consider a 4×4 matrix of the form

$$A = \begin{pmatrix} 1 & -0.172 & -0.234 & 0 \\ -0.365 & 1 & 0 & -0.204 \\ -0.165 & 0 & 1 & -0.215 \\ 0 & -0.236 & -0.372 & 1 \end{pmatrix}$$

Problem 2 Consider a 6× 6 matrix of the form

$$A = \begin{pmatrix} 1.0 & -0.1 & -0.1 & -0.4 & -0.2 & -0.1 \\ -0.5 & 1 & 0 & 0 & 0 & 0 \\ -0.3 & -0.1 & 1 & -0.2 & -0.1 & 0 \\ -0.2 & 0 & -0.1 & 1 & -0.1 & -0.3 \\ -0.2 & 0 & -0.1 & -0.1 & 1 & -0.2 \\ -0.1 & 0 & 0 & -0.1 & -0.1 & 1 \end{pmatrix}$$

By letting G, G_1 and G_2 be the itreration matrices of the classical Gauss-Seidel method, preconditioned Gauss-Seidel methods of (9) and (12) respectively, the spectral radii of these matrices are computed for Examples 1 and 2 and the results presented in Tables I and II.

3.0 Results and Discussion

Tables I and II depict the results of Problems 1 and 2 respectively. In the Tables, G, G_1, G_2, G_{SOR}, G_{GN}, G_M, and $G_{M\,\&\,N}$ represent the iteration matrices of the Gauss-Seidel, our first preconditioned Gauss-Seidel, our second preconditioned Gauss-Seidel, SOR, Gunawardena *et al.* (1991), Milaszewicz (1987) and Mayaki and Ndanusa (2019) respectively.

Table I: Comparison of spectral radii of G_1 and G_2 with various iteration matrices for Problem 1

Iteration matrix	Spectral radius
G_1	0.1601241711
G_2	0.06681777737
G	0.2277905779
G_{SOR}	0.1000000002
G_{GN}	0.08177303033
G_M	0.1682312333
$G_{M\,\&\,N}$	0.08177303033

Table 2: Comparison of spectral radii of G_1 and G_2 with various iteration matrices for Problem 2

Iteration matrix	Spectral radius
G_1	0.2807908647
G_2	0.1943430798
G	0.4206679675
G_{SOR}	0.2435217064
G_{GN}	0.3390264208
G_M	0.2663324128
$G_{M\,\&\,N}$	0.3384319902

It is well known that the spectral radius of the iteration matrix of an iterative method for linear systems is sufficient for convergence of the method. The method is known to converge when the spectral radius is less than 1 in absolute value; the closer it is towards 0 the faster the convergence. In Table I, the spectral radius of G_2 is seen to be smaller than that of the unpreconditioned Gauss-Seidel G. It is shown to be smaller that those of G_1, G_{GN}, G_M, $G_{M\,\&\,N}$ and even that of G_{SOR}. Although the spectral radius of G_1 outperforms those of G and G_M, it lags behind those of G_2, G_{SOR}, G_{GN} and $G_{M\,\&\,N}$. Similar trend is witnessed in Table II, with G_2 in the lead, followed by G_M, G_{SOR}, G_1, $G_{M\,\&\,N}$, G_{GN} and G, in that order.

4.0 Conclusion

Two preconditioned schemes of the Gauss-Seidel iterative method for solving linear systems are introduced, analysed and their convergence established. Numerical experiments reaffirmed their superiority over the unpreconditioned Gauss-Seidel method. More so, the performance of these methods, when compared to some other preconditioned methods in literature, showed significant improvement in the rate of convergence of the new methods over the existing ones.

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Conflict of Interest

The author declare no conflict of interest