On the Dynamical Analysis of a New Model for Measles Infection

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***Abstract*: Epidemiologists and other health workers all over the world have noted that vaccination is an effective means of preventing most childhood diseases. In this study a compartmental mathematical model was formulated with the inclusion of the vaccinated class in a bid to examine the dynamics of measles within a population. This new class allowed us to determine the required vaccination coverage and dosage that will guarantee eradication of measles. The model is expressed as a system of ordinary differential equations. The stability of the equilibrium states was examined with respect to the basic reproductive number R_0 ,and found that the disease-free equilibrium state is locally and globally stable when $R_0 \leq 1$, and the endemic equilibrium state is stable if $R_0 > 1$. It was also noted that the effective reproductive number R_{0v} under vaccination approaches zero as the proportion of successfully vaccinated individual increases. Lastly, the required vaccination dosage and coverage that can lead to measles eradication were studied.**

***Keywords*: Asymptotic expansion, Equilibrium, Immunity, Measles, Reproduction Number, Stability, Vaccination.**

I. INTRODUCTION

Mathematical models have become important tools in analyzing the spread, control and eradication of infectious diseases. Also, mathematical modeling is increasingly becoming a key method in determining effective control policy for a range of epidemic. The classical SIR model of Kermack and McKendrick has formed the foundation for most epidemiological models [1]. Over the years, the study of childhood infections using SIR family of models has yielded many interesting results [2], [3], [4], [5] and the extension of this family of models to include other classes (like passively immune, exposed etc) has helped in understanding the behavior of diseases with more complex dynamics [6], [7], [8]. The advantages of simple SIR models were noted by House et al. [9], but they also noted that as the dynamics of a disease become complex, there is the need for more sophisticated approach.
Measles is best known for causing rash and fever in childhood, but can lead to more severe health complications in adults. There are two types of measles, each caused by different viruses. The first type is rubella and the other, rubeola. Measles can cause significant birth defects if an infected pregnant woman passes the virus to her unborn child. It can also lead to pneumonia or inflammation of the brain (encephalitis). Measles are spread through the respiratory route. This means they are contagious through coughing and sneezing. In fact, the measles virus is one of the most contagious viruses known to man. As a result, it can spread rapidly in a susceptible population. Usually, people are immune to the virus (either through vaccination or by having had measles in the past) [10].

Worldwide, measles vaccination has been very effective, preventing an estimated 80 million cases and 4.5 million deaths annually [11], [12]. Although global incidence has been significantly reduced through vaccination, measles remains an important public health problem. Since vaccination coverage is not uniformly high worldwide, measles stands as the leading vaccine-preventable killer of children worldwide. Measles is estimated to have caused 1681 deaths daily in 2002, with more than half of these deaths occuring in sub-saharan Africa [13], [14]. In 2010, measles related deaths dropped to 380 per day, but the number increased to 430 deaths daily in 2011 [15]**.** It has been advised that the vaccine should be given whether the child is HIV-infected or not [16]. The vaccine is less effective in HIV-infected infants, but the risk of adverse reactions is low. A great advantage of measles vaccination programs is that other intervention programs like distribution of treated nets, vitamin A supplements etc, can be delivered along with it [17].
The World Health Assembly in 1989 and the World Summit for Children in 1990 set goals for measles morbidity and mortality reduction of 90% and 95%, respectively, compared with pre-vaccine levels. Therefore, vaccination against measles with one dose is one of the components of WHO's EPI (World Health Organization's Expanded Programme on Immunization) implemented from the 1980's in most Sub-Sahara African countries. The fundamental characteristic of vaccination is that it reduces the incidence of the disease in those immunized, and the susceptibles. Also, vaccination protects indirectly non-vaccinated susceptibles against infection by producing herd immunity. Measles is a disease that still has the tendency of killing people in developing countries.
Eradicating measles is becoming more challenging when compared to some other infectious diseases like small pox. Inability to eradicate measles is partly due to some factors which include but not limited to: high infectiousness of the measles virus,

inability to detect infected persons, the timeliness of administration vaccine and how it is delivered, patients resistance to vaccination and treatment, inadequate sensitization about the problems measles can cause etc.
It is of importance to examine vaccination scenario using mathematics. Immunization coverage and dosage had been linked in the past so it will be of importance to examine 'what optimum coverage is required to eradicate a disease in a population under certain dosage programme. Proposing proper vaccination strategy (mathematically) that will lead to eradication of measles in a population, is our goal in this study. In addition to the analysis of the mathematical model that will be formulated, a solution to the model using asymptotic expansion method which is a new method of solving epidemiological models will be provided. This method allows us to obtain a near-analytic solution for the model.

II. Methods and Materials

A. Model Description

In their article, Scherer et al. [18] proposed a model that considers vaccination as an epidemiology class but did not consider the exposed class. In this study we will follow their approach with inclusion of the exposed class.

In this study we will assume that newborns enter directly into the susceptible class $\bar{S}(t)$ at the rate $\mu\bar{N}(t)$ that is, all the newborns are assumed to be susceptible. When there is an adequate contact of a susceptible with an infective so that transmission occurs, then the susceptible enters the exposed class $\bar{E}(t)$ (the latent period), who are infected but not yet infectious. After the latent period ends, individuals enter the class $\bar{I}(t)$ of infectives, who are infectious in the sense that they are capable of transmitting the infection. When the infectious period ends, the individual enters the recovered class $\bar{R}(t)$ consisting of those with permanent infection-acquired immunity, otherwise they die. There is another class that we introduce here, this is called the vaccinated class $\bar{V}(t)$. These are the individuals that are vaccinated at birth.

Let the contact rate β be the average number of adequate contacts per person per unit time. The people in the exposed class will migrate to the infected class at the rate $\varepsilon\bar{E}(t)$, natural death will occur to the exposed class at the rate $\alpha\bar{E}(t)$, the recovery rate from the infectious class is $\gamma\bar{I}(t)$ and natural death will occur to individuals in the infectious class at the rate $\alpha\bar{I}$. More so, let κ be the average rate for vaccinated individuals to obtain immunity and move to recovered class. A fraction η of the newly born individuals are vaccinated at birth. This vaccination takes in a fraction e of the vaccinated individuals and protects them for an average period of $1/\tau$ years (proportion of those successively vaccinated at birth). In Nigeria for example, vaccination against measles consists of one dose of standard titer Schwarz vaccine given to infants at 9 months of age. Nevertheless, during epidemics, an early two-dose strategy was always implemented: one dose between 6 and 8 months and the other after 9 months. Before this age, we suppose that children gain protection from the maternal antibodies. Also it is worthy to note that natural death will occur to susceptible individuals and vaccinated individuals at the rate $\alpha\bar{S}(t)$ and $\alpha\bar{V}(t)$ respectively.

B. Model Formulation

Denote the cardinal number of set $\{\bar{S},\bar{V},\bar{E},\bar{I},\bar{R},\bar{N}\}$ at time t by $\{S(t),V(t),E(t),I(t),R(t),N(t)\}$, if the average number of contacts with infectives per unit time of a susceptible is given as $\beta I(t)$ then the number of new cases per unit time amongst the susceptibles is $\beta S(t)I)(t)$. Using this fact and the descriptions in section 2.1 above, we have the following system as the model:

$$\begin{aligned}
\frac{dS(t)}{dt} &= \mu(1-\upsilon\eta)N(t) - \beta I(t)S(t) - \alpha S(t)\\
\frac{dV(t)}{dt} &= \upsilon\eta\mu N(t) - \alpha V(t) - \kappa V(t)\\
\frac{dE(t)}{dt} &= \beta I(t)S(t) - (\varepsilon+\alpha)E(t)\\
\frac{dI(t)}{dt} &= \varepsilon E(t) - (\gamma+\alpha+\varphi)I(t)\\
\frac{dR(t)}{dt} &= \gamma I(t) - \alpha R(t) + \kappa V(t)
\end{aligned} \qquad (1)$$

in domain Ω.

The definition of the domain is;

$$\Omega=\left\{(S(t),V(t),E(t),I(t),R(t))\in R^5: S(t)+V(t)+E(t)+I(t)+R(t)=N(t)\right\}$$

III. Results

A. Model Solution

1) Existence of Solution for the Model: System (1) can be rewritten in the form:

$$\Psi_t = F\Psi + J(F) \qquad (2)$$

where

$$\Psi=\begin{bmatrix} S(t)\\ V(t)\\ E(t)\\ I(t)\\ R(t)\end{bmatrix};$$

$$F=\begin{pmatrix} -\alpha & 0 & 0 & 0 & 0\\ 0 & -(\alpha+\kappa) & 0 & 0 & 0\\ 0 & 0 & -(\varepsilon+\alpha) & 0 & 0\\ 0 & 0 & \varepsilon & -(\gamma+\alpha+\varphi) & 0\\ 0 & \kappa & 0 & \gamma & -\alpha\end{pmatrix}$$

and

$$J(\Psi)=\begin{pmatrix} \mu(1-\upsilon\eta)N(t)-\beta S(t)I(t)\\ \mu\upsilon\eta N(t)\\ \beta S(t)I(t)\\ 0\\ 0\end{pmatrix}$$ with F_t is the derivative of F w.r.t t .

If we set $W(\Psi)=F\Psi+J(\Psi)$ and using the fact that the second term on the right hand side of (2) satisfies

$$\begin{aligned}|J(\psi_1)-J(\psi_2)|\le L(&|S_1(t)-S_2(t)|+|V_1(t)-V_2(t)|\\ &+|E_1(t)-E_2(t)|+|I_1(t)-I_2(t)|\\ &+|R_1(t)-R_2(t)|)\end{aligned}$$

where L (non negative constant) is independent of $S(t),V(t),E(t),I(t)$ and $R(t)$, then $|W(\psi_1)-W(\psi_2)|\le H|\psi_1-\psi_2|$ (where $H=\max\{L,\|\mu N\|\}$). Now, with the restriction on $S(t),V(t),E(t),I(t)$ and $R(t)\ge 0$, we can see that the solution of system (1) exist and W is uniformly Lipschitz continuous.

2) Positivity of Solution:

Lemma 1

Let $S(0)=S_0\ge 0, V(0)=V_0\ge 0, E(0)=E_0\ge 0, I(0)=I_0\ge 0,$ and $R(0)=R_0\ge 0,$ be the initial data set of system (1) then $S(t),V(t),E(t),I(t)$ and $R(t)$ are non-negative $\forall t>0$.

Proof

Let $T=\sup[0,t]$ with $S(T)>0, V(T)>0, E(T)>0, I(T)>0$ and $R(T)>0$ $\forall t\ge 0$ and for convenience we set μN to a recruitment constant A.

From the first equation of system (1) we have

$$\frac{dS(t)}{dt}=A-\upsilon\eta\mu N(t)-\beta I(t)S(t)-\alpha S(t)$$

this is the same as

$$\frac{d}{dt}\left[S(u)\exp\left[\int_0^t \beta I(u)du-\alpha t\right]\right]=A-\upsilon\eta\mu N(t)\exp\left[\int_0^t \beta I(u)du-\alpha t\right]$$

$$\Rightarrow S(T)\exp\left[\int_0^T \beta I(u)du-\alpha T\right]-S(0)=\int_0^T A-\upsilon\eta\mu N(t)\exp\left[\int_0^u \beta I(\mathrm{v})d\mathrm{v}-\alpha u\right]du$$

which will lead to

$$S(t)=S_0\exp\left[-\int_0^T \beta I(u)du-\alpha T\right] \qquad (3)$$

$$+\left(\exp\left[-\int_0^T \beta I(u)du+(\mu-\alpha)T\right]\right)\left(\int_0^T A-\upsilon\eta\mu N(t)\exp\left[\int_0^u \beta I(\mathrm{v})d\mathrm{v}-\alpha u\right]du\right)$$

we can see that $S(t)\geq 0$.

From the second equation of system (1) we have

$\frac{dV(t)}{dt}=\upsilon\eta\alpha N(t)-(\alpha+\kappa)V(t)$ which leads to

$$V(t)=V_0e^{-(\alpha+\kappa)T}+\int_0^T \upsilon\eta\alpha N(u)e^{-(\alpha+\kappa)u}du\geq 0 \qquad (4)$$

From the third equation of system (1) we have

$\frac{dE(\mathrm{t})}{dt}=\beta S(t)I(t)-(\varepsilon+\alpha)E(t)$ on solving the equation we obtain

$$E(t)=E_0e^{-(\varepsilon+\alpha)\mathrm{T}}+\int_0^T \beta S(u)I(u)e^{(\varepsilon+\alpha)u}du\geq 0 \qquad (5)$$

From the fourth equation of system (1) we have

$$\frac{dI(t)}{dt}=\varepsilon E(t)-(\gamma+\alpha+\varphi)I(t)\geq-(\gamma+\alpha+\varphi)I(t)$$

$\Rightarrow\frac{dI(t)}{dt}\geq-(\gamma+\alpha+\varphi)I(t)$ which will lead to

$$I(t)\geq I_0e^{-(\gamma+\alpha+\varphi)T}\geq 0 \qquad (6)$$

From the fifth equation of system (1) we have

$$\frac{dR(t)}{dt}=\gamma I(t)-(\alpha-\kappa)R(t)\geq-(\alpha-\kappa)R(t)$$

$\Rightarrow\frac{dR(t)}{dt}\geq-(\alpha-\kappa)R(t)$ which will lead to

$$R(t)\geq R_0e^{-(\alpha-\kappa)\mathrm{T}}\geq 0 \qquad (7)$$

From (3)-(7), we can see that $S(t),V(t),E(t),I(t)$ and $R(t)$ are non-negative $\forall t>0$, which completes the proof of the lemma.

Lemma 2

If μN is set to a recruitment constant A in (1), then the feasible solution set Ω given as;

$\Omega=\left\{(S(t),V(t),E(t),I(t),R(t))\in R_+^5:S(t)+V(t)+E(t)+I(t)+R(t)=N(t)\leq\frac{A}{\alpha}\right\}$ is positively invariant. More so, if $F(t)$ is a solution of system (1) with the initial conditions, then $S(t)\leq\frac{A}{\alpha};V(t)\leq\frac{A}{\alpha};E(t)\leq\frac{A}{\alpha};I(t)\leq\frac{A}{\alpha};R(t)\leq\frac{A}{\alpha}$ $\forall t$ large enough.

Proof

Since the total population is given as $N(t)$ (by definition), then if we sum all the derivatives of the compartments we have;

$$\frac{dN(t)}{dt}=A-\alpha N(t)-\varphi I(t)\leq A-\alpha N(t) \qquad (8)$$

$$\Rightarrow\frac{dN(t)}{dt}\leq A-\alpha N(t)$$

this will lead to

$$N(t)\leq\frac{A}{\alpha}+e^{-\alpha t}\left(N_0-\frac{A}{\alpha}\right)$$

$\therefore \lim_{t\to\infty} N(t) \leq \frac{A}{\alpha}$

and so Ω is positively invariant.

Now, since $N(t)$ is continuous on $t\in[0,+\infty)$ $t\geq 0$ and

$\frac{dN(t)}{dt} = A - \alpha N(t) - \varphi I(t)$ which ultimately leads to $\lim_{t\to\infty} N(t) \leq \frac{A}{\alpha}$, so $N(t)$ is uniformly bounded.

If there exists a positive integer ξ, by definition of $N(t)$ we have;

$M(t) \leq \frac{A}{\alpha}; S(t) \leq \frac{A}{\alpha}; E(t) \leq \frac{A}{\alpha}; I(t) \leq \frac{A}{\alpha}; R(t) \leq \frac{A}{\alpha} \quad \forall\, t \geq \xi T$ where $T \in \square^{+}$.

3) Solution of the Model:

At this point we wish to solve the model analytically. Some authors have tried to solve various epidemiological models analytically in the past by using Homotopy Analysis Method (HAM). HAM has been noted to be a very good method in solving both weakly and strongly non linear problems. Awawdeh et al. [19] used Homotopy Analysis Method (HAM) to solve a SIR model analytically but it was noted that HAM gives solution in a series form rather than a proper analytic form. Also the time it takes to generate and evaluate the series is a little longer compared to the method we will be proposing.

In this section, we are going to employ the use of asymptotic expansion in solving the model. This will allow us to express our solution in terms of the parameter values.

We consider an asymptotic expansion of the form;

$$S = S_0 + \alpha S_1 + \alpha^2 S_2 + \ldots$$

$$V = V_0 + \alpha V_1 + \alpha^2 V_2 + \ldots$$

$$E = E_0 + \alpha E_1 + \alpha^2 E_2 + \ldots$$

$$I = I_0 + \alpha I_1 + \alpha^2 I_2 + \ldots$$

$$R = R_0 + \alpha R_1 + \alpha^2 R_2 + \ldots$$

$$\gamma = o(\alpha) = \alpha\gamma_0$$

$$\beta = o(\alpha) = \alpha\beta_0 \qquad (9)$$

this will lead to the following set of equations;

$o(1)$:

$$\frac{dS_0}{dt} = 0 \qquad \text{(i)}$$

$$\frac{dV_0}{dt} = -\kappa V_0 \qquad \text{(ii)}$$

$$\frac{dE_0}{dt} = -\varepsilon E_0 \qquad \text{(iii)}$$

$$\frac{dI_0}{dt} = \varepsilon E_0 - \varphi I_0 \qquad \text{(iv)}$$

$$\frac{dR_0}{dt} = \kappa V_0 \qquad \text{(v)}$$

$o(\alpha)$:

$$\frac{dS_1}{dt}=(1-\upsilon\eta)N-\beta_0 I_0 S_0-S_0 \quad \text{(vi)}$$

$$\frac{dV_1}{dt}=\upsilon\eta N-V_0-\kappa V_1 \quad \text{(vii)}$$

$$\frac{dE_1}{dt}=\beta_0 I_0 S_0-\varepsilon E_1-E_0 \quad \text{(viii)}$$

$$\frac{dI_1}{dt}=\varepsilon E_1-\gamma_0 I_0-I_0-\varphi I_1 \quad \text{(ix)}$$

$$\frac{dR_1}{dt}=\gamma_0 I_0-R_0+\kappa V_1 \quad \text{(x)}$$

Solutions of equations (i) to (x) will give us approximate solutions of system (1).
We used Maple13 to solve the equations and obtained the following algebraic solution:

$$S(t)=S_0+\alpha\left(\begin{array}{l}\dfrac{S_0\beta}{\alpha}\left(\dfrac{-E_0e^{-\varepsilon t}+\dfrac{\varepsilon E_0e^{-\varphi t}}{\varphi}+\dfrac{\varepsilon I_0e^{-\varphi t}}{\varphi}-I_0e^{-\varphi t}}{\varepsilon-\varphi}\right)\\ +((1-\upsilon\eta)N-S_0)t-\dfrac{(I_0+E_0)\beta S_0}{\varphi\alpha}\end{array}\right) \quad (10)$$

$$V(t)=V_0e^{-\kappa t}+\alpha\left(\frac{N\upsilon\eta e^{\kappa t}}{\kappa}-tV_0\frac{N\upsilon\eta}{\kappa}\right)e^{-\kappa t} \quad (11)$$

$$E(t)=E_0e^{-\varepsilon t}+\alpha\left(\begin{array}{l}\dfrac{-\beta S_0\varepsilon E_0t}{\alpha}+\dfrac{\beta S_0E_0\varepsilon e^{t(\varepsilon-\varphi)}}{\alpha(\varepsilon-\varphi)}\\ \dfrac{+\dfrac{\beta S_0\varepsilon I_0e^{t(\varepsilon-\varphi)}}{\alpha(\varepsilon-\varphi)}-t\varepsilon E_0+\varphi tE_0}{\varepsilon-\varphi}\\ \dfrac{\beta S_0(\varepsilon E_0+\varepsilon I_0-\varphi I_0)}{\alpha(\varepsilon-\varphi)^2}\end{array}\right)e^{-\varepsilon t} \quad (12)$$

$$I(t)=\left(-\frac{\varepsilon E_0e^{-t(\varepsilon-\varphi)}}{\varepsilon-\varphi}+\frac{\varepsilon E_0+\varepsilon I_0-\varphi I_0}{\varepsilon-\varphi}\right)e^{-\varphi t}+C+D \quad (13)$$

where

$$C=\alpha\left(\frac{1}{(\varepsilon-\varphi)^2}\right)\left(\begin{array}{l}\frac{\varphi\beta\varepsilon^2 S_0E_0((-\varepsilon+\varphi)te^{(-\varepsilon+\varphi)t}-e^{(-\varepsilon+\varphi)t})}{\alpha(\varepsilon-\varphi)^2}\\ -\frac{\varphi\beta\varepsilon S_0E_0I_0e^{(-\varepsilon+\varphi)t}}{\alpha(-\varepsilon+\varphi)}\\ -\varepsilon\varphi tE_0-\frac{\beta\varepsilon^2S_0E_0t}{\alpha}+\varepsilon^2E_0t-2\varphi\varepsilon tI_0\\ +\frac{\varepsilon^3E_0((-\varepsilon+\varphi)te^{(-\varepsilon+\varphi)t}-e^{(-\varepsilon+\varphi)t})}{(-\varepsilon+\varphi)^2}\\ +\frac{\gamma\varepsilon^2E_0t}{\alpha}+\frac{\gamma\varepsilon^2I_0t}{\alpha}+\frac{\gamma\varphi^2I_0t}{\alpha}-\frac{\varepsilon^2E_0e^{-(\varepsilon-\varphi)t}}{-\varepsilon+\varphi}\\ +\frac{\beta\varepsilon S_0I_0t}{\alpha}\\ +\frac{\varphi\beta\varepsilon^3S_0E_0((-\varepsilon+\varphi)te^{(-\varepsilon+\varphi)t}-e^{(-\varepsilon+\varphi)t})}{\alpha(-\varepsilon+\varphi)^2}\\ +\frac{\gamma\varepsilon\varphi E_0e^{-(\varepsilon-\varphi)t}}{\alpha(-\varepsilon+\varphi)}+\frac{\beta\varepsilon^2S_0E_0e^{-(\varepsilon-\varphi)t}}{\alpha(-\varepsilon+\varphi)}\\ +\frac{\beta\varepsilon^2S_0I_0e^{-(\varepsilon-\varphi)t}}{\alpha(-\varepsilon+\varphi)}\\ +\varepsilon^2tI_0-\frac{\beta\varepsilon^2S_0I_0t}{\alpha}-\frac{\gamma\varepsilon\varphi E_0t}{\alpha}\\ -\frac{2\gamma\varepsilon\varphi I_0t}{\alpha}-\frac{\gamma\varepsilon^2E_0e^{-(\varepsilon-\varphi)t}}{\alpha(-\varepsilon+\varphi)}\\ +\frac{\varphi\varepsilon E_0e^{-(\varepsilon-\varphi)t}}{(-\varepsilon+\varphi)}\\ -\frac{2\varphi\varepsilon^2E_0((-\varepsilon+\varphi)te^{(-\varepsilon+\varphi)t}-e^{(-\varepsilon+\varphi)t})}{(-\varepsilon+\varphi)^2}\\ +\frac{\varphi^2\varepsilon E_0((-\varepsilon+\varphi)te^{(-\varepsilon+\varphi)t}-e^{(-\varepsilon+\varphi)t})}{(-\varepsilon+\varphi)^2}+\varphi^2tI_0\end{array}\right)e^{-\varphi t}$$

$$D=-\left(\frac{\varepsilon(2\beta S_0\varepsilon E_0+\beta S_0\varepsilon I_0-\gamma\varepsilon E_0-\varphi\beta S_0I_0+\gamma\varphi S_0)}{(\varepsilon-\varphi)^3}\right)e^{-\varphi t}$$

$$R(t)=V_0-V_0e^{-\kappa t}+R_0+\alpha\left(\begin{array}{l}-\frac{(-\varepsilon+\varphi)(-V_0e^{-\kappa t}-V_0(-\kappa te^{-\kappa t}-e^{-\kappa t})+\upsilon\eta Ne^{-\kappa t}}{(\varepsilon-\varphi)\kappa}\\ -\frac{\gamma((E_0+I_0)\varepsilon-I_0\varphi)e^{-\varphi t}}{\alpha\varphi(\varepsilon-\varphi)}+\frac{\gamma E_0e^{-\varepsilon t}}{\alpha(\varepsilon-\varphi)}+\upsilon\eta N(\varepsilon-\varphi)t\\ +(-V_0-R_0)t-\frac{N\upsilon\eta\varphi-\gamma E_0\kappa-\gamma I_0\kappa}{\kappa\varphi}\end{array}\right)$$

with $S_0, V_0, E_0, I_0 E$ and R_0 being the initial values of the susceptible, vaccinated, exposed, infective and recover classes respectively.

B. Model Analysis

1) The basic reproduction number of the Model: The basic reproduction number (sometimes called basic reproductive ratio) of an infection is the number of cases produced on the average over the course of its infectious period [3]. It is denoted by $\hat{R}_0$, it can be referred to as the average number of secondary infection due to introduction of an infected individual into a disease free population. This is a useful tool because it helps determining whether or not an infectious disease can spread through a population.

From system (1) the basic reproduction number $\hat{R}_0$ is the same as the contact number σ given by the product of the contact rate β and the average death-adjusted infectious period $\frac{1}{\gamma+\alpha+\varphi}$ multiplied by the fraction $\frac{\varepsilon}{\varepsilon+\alpha}$ of exposed people surviving the latent class $E(t)$. Thus, the reproductive number R_0 is given as:

$$\hat{R}_0=\sigma=\frac{\beta\varepsilon}{(\gamma+\alpha+\varphi)(\varepsilon+\alpha)} \qquad (14)$$

This shows that R_0 for the model is still the average number of secondary infections due to an infective during the infectious period, when everyone in the population is susceptible.

Now, the effective reproduction number in the presence of vaccination is;

$$\hat{R}_{0V}=(1-e\eta)\frac{\kappa\beta\varepsilon}{(\gamma+\alpha+\varphi)(\varepsilon+\alpha)}$$

$$\Rightarrow \hat{R}_{0V}=\kappa(1-e\eta)\hat{R}_0$$

2) Equilibra and Stability: In this section two types of equilibra will be considered, firstly we are going to study the disease-free equilibrium (which is the equilibrium point when the disease does not exist in the population) and later discuss the endemic equilibrium (which is equilibrium point in the presence of infection in the population).

Disease free equilibrium

The disease free equilibrium $\left(S^0,V^0,E^0,I^0,R^0\right)$ of system (1) is when the disease does not exist in the population, that is the vaccinated class, exposed class, infected class and recovered class are all zero i.e $V^0=0, E^0=0, I^0=0$ and $R^0=0$ so the populace remain totally susceptible, i.e $S^0=N(t)$. Therefore the disease-free equilibrium is given as $\left(N(t),0,0,0,0\right)$.

We can see that this disease-free equilibrium is locally stable if $\hat{R}_0<1$ and unstable if $\hat{R}_0>1$ which is consistent with other studies, see [3], [20].

Endemic equilibrium

At endemic equilibrium point, system (1) becomes:

$$\begin{aligned}
&\mu(1-\upsilon\eta)N(t)-\beta I(t)S(t)-\alpha S(t)=0\\
&\upsilon\eta\mu N(t)-\alpha V(t)-\kappa V(t)=0\\
&\beta I(t)S(t)-(\varepsilon+\alpha)E(t)=0\\
&\varepsilon E(t)-(\gamma+\alpha+\varphi)I(t)=0\\
&\gamma I(t)-\alpha R(t)+\kappa V(t)=0
\end{aligned} \qquad (15)$$

solving (15) we have the following endemic equilibrium points;

$$S_1^e(t)=\frac{\mu N(1-\upsilon\eta)}{\alpha}, V_1^e=\frac{\upsilon\eta\mu N}{(\alpha+\kappa)}; E_1^e=0; I_1^e=0, R_1^e=\frac{\mu\kappa\eta\upsilon N}{\alpha(\alpha+\kappa)} \qquad (16)$$

and

$$S^e = \frac{(\gamma+\alpha+\varphi)(\varepsilon+\alpha)}{\beta\varepsilon}$$

$$V^e = \frac{\upsilon\eta\mu N}{\alpha+\kappa}$$

$$E^e = \frac{\mu N\beta\varepsilon - \mu\upsilon\eta N\beta\varepsilon - \alpha(\gamma+\alpha+\varphi)(\varepsilon+\alpha)}{\varepsilon\beta(\varepsilon+\alpha)}$$

$$I^e = \frac{\mu N\beta\varepsilon - \mu\upsilon\eta N\beta\varepsilon - \alpha(\gamma+\alpha+\varphi)(\varepsilon+\alpha)}{\beta((\gamma+\alpha+\varphi)(\varepsilon+\alpha))}$$

$$R^e = \frac{1}{\beta((\gamma+\alpha+\varphi)(\varepsilon+\alpha))(\alpha+\kappa)} \begin{pmatrix} \gamma\kappa N\beta\varepsilon + \kappa\upsilon\eta N\beta\varepsilon\alpha \\ +\kappa\mu\upsilon\eta N\beta\varepsilon\varphi + \kappa\mu\upsilon\eta N\beta\gamma\alpha \\ +\kappa\upsilon\eta N\beta\alpha^2 \\ +\kappa\upsilon\mu\eta N\beta\alpha\varphi + \gamma\mu\alpha N\beta\varepsilon \\ -\gamma\alpha\upsilon\eta\mu N\beta\varepsilon - \gamma^2\alpha^2 \\ -\gamma^2\alpha\varepsilon - \gamma\alpha^2\varepsilon - \gamma\alpha\varepsilon\varphi \\ -\gamma\alpha^3 - \gamma\alpha^2\varphi - \gamma^2\kappa\alpha - \gamma^2\kappa\varepsilon \\ -\gamma\kappa\varepsilon\alpha - \gamma\kappa\varepsilon\varphi - \gamma\kappa\alpha^2 - \gamma\kappa\varepsilon\varphi \end{pmatrix} \qquad (17)$$

where $S_i^e, V_i^e, E_i^e, I_i^e$ and R_i^e ($i = 1, 2$)are the susceptible, vaccinated, exposed , infective and recovered individuals at endemic equilibrium.

We can see from (16) that $I_1^e = 0$ and this is not possible because there must exist at least an infective in the population for the disease to exist, so the endemic equilibrium is given as (17).

We can divide (1) by the total population in other to covert the classes to fractions and eliminate s by setting $s = 1 - v - e - i - r$, where s, v, e, i, r are the proportions (fraction) of the population that are in the susceptible, vaccinated, exposed, infected and recovered class respectively. This we lead us to;

$$\begin{aligned} \frac{dv}{dt} &= \mu\upsilon\eta - (\alpha+\kappa)v \\ \frac{de}{dt} &= \beta i(1-v-e-i-r) - (\varepsilon+\alpha)e \\ \frac{di}{dt} &= \varepsilon e - (\gamma+\alpha+\varphi)i \\ \frac{dr}{dt} &= \gamma i - \alpha r + \kappa v \end{aligned} \qquad (18)$$

in domain Ω defined by

$\Omega = \{(v, e, i, r) \in \mathbb{R}^4 : (v + e + i + r) \le 1\}$ with

$v(0)=v_0, e(0)=e_0, i(0)=i_0$ and $r(0)=r_0$ (19)

The disease-free equilibrium is locally asymptotically stable if $\hat{R}_0 < 1$ and is an unstable hyperbolic equilibrium with a stable manifold outside Ω and an unstable manifold tangent to a vector into Ω when $\hat{R}_0 > 1$. On using the Lyapunov functional $F = \varepsilon e + (\varepsilon+\alpha)i$ the disease-free equilibrium can be shown to be globally asymptotically stable in domain Ω if $\hat{R}_0 \le 1$. The Lyapunov derivative of F is $\dot{F} = [\beta\varepsilon s - (\varepsilon+\alpha)(\gamma+\alpha+\varphi)]i \le 0$ since $\beta\varepsilon \le (\varepsilon+\alpha)(\gamma+\alpha+\varphi)$ and $0 \le s \le 1$. Now, the set where $\dot{F} = 0$ is the face of Ω with $i = 0$, but $\frac{di}{dt} = \varepsilon e$ on this face, so that i moves off the face unless $e = 0$. Now, $\frac{dv}{dt} = \upsilon\eta\mu - \alpha v - \kappa v \le -(\alpha+\kappa)v$, so $v \to 0$. When $e = i = v = 0$, $\frac{dr}{dt} = -\alpha r$, so that $\to 0$. By definition of Ω the origin is the only positively invariant subset of the set and so $\dot{F} = 0$, so by Lyapunov-Lasalle theorem all paths in Ω approach the origin and so the disease-free equilibrium is globally asymptotically stable in Ω if $\hat{R}_0 \le 1$.

Also, the characteristic polynomial of the Jacobian at the endemic equilibrium is of order five and it can be analyzed to show that the Routh-Hirwitz criteria are satisfied if $\hat{R}_{0V}>1$, therefore the endemic equilibrium is locally asymptotically stable. More so, if $\hat{R}_{0V}>1$, then the disease-free equilibrium is unstable and the endemic equilibrium is locally asymptotically stable [20].

C. Immunity

1) Waning immunity: The mathematical model represented in system (1) was formulated based on the assumption that vaccine-induced protection is life-long, i.e., there is no waning of vaccine-induced immunity. Until the 1990s, this was a universal assumption of mathematical models of vaccination [20]. This assumption was routinely made because for most of the major vaccines against childhood infectious disease, it is approximately correct. Sensitivity of model predictions to this assumption has been studied by some authors, see [18], [21] for examples.
Assuming a waning immunity after vaccination then we have the following vaccine-induced basic reproduction number:

$$R_{0V}=(1-e\eta\frac{\alpha}{\alpha+\tau})R_0 \qquad (20)$$

The term $\frac{\alpha}{\alpha+\tau}$ in the equation is the fraction of a lifetime for which an individual is protected by a vaccine that gives immunity that wanes at rate τ in a population with fixed death rate α. For example in Nigeria the life expectancy is 48 years, then the fixed death rate is approximately 0.02 and therefore a vaccine with immunity that wanes at the same rate is only as good as a vaccine that gives protection that does not wane with 50% of recipients fully protected. A graph showing the relationship between vaccine-induced basic reproduction number and proportion successfully vaccinated is presented in Fig. 1. We can see from the figure that, as the vaccination coverage increases the basic reproduction number decreases.

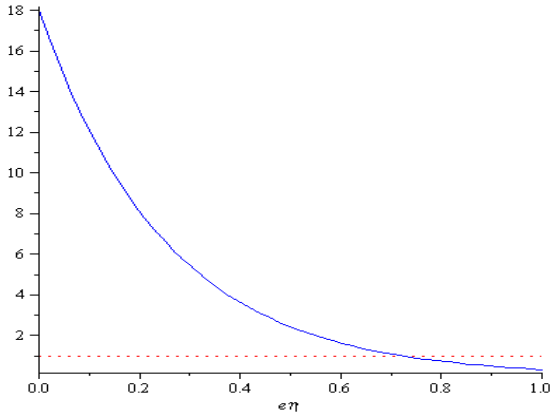

Figure 2: A graph of the Reproductive ratio R_{0V} under vaccination against the proportion successfully vaccinated($\upsilon\eta$)

*2) Herd immunity:*Herd immunity otherwise referred to as life-long immunity is the level of immunity in a population which prevents total outbreak of a disease in a society, though some cases may still be recorded.
Making similar assumption to that of [22] we have the following scenario; for example in a population with 10^5 newborns and 95% vaccine coverage, this will lead to 950000 individuals will be vaccinated and 50000 unvaccinated. If vaccine efficacy is 85% as estimated in [3], [22] (for vaccination at 9 months of age and a single dose), we have 807500 immuned and 142500 vaccinated but non-immuned. These will give us 807500 immuned and 192500 susceptibles, and so the percentage of those with herd immunity will be 80.75%.
As we know the higher the reproductive number of a disease, the higher the proportion of the population will have to be vaccinated to achieve herd immunity. This was almost the idea followed by WHO's Technical Working Group, when devising strategies to control a full range of diseases. For instance, this procedure has succeeded during the worldwide campaign for smallpox eradication in the 1960s.

D. Measles vaccination dosage

For outbreak of an infectious disease to go into decline, then it is required that each case should generate, on average, less than one other case. As we know epidemics often peak and go into decline as R_0 falls below 1, because the pool of susceptible individuals has been temporarily exhausted. For the trajectory of incidence to remain on a downward course until the agent is eradicated we require that the effective reproductive ratio should remain below 1, even when the number of susceptible

individuals is at its maximum. Two measles vaccination dosage strategies will be examined in this section and we shall discuss their theoretical results and their implication on possible eradication of measles.

1) Single dose strategy: Let R_{0v} be the basic reproductive number under vaccination, which is the number of secondary cases caused by one primary case introduced into a population in which a proportion η has been vaccinated. If we have perfect vaccine with life-long protection then

$R_{0v}=(1-\eta)R_0$.

So, to reach the critical proportion η_c that will achieve eradication we need the control condition $R_0(1-\eta_c)<1$. Which means that $R_{0v}=1$

$$\therefore R_0(1-\eta_c)=1$$

$$\Rightarrow 1-\eta_c=\frac{1}{R_0}$$

$$\Rightarrow \eta_c=\frac{R_0-1}{R_0} \qquad (21)$$

Using equation (19) and the fact that the R_0 (basic reproduction number of measles) for most countries is $\cong 18$, so, $\eta_c \cong 0.94$.

Under a single dose schedule with 85% vaccine efficacy and the fact that critical proportion η_c is equal to coverage multiplied by efficacy, then the coverage that will be required for eradication is

$$\text{coverage}=\frac{\eta_c}{efficacy}$$

$$=\frac{0.94}{0.85}=1.105$$

$\cong 1$ (*to* the nearest whole number)

So for eradication to be achieved under a single dose strategy the whole population of newborns will be required to be vaccinated (which is not feasible in reality) at 9 months.

2) Two-dose strategy: It has been stated by WHO that by 2020 measles must be eradicated in at least five WHO regions and the first core strategy mapped out to achieve this is to maintain high vaccination coverage with two doses of measles-containing vaccine [17], [15]. It was also noted that in 2011, about 84% of the world's children received one dose of measles vaccine by their first birthday through normal immunization centres. But two doses of measles vaccine are recommended in order to ensure immunity and prevent outbreaks, because about 15% of vaccinated children fail to develop immunity from the first dose [14], [11], [12], [23].

Measles vaccine strategy (schedule) of WHO is that the first dose should be given at 9 or 12 to 15 months and a second opportunity to receive a dose of measles vaccine either through routine 18 months or four to six years or supplemental immunization activities should be provided for all children [10], [17].

Now, if the proportion that is successfully vaccinated is denoted as ς satisfying:

$\varsigma+\varsigma(1-\varsigma)=$ critical proportion vaccinated

for $\varsigma<1$

$$\therefore \varsigma+\varsigma(1-\varsigma)=0.94$$

$$\Rightarrow \varsigma^2-2\varsigma+0.94=0$$

giving $\varsigma = 0.755$

Again, using the fact that coverage x efficacy= proportion successfully vaccinated, then we can determine the proportion that will require vaccination under the two-dose strategy to achieve eradication as follows:

coverage $\times$ efficacy=0.755

and since it is a two-dose strategy the efficacy is $\cong 0.98$

So

$$\Rightarrow \text{coverage}=\frac{0.755}{0.98}$$

$\cong 0.77$

We can conclude that to eradicate measles, we require only a 77% coverage under a two-dose strategy as compared to 100% coverage under a single-dose strategy.

IV. CONCLUSION

Formulating a model with vaccinated class allowed us to determine the required vaccination coverage and dosage that can lead to eradication of measles in a population. We have also been able to show existence and positivity of the model solution, also we have demonstrated that the reproduction number is a good tool in gaining insight into the dynamics of measles infection under different scenarios including vaccination. We were able to determine the equilibra states (disease-free and endemic) and their stability. Also we were able to show that under vaccination, the reproductive number R_{0v} will have values lower than one as the proportion successfully vaccinated increases, which is consistent with other studies [24], [25], [26]. Lastly we proposed a method of solution to coupled nonlinear system of differential equations in an epidemiological sense.

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