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A Finite Element Technique to the Optimal Control of Two Dimensional Diffusion Equations

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Abstract
This paper dealt with the study of the optimal control $(u(x,t))$ and optimal state $(z(x,t))$ for two dimensional diffusion equations using the finite element method (FEM). The finite element method is a very powerful tool for getting numerical solution of a wide range of engineering problems. The optimal control profile u_e and optimal state profile z_e against the number of element e of the two dimensional diffusion solutions with varying space dimension n are obtained. It was discovered that the nodal temperature shows cooling effect from the left to the right side of the rod, and subsequently the output decreases. It is symmetric in nature.
Key words: Finite element, boundary-value problem, optimization problem, and diffusion equation.

Introduction

The related applications of optimization methods to equations of mathematical physics have become obvious in most literatures. FEM is a very powerful method for Heat Transfer analysis in the design stages like Engine Piston where the temperature distribution throughout the piston can be found and fins where the temperature distribution and heat transfer is found. More on the control of diffusion problem have also been solved by Ibiejugba (1980) via a Ritz penalty technique and Reju (1991).

Diffusion Model

The governing equation is given by

$$\frac{\partial z_n(x,y,t)}{\partial t} = \frac{\partial^2 z_n(x,y,t)}{\partial x^2} + u_n(x,y,t) \quad 1.1$$

where u is the control function, which is diffusion process with source.
Considering the problem as follows:

$$Min \;\; J(z,u) = \int_0^1\int_0^1\int_0^1 [z_n^2(x,y,t) + u_n^2(x,y,t)]dxdydt \quad 1.2a$$

Subject to

$$\frac{\partial z_n(x,y,t)}{\partial t} = \frac{\partial^2 z_n(x,y,t)}{\partial x^2} + u_n(x,y,t)$$

$$\left.\begin{array}{ll} z(x,0,t) = z(1,y,t) & 0 \le x \le 1 \\ z(0,y,t) = z(1,y,t) & 0 \le y \le 1 \\ z(x,y,0) = z(x,y,1) & 0 \le t \le 1 \end{array}\right\} \quad 1.2b$$

Laying emphasis on quadratic functional, we easily see that the functional can be written as:

$$\int_0^1\int_0^1\int_0^1 [z_n^2(x,y,t) + u_n^2(x,y,t)]dxdydt = \int_0^1\int_0^1\int_0^1 [u_n(x,y,t) + iz_n(x,y,t)][u_n(x,y,t) - iz_n(x,y,t)]dxdydt \quad 1.3$$

where $i = \sqrt{(-1)}$ is the complex unit.
Setting

$$\left.\begin{array}{l} w(x,t) = u(x,y,t) + z(x,y,t) \\ or\; w(x,t) = z(x,y,t) + iu(x,y,t) \end{array}\right\} \quad 1.4$$

we have the following equation

$$\int_0^1\int_0^1\int_0^1 [z^2(x,y,t) + u^2(x,y,t)]dxdydt = \int_0^1\int_0^1 [w(x,y,t)\vec{w}(x,y,t)]dxdt = \int_0^1\int_0^1\int_0^1 w^2(x,y,t)dxdydt \quad 1.5$$

$$u^2(x,y,t) + z^2(x,y,t) = w^2(x,y,t) \quad 1.6$$

Following the Pythagorean principle, equation (1.6) suggests that $u(x,y,t)$ and $z(x,y.t)$ are orthogonal planes in

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(x, y, t) while the set $\{u(x,y,t), z(x,y,t), w(x,y,t)\}$ forms a Pythagorean triple. The function $u(x,y,t)$ is a source (negative sink) term that adds heat to the unit rod at a rate $u(x,y,t)$ per unit time, per unit length. Thus, according to Boyce and Diprima (1977), we have

$u(x,y,t) = u(z,x,y,t) > 0$ 1.7

This shows that $u(x,y,t)$ also depends on $z(x,y,t)$. Since $u(x,y,t)$ and $z(x,y,t)$ are planes in (x,y,t), it is then easy to see that $u(x,y,t)$ and $z(x,y,t)$ are parabolic surfaces instead of curves.

We see that with $z^2(x,y,t) + u^2(x,y,t) = [z(x,y,t) + iu(x,y,t)][z(x,y,t) - iu(x,y,t)]$ the roots of cost integrand are the imaginary planes $\pm iu(x,y,t)$.

Methodology

Applying the Hamiltonian principle for this problem which is related to that of Singh and Titli we obtain

$$H = z^2(x,y,t) + u^2(x,y,t) + \lambda^T\left[\frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2} + u(x,y,t)\right] \quad 2.1$$

where

$\lambda^T = \lambda^T(t)$ 2.2

Setting

$$f(z,u) = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + u \quad and \quad g(z,u) = z^2 + u^2 \quad 2.3$$

We then have the first order necessary conditions for optimality as:

$$\frac{\partial z}{\partial t} = \frac{\partial H}{\partial \lambda} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + u = f(z,u) \quad 2.4$$

$$\frac{\partial \lambda}{\partial t} = \frac{\partial H}{\partial z} = -\left(\frac{\partial f}{\partial z}\right)^{\lambda^T} = -\frac{\partial g}{\partial z} = -2z(x,y,t) \quad 2.5$$

$$\frac{\partial H}{\partial u} = 0 \text{ or } \left(\frac{\partial f}{\partial u}\right)^{\lambda^T} + \frac{\partial g}{\partial u} = 0 \quad 2.6$$

where

$H = g(z,u) + \lambda^T f(z,u)$ 2.7

Equation (2.6) gives $\lambda + 2u = 0$ or $\lambda = -2u$ 2.8

By virtue of (2.5) and (2.6), we have

$$\frac{\partial \lambda}{\partial t} = -2\frac{\partial u}{\partial t} = -2z$$

$$\Rightarrow z(x,y,t) = \frac{\partial}{\partial t}u(x,y,t) \quad 2.9$$

Assuming that equation (2.6) admits the Fourier solution proposed and Duchateau and Zachmann (1986).

$$z(x,y,t) = \sum_{i=1}^{\infty} \alpha_i(t) sin n\pi ix sin n\pi iy \quad 2.10$$

$$u(x,y,t) = \sum_{i=1}^{\infty} u_i(t) sin n\pi ix sin n\pi iy \quad 2.11$$

We then have our new solution

$$z(x,y,t) = \frac{\partial}{\partial t}\sum_{i=1}^{\infty} u_i(t) sin n\pi ix sin n\pi iy = \sum_{i=1}^{\infty} u_{it}(t) sin n\pi ix sin n\pi iy \quad 2.12$$

Hence, it immediately follows that

$\alpha_i(t) = u_{it}(t)$ 2.13

and

$$z_t(x,y,t) = \sum_{i=1}^{\infty} u_{itt}(t) sin n\pi ix sin n\pi iy \quad 2.14$$

$$z_{tt}(x,y,t) = \sum_{i=1}^{\infty} u_{ittt}(t) sin n\pi ix sin n\pi iy \quad 2.15$$

$$z_{xx} = \sum_{i=1}^{\infty} i^2(-\pi^2)u_{it} sin n\pi ix sin n\pi iy \quad 2.16$$

$$z_{yy} = \sum_{i=1}^{\infty} i^2(-\pi^2)u_{it} sin n\pi ix sin n\pi iy \quad 2.17$$

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$$z(x,y,t) = \sum_{i=1}^{n} u_{it}(t)\sin n\pi x \sin n\pi y \quad 2.18$$

From the constrained equation it follows that

$$u_{itt}(t) = -u_{itt}(t) - 2i^2\pi^2 u_{it}(t) + u_i(t) \quad 2.19$$

The problem now becomes

$$\left[\int_0^1 (u_1^2 + u_2^2 + \dots\dots + u_n^2) + (u_{1t}^2 + u_{2t}^2 + \dots\dots + u_{nt}^2)\right]$$

Finite Element Formulation

The stepwise solution procedure is as follows.

STEP 1: Solution continuum discretization.

We discretize the domain $(0 \le t \le 1)$ with elements of equal length.

STEP 2: Interpolation model.

Consider a one-dimensional rod of length l. Let the nodes be donated by i and j and the nodal values of the field variable u by u_i and u_j. Let us assume our interpolation model for each element as:

$$u(t) = \alpha_1 + \alpha_2(t) \quad 3.1$$

That is, we assume that the interpolation model is linear where α_1 and α_2 are the unknown coefficients.

Using the nodal conditions

$$\left.\begin{aligned} u(t) &= u_i \quad at \quad t = t_i \\ u(t) &= u_j \quad at \quad t = t_j \end{aligned}\right\} \quad 3.2$$

From equation (3.1), we obtain

$$u_i = \alpha_1 + \alpha_2 t_i \quad 3.3$$

$$u_j = \alpha_1 + \alpha_2 t_i \quad 3.4$$

Solving (3.3) and (3.4) simultaneously, we have

$$\left.\begin{aligned} \alpha_1 &= \frac{u_i t_j - u_j t_i}{l} \\ \alpha_2 &= \frac{u_j - u_i}{l} \end{aligned}\right\} \quad 3.5$$

where t_i and t_j denote the global co-ordinate of nodes i and j respectively.

Substituting solution (3.5) into (3.1), we have

$$u(t) = \frac{(u_i t_j - u_j t_i)}{l} + \frac{(u_j - u_i)t}{l}$$

$$= \frac{(t_j - t)u_i}{l} + \frac{(t - t_i)u_j}{l}$$

$$= \emptyset_i(t)u_i + \emptyset_j(t)u_j$$

$$= [\emptyset(t)]\vec{u}^{(e)} \quad 3.6$$

where $\emptyset_i(t) = \frac{(t_j - t)}{l}$, $\emptyset_j(t) = \frac{(t - t_i)}{l}$ and $\vec{u}^{(e)} = \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$

STEP 3: Element Characteristic Matrices and Vectors

The necessary conditions for optimality lead to the functional

$$J = \frac{1}{2}\int\left[-\left(\frac{\partial u_n}{\partial t}\right)^2 + 4\pi^2 n^2 u_n - u_n^2\right]dt \quad 3.7$$

The element characteristics matrices and vectors are obtained by expressing the functional J in matrix form as follows:

Evaluating the functional J over the length of element e, we obtain

$$J^{(e)} = \frac{1}{2}\int_0^{(e)}\left[-\left(\frac{\partial u_n}{\partial t}\right)^2 + 4\pi^2 n^2 - u_n^2\right]dt \quad 3.8$$

The functional J is expressed as the sum of E elemental quantities $J^{(e)}$ as

$$J = \sum_{e=1}^{E} J^{(e)} \quad 3.9$$

substituting (3.6) in (3.8) we obtain

$$J^{(e)} = \frac{1}{2}\int_{t_i}^{t_j}\left[-\vec{u}_n^{(e)T}\left[\frac{\partial\emptyset}{\partial t}\right]^T\left[\frac{\partial\emptyset}{\partial t}\right]\vec{u}_n^{(e)} - \vec{u}_n^{(e)T}[\emptyset]^T[\emptyset]\vec{u}_n^{(e)} + 4\pi^2 n^2[\emptyset]\vec{u}_n^{(e)}\right] \quad 3.10$$

Using the equivalent minimization problem (2.19) subject to (2.10), the conditions for minimum are;

$$\frac{\partial J}{\partial u_i} = \frac{\partial}{\partial u_i}\sum_{e=1}^{E} J^{(e)}$$

$$= \sum_{e=1}^{E}\frac{\partial J^{(e)}}{\partial u_i} = 0 \qquad 3.11$$

$i = 1,2 \ldots \ldots \ldots m,$
where E is the number of nodal degrees of freedom.
Equation (3.11) can also be expressed as;

$$\sum_{e=1}^{E}\frac{\partial J^{(e)}}{\partial u^{(e)}} = \vec{0} \qquad 3.12$$

$$i.e \quad \sum_{e=1}^{E}\int_{t_i}^{t_j}\left[-\left[\frac{\partial \emptyset}{\partial t}\right]^T\left[\frac{\partial \emptyset}{\partial t}\right]u_n^{(e)} - [\emptyset]^T[\emptyset]\vec{u}_n^{(e)} + 4\pi^2 n^2[\emptyset]^T\right]dt = \vec{0} \qquad 3.13$$

$$i.e \quad \sum_{e=1}^{E}[k^{(e)}]u_n^{(e)} = \sum_{e=1}^{E}\vec{p}^{(e)} \qquad 3.14$$

J has been replaced by the sum of the element functional $J^{(e)}$,
e = 1, 2… m in the assembling.
Equation (3.11) is stated in a linear matrix form as;

$$\sum_{e=1}^{m}\frac{\partial J^{(e)}}{\partial u^{(e)}} = \sum_{e=1}^{m}\left[k^{(e)}u_n^{(e)} - p^{(e)}\right] = 0 \qquad 3.15$$

This is necessary to form our governing finite element equations, where
$[k^{(e)}]$ = element characteristics matrix

$$= \int_{t_i}^{t_j}\left[\left[\frac{\partial \emptyset}{\partial t}\right]^T\left[\frac{\partial \emptyset}{\partial t}\right] + [\emptyset]^T[\emptyset]\right]dt \qquad 3.16$$

$p^{(e)}$ = element characteristic vector

$$= \int_{t_i}^{t_j}\pi^2 n^2[\emptyset]^T\,dt \qquad 3.17$$

$\vec{u}^{(e)}$, the assembled nodal temperature vector is given as;

$$= \begin{Bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_m \end{Bmatrix} \qquad 3.18$$

substituting
$[\emptyset(t)] = [\emptyset_i(t) \quad \emptyset_j(t)]$

$$= \left[\frac{t_{j-t}}{l^{(e)}} \quad \frac{t-t_i}{l^{(e)}}\right]$$

into (3.16) and (3.17) we have;

$$[k^{(e)}] = \int_{t_i}^{t_j}\left[\begin{Bmatrix}\frac{-1}{l^{(e)}} \\ \frac{1}{l^{(e)}}\end{Bmatrix}\left\{\frac{-1}{l^{(e)}} \quad \frac{1}{l^{(e)}}\right\} + \begin{Bmatrix}\frac{t_{j-t}}{l^{(e)}} \\ \frac{t-t_i}{l^{(e)}}\end{Bmatrix}\left\{\frac{t_{j-t}}{l^{(e)}} \quad \frac{t-t_i}{l^{(e)}}\right\}\right]$$

$$= \frac{1}{l^{(e)}}\begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix} + \frac{l^{(e)}}{6}\begin{bmatrix}2 & 1 \\ 1 & 2\end{bmatrix} \qquad 3.19$$

$$\vec{p}^{(e)} = \pi^2 n^2\int_{t_i}^{t_j}\begin{Bmatrix}\frac{t_j-t}{l^{(e)}} \\ \frac{t-t_i}{l^{(e)}}\end{Bmatrix}dt$$

$$= \frac{\pi^2 n^2}{2}\begin{Bmatrix}t_j - t_i \\ t_j - t_i\end{Bmatrix} \qquad 3.20$$

STEP 4: Assembly of element matrices and vectors and derivation of governing equations.
The element characteristic matrices and vectors are assembled and the overall equations are obtained as:

$$[k]\vec{u} = \vec{p} \qquad 3.21$$

STEP 5: Solution.
The system equation (3.21) is solved after incorporating the boundary value conditions.

4. Results and Conclusion

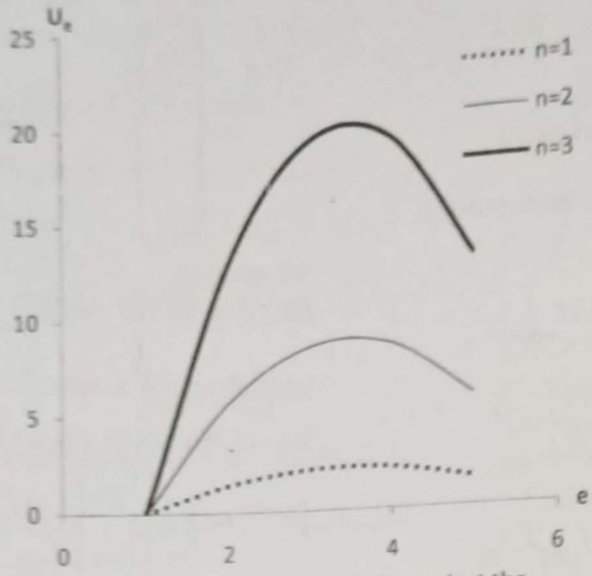

Fig. 1 Optimal Control Profile U_e against the number of element e for various values of n when E = 5

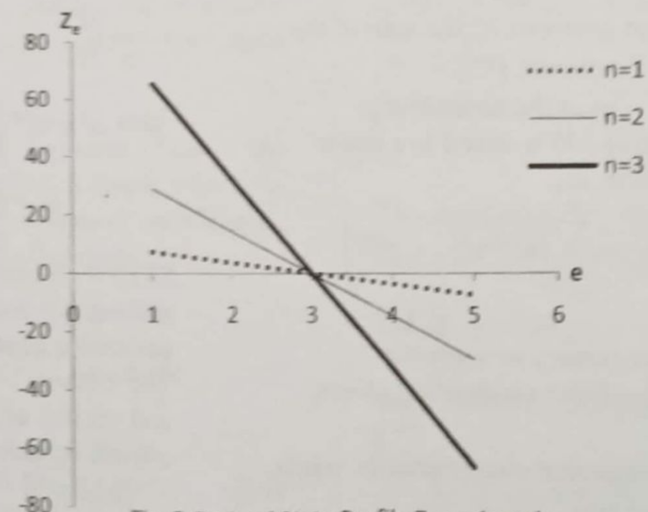

Fig. 2 Optimal State Profile Z_e against the number of element e for various values of n when E= 5

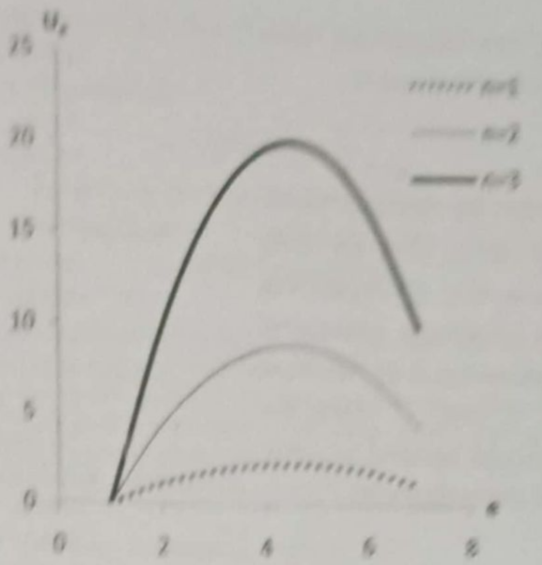

Fig. 3 Optimal Control Profile U_e against the number of element e for various values of n when E = 7

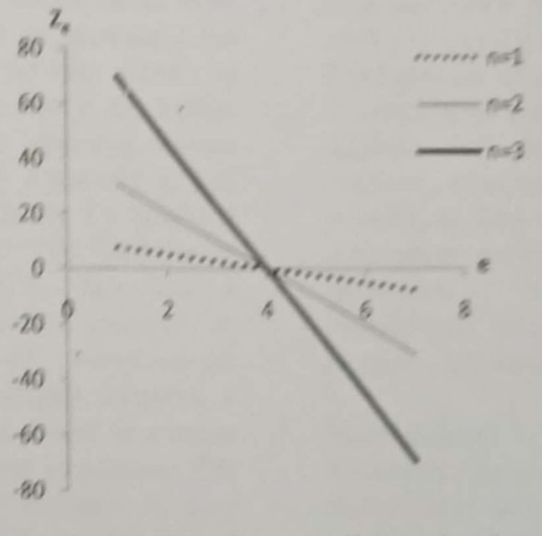

Fig. 4 Optimal State Profile Z_e against the number of element e for various value of n when E = 7

Figures 1 and 3 show the optimal control profile U_e against the number of element, e for various values of n when E = 5 and E = 7 respectively. Generally, the nodal temperatures show cooling process from the left side of the rod to the right side

Figures 2 and 4 show the optimal state profile Z_e against the number of element, e for various values of n when E = 5 and E = 7 respectively. It was discovered that the output for each number of element is symmetric.

Conclusion

Increase in number of elements yields better results. However, there are more computations as the characteristics matrix becomes larger and impossible to be solved manually; it is therefore recommended that computer programming should be used to solve large systems of finite elements.

References

Boyce, W.E. and Diprima, R.C. (1977) "Elementary Differential Equations and Boundary Value Problems", John Wiley and sons, USA.

Duchateau, P. and Zachmann, D.W. (1986) - Partial Differential Equations", McGraw-Hill Publishing Company, USA.

Ibiejugba M. A. and K. R. Adeboye, (1984)"On the convergence of diffusion equation". Advances in modeling and simulation, 2:47-56.

Krasnov et al (1984) "Problems and Exercises in the Calculus of Variations", Mir Publishers, Moscow.

Reju, S.A.,Ibiejugba, M.A. andEvans D.J. (1999) "On Navier- Stokes Equation with the Extended Conjugate Gradient Algorithm" inter. J., Math Computer , vol. 76 No 1-2.

Singh, M.G. and Titli, A. (1978) "Systems, Decomposition, Optimization and Control", Pergamon Press.