Three Step Continuous Hybrid Block Method for the Solution of $y' = f(x, y)$

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Abstract

In this paper, we present a block method for the direct solution of first order initial value problems of ordinary differential equations. Collocation and interpolation approach was adopted to generate a continuous linear multistep method which was then solved for the independent solution to give a continuous block method. We evaluated the result at selected grid points to give a discrete block which eventually gave simultaneous solutions at both grid and off grid points. The three step block method is zero stable, consistent and convergent. Numerical experiments on some selected problems compared with the exact solution proved the efficiency and accuracy of the derived method.

Keywords: consistent, convergent, collocation, off grid points, interpolation, zero stable

1.0 Introduction

We consider the first order differential equation of the form:

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{1}$$

With the advancement in Science and Technology, mathematical modeling of physical phenomenon into ordinary differential equation has been a subject of research.Ordinary differential equations (ODEs) are an indispensable tool for modeling such behaviors mathematically and first order ordinary differential equation is not an exemption. In order to understand the physical phenomenon, solution to the model is required and since most ODEs are not solvable analytically, numerical methods are implored to produce an approximate solution(Badmus, 2013).

Recent researches on numerical solutions of first order differential equation have been directed to increasing the efficiency and accuracy. Researchers developed direct methods for higher order ODEs among whom are Awoyemi *et al.* (2011), ObaruahandKayode (2013) and Adesanya *et al.* (2013) to address the challenges associated with method of reduction to system of first order.

In order to avoid these challenges, Jator (2010), Adesanya *et al.* (2012)andAnake *et al.* (2012) developed block methods in which approximations are simultaneously generated at different grid points in the interval of integration and is less expensive in terms of the number of function evaluations compared to linear multistep methods.

In this paper, we developed a three step linear multistep method with two off-grid points implemented in block method.

2.0 Development of Method

In this section, we want to derive the Hybrid Linear Multistep Method of the form:

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} + h\beta_v f_{n+v} \qquad (2)$$

Where v_j is not an integer and α_j and β_j are continuous coefficient defined by;

$$\left.\begin{aligned} \alpha_j(x) &= \sum_{j=0}^{k}\alpha_j x^j \\ \beta_j(x) &= \sum_{j=0}^{k}\beta_j x^j \end{aligned}\right\} \qquad (3)$$

Now, the general form of the power series is given by:

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \qquad (4)$$

Discretizing equation (4) at one interpolation point, x_n and collocation points, $x_n, x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}, x_{n+1}, x_{n+2}, x_{n+3}$ where m, $1 \le m < k$ and $t, t > 0$ are the numbers of collocation and interpolation points respectively gives:

$$\sum_{j=0}^{m+t-1}\alpha_j x^j = y_{n+i},\ i = 0 (5)$$

$$\sum_{j=0}^{m+t-1}\beta_j x^j = f_{n+i},\ i = 0, \frac{1}{3}, \frac{2}{3}, 1, 2, 3 \qquad (6)$$

Equation (5) and (6) give a system of $(s+m)$ equations which is solved by Gaussian elimination method to obtain $\alpha_j(x)$, $\beta_j(x)$ and $\beta_k(x)$.

The general form of the proposed method with the addition of the two off grid points is expressed as:

$$\left.\begin{aligned} y(x) &= \alpha_0(x)y_n + \\ &h\left(\beta_0(x)f_n + \beta_{n+\frac{1}{3}}(x)f_{n+\frac{1}{3}} + \beta_{n+\frac{2}{3}}(x)f_{n+\frac{2}{3}} + \beta_{n+1}(x)f_{n+1} + \beta_{n+2}(x)f_{n+2} + \beta_{n+3}(x)f_{n+3}\right)\end{aligned}\right\} \quad (7)$$

In this derivation, we use $s=1$, $m=6$, $k=3$ $v=\frac{1}{3},\frac{2}{3}$ and also express $\alpha_j(x)$, $\beta_j(x)$ and $\beta_k(x)$ as a function of t, where $t=x-x_n$ to obtain the continuous form as follows:

$$\left.\begin{aligned}
\alpha_0 &= 1 \\
\beta_0 &= -\frac{t}{240}\left(240h - \frac{760t}{h} + \frac{1100t^2}{h^2} - \frac{775t^3}{h^3} + \frac{252t^4}{h^4} - \frac{30t^5}{h^5}\right) \\
\beta_{\frac{1}{3}} &= \frac{27t}{320}\left(\frac{72t}{h} - \frac{160t^2}{h^2} + \frac{125t^3}{h^3} - \frac{48t^4}{h^4} + \frac{6t^5}{h^5}\right) \\
\beta_{\frac{2}{3}} &= -\frac{27t}{1120}\left(\frac{180t}{h} - \frac{580t^2}{h^2} + \frac{585t^3}{h^3} - \frac{228t^4}{h^4} + \frac{30t^5}{h^5}\right) \\
\beta_1 &= \frac{t}{240}\left(\frac{360t}{h} - \frac{1280t^2}{h^2} + \frac{1515t^3}{h^3} - \frac{648t^4}{h^4} + \frac{90t^5}{h^5}\right) \\
\beta_2 &= -\frac{t}{4800}\left(\frac{36t}{h} - \frac{140t^2}{h^2} + \frac{195t^3}{h^3} - \frac{108t^4}{h^4} + \frac{18t^5}{h^5}\right) \\
\beta_3 &= \frac{t}{6720}\left(\frac{40t}{h} - \frac{160t^2}{h^2} + \frac{225t^3}{h^3} - \frac{144t^4}{h^4} + \frac{30t^5}{h^5}\right)
\end{aligned}\right\} \quad (8)$$

Substituting (8) into (7) and evaluating at $x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}, x_{n+1}, x_{n+2}, x_{n+3}$ respectively, we have:

$$y_{n+\frac{1}{3}} = y_n + h\left(\frac{6763}{58320}f_n + \frac{863}{2880}f_{n+\frac{1}{3}} - \frac{1177}{10080}f_{n+\frac{2}{3}} + \frac{709}{19440}f_{n+1} - \frac{13}{7776}f_{n+2} + \frac{211}{1632960}f_{n+3}\right) \quad (9)$$

$$y_{n+\frac{2}{3}} = y_n + h\left(\frac{389}{3645}f_n + \frac{83}{180}f_{n+\frac{1}{3}} + \frac{4}{45}f_{n+\frac{2}{3}} + \frac{13}{1215}f_{n+1} - \frac{1}{1218}f_{n+2} + \frac{83}{180}f_n\right) \quad (10)$$

$$y_{n+1} = y_n + h\left(\frac{9}{80}f_n + \frac{27}{64}f_{n+\frac{1}{3}} + \frac{351}{1120}f_{n+\frac{2}{3}} + \frac{37}{240}f_{n+1} - \frac{1}{480}f_{n+2} + \frac{1}{6720}f_n\right) (11)$$

$$y_{n+2} = y_n - h\left(\frac{1}{15}f_n - \frac{27}{20}f_{n+\frac{1}{3}} + \frac{54}{35}f_{n+\frac{2}{3}} - \frac{29}{15}f_{n+1} - \frac{1}{3}f_{n+2} + \frac{1}{140}f_{n+3}\right) (12)$$

$$y_{n+3} = y_n + h\left(\frac{57}{80}f_n - \frac{729}{320}f_{n+\frac{1}{3}} + \frac{729}{160}f_{n+\frac{2}{3}} - \frac{153}{80}f_{n+1} + \frac{261}{160}f_{n+2} + \frac{93}{320}f_n\right) (13)$$

3.0 Basic Properties of the Developed Method

3.1 Order of the Block

Let the linear operator $L\{y(x):h\}$ on (2) be:

$$L\{y(x):h\} = \sum_{j=0}^{k}\alpha_j y(x+jh) - h\sum_{j=0}^{k}\beta_j y'(x+jh) + h\beta_v y'(x+jh) \quad (14)$$

where the function $y(x)$ is assumed to have continuous derivatives of sufficiently high order (Lambert, 1973). Expanding (14) in Taylor series about the point x gives the expression:

$$L\{y(x):h\} = C_0 y(x) + C_1 y'(x) + \ldots + C_p y^p(x) + C_{p+1} y^{p+1}(x) + \ldots (15)$$

Definition: The linear operator L and associated block method are said to be of order p if $C_0 = C_1 = C_2 = \ldots = C_p = 0$, $C_{p+1} \neq 0$ *is called the error constant and implies that the truncation error is given by* $t_{n+k} = C_{p+1}h^{p+1}y^{p+1}(x) + O(h^{p+2})$.

Comparing the coefficient of h, the order ofthe method is six with error constant

$$C_6 = \left[-\frac{281}{14696640}, -\frac{31}{2755620}, -\frac{11}{544320}, \frac{17}{34020}, -\frac{13}{4032}\right]$$

3.2 Consistency

From (15),

$$\left.\begin{aligned} C_0 &= \sum_{j=0}^{k} \alpha_j = 0 \\ C_1 &= \sum_{j=0}^{k} \left(j\alpha_j - \beta_j \right) = 0 \end{aligned}\right\} \quad (16)$$

Since $C_0 = C_1 = 0$, then the method is consistent.

3.3 Zero Stability

The linear multistep method (2) is said to be zero stable if the zeros of the first characteristic polynomial are such that none is larger than one in magnitude and any zero equal to one in magnitude is simple (that is, not repeated).

For the developed method, we have for schemes (9)–(13), we have

$$\left.\begin{aligned} \rho(z) &= z^{\frac{1}{3}} - 1 \Rightarrow z = 1 \\ \rho(z) &= z^{\frac{1}{2}} - 1 \Rightarrow z = 1 \\ \rho(z) &= z - 1 \Rightarrow z = 1 \\ \rho(z) &= z^2 - 1 \Rightarrow z = 1, z - 1 \\ \rho(z) &= z^3 - 1 \Rightarrow z = 1 \end{aligned}\right\} \quad (17)$$

Equation (17) shows that the schemes (9)-(13) are zero stable.

3.4 Convergence

The schemes are convergent sine they are consistent and zero stable.

4.0 Numerical Examples

Problem 1:

$y' = x + y, \quad y(0) = 1, \quad h = 0.01$

Exact solution: $y(x) = 2e^x - (x + 1)$

Problem 2:

$$y' = -y + 20, \quad y(0) = 1, \quad h = 0.01$$

Exact solution: $y(x) = 5 - 4e^{-4x}$

Table 4.1: Comparison of result of Problem 1with exact solution.

x	Exact Result	Computed Result	Error
0.01	1.010100334168340	1.010100334168330	1.0 E−14
0.02	1.020402680053510	1.020402680053510	0.0 E+00
0.03	1.030909067907030	1.030909067907030	0.0 E+00
0.04	1.041621548384780	1.041621548384770	1.0 E−14
0.05	1.052542192752050	1.052542192752040	1.0 E−14
0.06	1.063673093090720	1.063673093090710	1.0 E−14
0.07	1.075016362508430	1.075016362508430	0.0 E+00
0.08	1.086574135349920	1.086574135349910	1.0 E−14
0.09	1.098348567410420	1.098348567410420	0.0 E+00
0.1	1.110341836151300	1.110341836151290	1.0 E−14

Table 4.2: Comparison of result of Problem 2 with exact solution.

x	Exact Result	Computed Result	Error
0.01	1.00025002500166	1.00025002500166	0.0 E+00
0.02	1.00100040010668	1.00100040010668	0.0 E+00
0.03	1.00225202621554	1.00225202621556	2.0 E−14
0.04	1.00400640683213	1.00400640683214	1.0 E−14
0.05	1.00626565107425	1.00626565107425	0.0 E+00

0.06	1.00903247790016	1.00903247790023	7.0 E−14
0.07	1.01231022156300	1.01231022156307	7.0 E−14
0.08	1.01610283830835	1.01610283830840	5.0 E−14
0.09	1.02041491433388	1.02041491433402	1.4 E−14
0.1	1.02525167503344	1.02525167503359	1.5 E−14

5.0 Conclusion

In this paper, we have developed a three step hybrid linear multistep method for solving first order differential equation. The method was found to be zero stable, consistent and hence convergent. Numerical examples are used to show that our new method favorable performs well in comparison with the exact solution.

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