A06: Hybrid Block Method for Direct Solution of Non-Linear Second Order Initial Value Problems in Ordinary Differential Equations

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Abstract:
This article presents a linear multistep method using both grid and off-grid points of 3 steps using power series as basis function for finding approximate solution to general second order ordinary differential equations. The method of collocation and interpolation is applied at the various points to obtain a new continuous hybrid block method. The basic properties are established and the method found to be convergent. Some nonlinear problems were solved using the new method and the numerical results proved the efficiency of the method.

Keywords: Collocation, Consistency, Convergence, Hybrid block method, Initial Value, Interpolation, Non-linear, Problems, Stability

1. Introduction

The need for the exact solution of non-linear differential equations encountered in various discipline of applied sciences including the theory of heat and mass transfer, nonlinear mechanics, combustion theory, engineering, etc. plays a vital role in understanding the qualitative features of these phenomena and processes. However, a few of these equations have known exact solutions. As a result several methods like semi-analytic, asymptotic or numerical methods are sought to obtain approximate solution to the problems appearing in real-world (Duarte and Mota, 1997; Butcher, 2008; Polyanin and Zaitsev, 2003).
In this paper, a general non-linear second order ordinary differential equation of the form (1) below will be addressed;

$$\frac{d^2y}{dx^2} = f\left(x, y(x), \frac{dy}{dx}\right), \qquad y(x_0) = A, \quad \frac{dy(x_0)}{dx} = B \tag{1}$$

The conventional method of solving (1) is to first reduce the differential equation into system of first order ordinary differential equations which has serious drawbacks such as wastage of human and computer time due to complicated computational work and lengthy execution time. In recent years, many researchers have used various approaches like symmetry method (Duarte and Mota, 1997), implicit method by (Hasan *et. al.*, 2014), integrator block off-grid points collocation method (Studies, 2019), Adams type hybrid block methods associated with Chebyshev polynomials (Badmus *et al.*, 2015), and numerical integration in handling this kind of problem. Other researchers have also proffer solution to (1) using Linear Multistep Method (LMM) which are self-starting including; (Badmus *et al.*, 2015; Ehigie *et al.*, 2011; Majid *et al.*, 2012; Okedayo *et al*, 2018; Ukpebor, 2019) and proposed methods used one or combined basis functions such as power series, Chebyshev, Legendre and Laguerre polynomials.

In this paper, a hybrid block LMM using both grid and off-grid points at collocation will be developed for the direct solution of general nonlinear (1) choosing power series as basis function. Some basic properties like order and error constants, consistency, stability and convergence of the method will be discussed and numerical examples with non-linear general second order ordinary differential equations will be used to validate the method.
Development of the method will be discussed in the next section; basic properties of the method will be fully addressed in section 3; numerical experiments will be performed in section 4 and section 5 will discuss the result and conclusion will follow.

2. Development of the Method

In the development of the hybrid-block method, power series of the form (2) is considered as basis function;

$$y(x)=\sum_{n=0}^{c+i-1} a_j x^j \qquad (2)$$

Where $a_j \in \square$ are unknown to be determined, $y \in C^m$, c are the collocation points, and i are the interpolating point, to obtain the approximate solution of (1), the first and second derivatives of (2) are;

$$\left.\begin{aligned} y'(x) &= \sum_{n=0}^{c+i-1} a_j j x^{j-1} \\ y''(x) &= \sum_{n=0}^{c+i-1} a_j j(j-1)x^{j-2} \end{aligned}\right\} \qquad (3)$$

Therefore; (1) becomes;

$$\sum_{n=0}^{c+i-1} a_j j(j-1)x^{j-2} = f(x,y,y') \qquad (4)$$

Discretizing (2) and (4) gives;

$$\sum_{n=0}^{c+i-1} a_j x_n^j = y_n \qquad (5)$$

$$\sum_{n=0}^{c+i-1} a_j j(j-1)x_n^{j-2} = f_n \qquad (6)$$

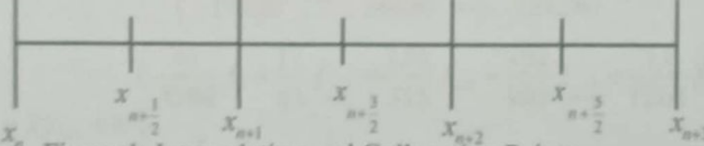

Figure 1. Interpolating and Collocating Points

Interpolating (2) at $x = x_{n+j}; j = 0,1$ and Collocating (4) at $x = x_{n+j}; j = 0,\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},3$ as illustrated above gives the system of nonlinear equations in matrix form;

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x_{n+\frac{1}{2}}^2 & 20x_{n+\frac{1}{2}}^3 & 30x_{n+\frac{1}{2}}^4 & 42x_{n+\frac{1}{2}}^5 & 56x_{n+\frac{1}{2}}^6 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 \\ 0 & 0 & 2 & 6x_{n+\frac{3}{2}} & 12x_{n+\frac{3}{2}}^2 & 20x_{n+\frac{3}{2}}^3 & 30x_{n+\frac{3}{2}}^4 & 42x_{n+\frac{3}{2}}^5 & 56x_{n+\frac{3}{2}}^6 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 \\ 0 & 0 & 2 & 6x_{n+\frac{5}{2}} & 12x_{n+\frac{5}{2}}^2 & 20x_{n+\frac{5}{2}}^3 & 30x_{n+\frac{5}{2}}^4 & 42x_{n+\frac{5}{2}}^5 & 56x_{n+\frac{5}{2}}^6 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \end{pmatrix} \quad (7)$$

Solving the system of nonlinear equations above using Maple 2017 software package and substituting the results of $a_j \in \mathbb{R}$ into (2) produces a continuous hybrid-block linear multistep method of the form:

$$y(x) = \alpha_0 y_n + \alpha_1 y_{n+1} + \beta_0 f_n + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_1 f_{n+1} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2} + \beta_{\frac{5}{2}} f_{n+\frac{5}{2}} + \beta_3 f_{n+3} \quad (8)$$

Evaluating (8) at $x = x_{n+j}, j = \frac{1}{2}, \frac{3}{2}, 2, \frac{5}{2}, 3$ yields the following discrete schemes:

$$y_{n+\frac{1}{2}} = \frac{1}{2}y_n + \frac{1}{2}y_{n+1} + h^2\begin{pmatrix} -\frac{863}{96768}f_n - \frac{8999}{80640}f_{n+\frac{1}{2}} + \frac{769}{161280}f_{n+1} - \frac{1987}{120960}f_{n+\frac{3}{2}} \\ +\frac{1609}{161280}f_{n+2} - \frac{263}{80640}f_{n+\frac{5}{2}} + \frac{221}{483840}f_{n+3} \end{pmatrix} \quad (9)$$

$$y_{n+\frac{3}{2}} = -\frac{1}{2}y_n + \frac{3}{2}y_{n+1} + h^2\begin{pmatrix} \frac{1291}{161280}f_n + \frac{12117}{8960}f_{n+\frac{1}{2}} + \frac{10711}{53760}f_{n+1} + \frac{1567}{40320}f_{n+\frac{3}{2}} \\ -\frac{163}{17920}f_{n+2} + \frac{67}{26880}f_{n+\frac{5}{2}} - \frac{53}{161280}f_{n+3} \end{pmatrix} \quad (10)$$

$$y_{n+2} = -y_n + 2y_{n+1} + h^2\begin{pmatrix} \frac{61}{3780}f_n + \frac{17}{63}f_{n+\frac{1}{2}} + \frac{134}{315}f_{n+1} + \frac{262}{945}f_{n+\frac{3}{2}} + \frac{11}{1260}f_{n+2} \\ +\frac{1}{315}f_{n+\frac{5}{2}} - \frac{1}{1890}f_{n+3} \end{pmatrix} \quad (11)$$

$$y_{n+\frac{5}{2}} = -\frac{3}{2}y_n + \frac{5}{2}y_{n+1} + h^2\begin{pmatrix} \frac{787}{32256}f_n + \frac{2167}{5376}f_{n+\frac{1}{2}} + \frac{7015}{10752}f_{n+1} + \frac{4339}{8064}f_{n+\frac{3}{2}} \\ +\frac{2479}{10752}f_{n+2} + \frac{151}{5376}f_{n+\frac{5}{2}} - \frac{53}{32256}f_{n+3} \end{pmatrix} \quad (12)$$

$$y_{n+3}=-2y_n+3y_{n+1}+h^2\begin{pmatrix}\frac{2}{63}f_n+\frac{19}{35}f_{n+\frac{1}{2}}+\frac{361}{420}f_{n+1}+\frac{262}{315}f_{n+\frac{3}{2}}\\+\frac{31}{70}f_{n+2}+\frac{29}{105}f_{n+\frac{5}{2}}+\frac{19}{1260}f_{n+3}\end{pmatrix} \quad (13)$$

Differentiating (8) and evaluating at all the collocation points taking $y'(x)=z(x)$ yields;

$$z_n=-y_n+y_{n+1}-\frac{1}{7560}h\begin{pmatrix}1027f_n+3492f_{n+\frac{1}{2}}-1680f_{n+1}+1576f_{n+\frac{3}{2}}\\-873f_{n+2}+276f_{n+\frac{5}{2}}-38f_{n+3}\end{pmatrix} \quad (15)$$

$$z_{n+\frac{1}{2}}=-y_n+y_{n+1}-\frac{1}{40320}h\begin{pmatrix}885f_n+3080f_{n+\frac{1}{2}}-6527f_{n+1}+4096f_{n+\frac{3}{2}}\\-2081f_{n+2}+632f_{n+\frac{5}{2}}-85f_{n+3}\end{pmatrix} \quad (16)$$

$$z_{n+1}=-y_n+y_{n+1}-\frac{1}{7560}h\begin{pmatrix}112f_n+2148f_{n+\frac{1}{2}}+1713f_{n+1}-248f_{n+\frac{3}{2}}\\+66f_{n+2}-12f_{n+\frac{5}{2}}+f_{n+3}\end{pmatrix} \quad (17)$$

$$z_{n+\frac{3}{2}}=-y_n+y_{n+1}-\frac{1}{120960}h\begin{pmatrix}2063f_n+31608f_{n+\frac{1}{2}}+58227f_{n+1}+33536f_{n+\frac{3}{2}}\\-5715f_{n+2}+1416f_{n+\frac{5}{2}}-175f_{n+3}\end{pmatrix} \quad (18)$$

$$z_{n+2}=-y_n+y_{n+1}-\frac{1}{2520}h\begin{pmatrix}39f_n+692f_{n+\frac{1}{2}}+1072f_{n+1}+1480f_{n+\frac{3}{2}}\\+523f_{n+2}-28f_{n+\frac{5}{2}}+2f_{n+3}\end{pmatrix} \quad (19)$$

$$z_{n+\frac{5}{2}}=-y_n+y_{n+1}-\frac{1}{120960}h\begin{pmatrix}2143f_n+31128f_{n+\frac{1}{2}}+58755f_{n+1}+54784f_{n+\frac{3}{2}}\\+72093f_{n+2}+23784f_{n+\frac{5}{2}}-767f_{n+3}\end{pmatrix} \quad (20)$$

$$z_{n+3}=-y_n+y_{n+1}-\frac{1}{7560}h\begin{pmatrix}80f_n+2340f_{n+\frac{1}{2}}+2409f_{n+1}+5768f_{n+\frac{3}{2}}\\+1602f_{n+2}+5556f_{n+\frac{5}{2}}+1145f_{n+3}\end{pmatrix} \quad (21)$$

Combined (9) to (21) give the hybrid block method for solving general second order ODE.

3. Analysis of the Properties of the Hybrid Block Method

This section seeks to analyze some basic properties of the developed method by obtaining the orders of the method, error constants, consistency, stability and convergence.

3.1 Order and Error Constant

Following (Lambert, 1973) with the proposed methods developed in section 2 of this paper, we associate a linear difference operator ψ by;

$$\psi\{y(x),h\}=\sum_{j=0}^{k}\{a_j y(x+jh)-h^2 b_j y''(x+jh)\} \quad (22)$$

where $y(x)$ is an arbitrary function continuously differentiable on $[a,b]$ and α_j, β_j are non-zero. Assuming the operator ψ operates on $y(x)$ allowing it to have as many higher derivatives as needed. Expanding the test function $y(x+jh)$ and $y''(x+jh)$ as Taylor's series about x and collecting the like terms in equation (22) produces;

$$\psi\{y(x),h\} = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + \ldots + c_k h^k y^{(k)}(x) + \ldots \tag{23}$$

where c_k are constants.

Definition 1: The difference operator (22) with the associated methods are said to be of order p if in (23); $c_0 = c_1 = c_2 = \ldots = c_p = c_{p+1} = 0$, $c_{p+2} \neq 0$ and c_{p+2} is the error constant.

Consider equation (9), to obtain the order and error constants as discussed above yield;

$$c_0 = 1 - \frac{1}{2} - \frac{1}{2} = 0$$

$$c_1 = \frac{1}{2} - \frac{1}{2} = 0$$

$$c_2 = \frac{1}{8} - \frac{1}{4} + \frac{863}{96768} + \frac{8999}{80640} - \frac{769}{161280} + \frac{1987}{120960} - \frac{1609}{161280} + \frac{263}{80640} - \frac{221}{483840} = 0$$

$$c_3 = \frac{1}{48} - \frac{1}{12} + \frac{8999}{161280} - \frac{769}{161280} + \frac{1987}{80640} - \frac{1609}{80640} + \frac{263}{32256} - \frac{221}{161280} = 0$$

$$c_4 = \frac{1}{384} - \frac{1}{48} + \frac{8999}{645120} - \frac{769}{322560} + \frac{1987}{107520} - \frac{1609}{80640} + \frac{1315}{129024} - \frac{221}{107520} = 0$$

$$c_5 = \frac{1}{3840} - \frac{1}{240} + \frac{8999}{3870720} - \frac{769}{967680} + \frac{1987}{215040} - \frac{1609}{120960} + \frac{6575}{774144} - \frac{221}{107520} = 0$$

$$c_6 = \frac{1}{46080} - \frac{1}{1440} + \frac{8999}{30965760} - \frac{769}{3870720} + \frac{1987}{573440} - \frac{1609}{241920} + \frac{1315}{6193152} - \frac{221}{143360} = 0$$

$$c_7 = \frac{1}{645120} - \frac{1}{10080} + \frac{8999}{309657600} - \frac{769}{19353600} + \frac{5961}{5734400} - \frac{1609}{604800} + \frac{32875}{12386304} - \frac{663}{716800} = 0$$

$$c_8 = \frac{1}{10321920} - \frac{1}{80640} + \frac{8999}{3715891200} - \frac{769}{116121600} + \frac{5961}{22937600} - \frac{1609}{1814400} + \frac{164375}{148635648} - \frac{663}{1433600} = 0$$

$$c_9 = \frac{1}{185794560} - \frac{1}{725760} + \frac{8999}{52022476800} - \frac{769}{812851200} + \frac{17883}{321126400} - \frac{1609}{6350400} + \frac{821875}{208899072} - \frac{1989}{10035200} = -\frac{19}{6193152}$$

Observe that $c_0 = c_1 = c_2 = \ldots = c_8 = 0$, $c_9 \neq 0$. Therefore, the method is of order 7 and $c_{p+2} = -\frac{19}{6193152}$ is the error constant. Following the same approach, we obtain the orders

$\{7,7,7,7\}$ and error constants $\left\{\frac{1}{483840},\frac{1}{241920},\frac{223}{30965760},\frac{1}{241920}\right\}$ of the other methods from (10) to (13).

3.2 Consistency of the Method

Definition 2: A linear multistep is said to be consistent if its order $p \geq 1$ and it follows that;

i.) $\sum_{j=0}^{k} a_j = 0; \quad j = 0,1,2,3.$

ii.) $\rho'(1) = 0$

iii.) $2!\sigma(1) = \rho''(1)$

where $\rho(r)$ and $\sigma(r)$ are the first and second characteristic polynomials respectively.

From section 3.1, the order $p = 7$, then, the first part of the definition is satisfied. Also, for the scheme in equation (9),

$\sum_{j=0}^{k} a_j = a_0 + a_{\frac{1}{2}} + a_1 + a_{\frac{3}{2}} + a_2 + a_{\frac{5}{2}} + a_3 = \frac{1}{2} - 1 + \frac{1}{2} + 0 + 0 + 0 = 0$, which prove the first condition.

Also, the first characteristic polynomials; $\rho(r) = \frac{1}{2} - r^{\frac{1}{2}} + \frac{1}{2}r$

Then,

$$\rho(1) = \frac{1}{2} - 1^{\frac{1}{2}} + \frac{1}{2}(1) = 0$$

which satisfied the second condition.

The second polynomial,

$$\sigma(r) = \frac{863}{96768} + \frac{8999}{80640}r^{\frac{1}{2}} - \frac{769}{161280}r + \frac{1987}{120960}r^{\frac{3}{2}} - \frac{1609}{161280}r^2 + \frac{263}{80640}r^{\frac{5}{2}} - \frac{221}{483840}r^3$$

$\sigma(1) = \frac{1}{8}$

$\rho''(r) = \frac{1}{4}r^{-\frac{3}{2}}$; $\qquad \rho''(1) = \frac{1}{4}$

Then,

$2!\sigma(1) = 2 \times \frac{1}{8} = \frac{1}{4} = \rho(1)$, which satisfied the third condition.

Hence, the method is consistent.

3.3 Stability

Definition 3: (Lambert, 1973): A linear multistep method is said to be zero-stable if no root of the first characteristic polynomial has modulus greater than one, and if every root with modulus one is simple. i.e. $|r| \leq 1$.

Then;

$\frac{1}{2} - r^{\frac{1}{2}} + \frac{1}{2}r = 0$

which gives;

$|r| \leq 1.$

Therefore, the method is zero-stable.

3.4 Convergence

Theorem (Lambert, 1973): The necessary and sufficient conditions for a linear multistep method to be convergent are that it be consistent and zero-stable.

4. Numerical Implementation

In this section, the efficiency of the new method is tested with three non-linear problems. The nonlinear differential equations appear in mathematical modeling of oscillators and seismology. The performance is tabulated in the tables below;

Problem 1:

$$\frac{d^2}{dx^2}y(x) = \varepsilon y(x)\left(\frac{dy(x)}{dx}\right)^2 \quad :$$

$$y(0)=1, \quad y'(0)=-1, \quad h=0.1, \quad \varepsilon=\frac{1}{1000}$$

Exact Solution: $y(x) = 20\sqrt{5}\, RootOf\left(50 erf(\_Z)\sqrt{\pi}e^{\frac{1}{2000}} - 50 erf\left(\frac{\sqrt{5}}{100}\right)\sqrt{\pi}e^{\frac{1}{2000}} + \sqrt{5}x\right)$

Problem 2:

$$\frac{d^2}{dx^2}y(x) = \frac{1}{x+y(x)}\left(2\frac{d}{dx}y(x)+1\right)\frac{d}{dx}y(x) \quad :$$

$$y(0)=1, \quad y'(0)=-1, \quad h=0.1$$

Exact Solution: $y(x) = \frac{2-x^2}{2(x+1)}$

Problem 3:

$$\frac{d^2}{dx^2}y(x) = -x\left(\frac{d}{dx}y(x)\right)^2 \quad :$$

$$y(0)=0, \quad y'(0)=1, \quad h=0.1$$

Exact Solution: $y(x) = \sqrt{2}\arctan\left(\frac{\sqrt{2}}{2}x\right)$

5.0 Discussion of Results and Conclusion

5.1 Discussion of Results

Table 4.1: Exact and Numerical Results of Problem 1

n	x_n	y_n (Exact)	y_n (Computed)	Error
0	0.0	1.00000000000000000000	1.00000000000000000000	0.00
1	0.1	0.90000483150234666072 36	0.90000483150234666072 28	$2.00\times10^{[illegible]}$
2	0.2	0.80001366644482882400	0.80001366644482882400	$9.00\times10^{[illegible]}$
3	0.3	0.70006649122196635219	0.70006649122196635205	$5.00\times10^{[illegible]}$
4	0.4	0.60000911887248851276	0.60000911887248851270	$6.00\times10^{[illegible]}$
5	0.5	0.50010416141245397470	0.50010416141245397462	$8.00\times10^{[illegible]}$
6	0.6	0.40014996127411381318	0.40014996127411381314	$4.00\times10^{[illegible]}$
7	0.7	0.30018777923935526284	0.30018777923935526277	$7.00\times10^{[illegible]}$
8	0.8	0.20023459636074145034	0.20023459636074145026	$8.00\times10^{[illegible]}$
9	0.9	0.10028341393660784297	0.10028341393660784286	$1.10\times10^{[illegible]}$
10	1.0	0.00033323335119432898	0.00033323335119432877	$1.27\times10^{[illegible]}$

Table 4.2: Exact and Numerical Results of Problem 2

n	x_n	y_n (Analytical Solution)	y_n (Computed Solution)	Error
0	0.0	1.00000000000000000000	1.00000000000000000000	0.00
1	0.1	0.90454545454545454545	0.904545454606201553227	1.59×10^{-9}
2	0.2	0.81666666666666666667	0.81666669669900823243	2.86×10^{-9}
3	0.3	0.73461538461538461538	0.7346153988260000540913	3.66×10^{-9}
4	0.4	0.65714285714285714286	0.657142860038505438185	3.24×10^{-9}
5	0.5	0.58333333333333333333	0.583333339625011896913	2.92×10^{-9}
6	0.6	0.51250000000000000000	0.51250000261895603954	2.62×10^{-9}
7	0.7	0.44411764705882352941	0.44411764931800066808	2.26×10^{-9}
8	0.8	0.37777777777777777778	0.37777777972561134430	1.95×10^{-9}
9	0.9	0.31315789473684210526	0.31315789640803513692	1.67×10^{-9}
10	1.0	0.25000000000000000000	0.25000000141137523406	1.41×10^{-9}

Table 3: Result of Problem 3

n	x_n	y_n (Analytical Solution)	y_n (Computed Solution)	Error
0	0.0	0.00000000000000000000	0.00000000000000000000	0.00
1	0.1	0.099833831554535202646	0.099833831504256147977	5.03×10^{-11}
2	0.2	0.19868244159357967721	0.19868244148748173312	1.06×10^{-10}
3	0.3	0.29561772648220914426	0.29561772632702687133	1.55×10^{-10}
4	0.4	0.38981778502212646891	0.38981778490141945207	1.21×10^{-10}
5	0.5	0.48060196634497672857	0.48060196626176618838	8.32×10^{-11}
6	0.6	0.56744914996438796273	0.56744914990920227261	5.52×10^{-11}
7	0.7	0.64999989137787509177	0.64999989131296949661	6.49×10^{-11}
8	0.8	0.72804556372087131920	0.72804556364860842756	7.23×10^{-11}
9	0.9	0.80150882570643781567	0.80150882562686068419	7.96×10^{-11}
10	1.0	0.87041975136710319748	0.87041975126617574675	1.01×10^{-10}

Table 1 – 3 present the numerical result of the new method and the comparison with the exact solution of the non-linear second order ordinary differential equations presented above. The new hybrid block method produces a very close approximate solution to the exact solution and this is apparent from the error produced.

5.2 Conclusion

In this paper, a new hybrid block method is developed through interpolation and collocation method for solving general second order initial value problems of ordinary differential equations. Some basic properties are also established which shows that the new method is convergent. Furthermore, some non-linear problems are tested with the new method and comparison made with the exact solution and error incurred. From the tabulated results, the new method produces favourable numerical results in terms of accuracy and rate of convergence. Therefore, the new method can be implored for solving all kinds of non-linear second order ordinary differential equations with exact or without exact solution.