A SEQUENTIAL STATE VARIABLE APPROACH TO THE CONTROL OF NAVIER-STOKES' EQUATION

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Abstract
This paper dealt with the solution to an optimization problem for the Navier- Stokes' equation using the finite element method. The finite element method is a numerical method which can be used for the accurate solution of a complex boundary-value problem and other problems in Engineering. It was discovered that as the viscosity increases the velocity increases and consequently increase in external forces. The flow is symmetric about the co-ordinate axis.

Key words: Finite element, boundary-value problem, optimization problem, and Navier- Stokes' equation.

1. Introduction

The solution of an optimization problem for the Navier- Stokes' equation which gives the standard equation of motion for the viscous incompressible fluids is sought for.
Among the various modified forms of the Navier-Stokes' equation, we consider in particular the linear non-stationary problem.

$$\frac{\partial v(x,t)}{\partial t} - v\Delta v(x,t) = -\text{grad } p(x,t) + f(x,t)$$
$$div\ v(x,t) = 0$$
$$v(x,t)|_{\partial\Omega} = 0, \quad v(x,t)|_{t=0} = a(x)$$
1.1

The above boundary-value problem has been given an extensive functional treatment by Ladyszhenkaya [6] and her colleagues with the assumption that:
$v(x,t) \in L_2(\Omega), \quad f(x,t) \in L_2(\Omega)$
where Ω is the space occupied by the fluid with kinematic viscosity v and $\Omega_T = \Omega \times [0,T]$ $for\ t \in [0,T]$
Reju [7] has also extended the application of ECGM to the same equation as partly presented below.

Peculiar to the above boundary -value problem is the orthogonal space decomposition: $L_2(\Omega_T) = G(\Omega_T) \ominus J(\Omega_T)$
where $J(\Omega_T)$ is the closure of the set of infinitely differentiable solenoidal vectors of compact support in Ω_T, while $G(\Omega_T)$ is the set of elements of $L_2(\Omega_T)$ that are orthogonal to $J(\Omega_T)$.
Thus, $-\text{grad } p \in G(\Omega_T)$ and without loss of generality, we can assume that $f(x,t) \in J(\Omega_T)$ since its gradient can be incorporated in $-grad\ p$.
Basic to our formulation in the sequel is also a well known result: "if in the course of time the external forces $f(x,t)$ die out, and if the boundary conditions correspond to a state of rest$\left(i.e\ v|_{\partial\Omega} = 0\right)$, then the motion given by $v(x,t)$ also dies out, regardless of what the motion was at $t = 0$.
If as $t \to \infty$, the values $f(x,t)$ of the external forces approach stationary value $f_0(x)$ for which the corresponding boundary-value problem has a solution $v_0(x,t)$ with small Reynolds number, then the solution of the non-stationary problem corresponding to arbitrary

initial-value requires $v(x,t)$ approach $v_0(x)$ as $t \to \infty$".
Finally, for a finite interval, the solution $v(x,t)$ depend continuously on the initial-values $v(x,0)$ and on the external forces $f(x,t)$.

PROBLEM FORMULATION

Based on the foregoing fundamental remarks, the elliptic control problem is formulated as.

$$Min\, J(v,f) = Min \int_0^T \int_\Omega [v^2(x,t) + f^2(x,t)]dxdt \quad 1.2$$

Subject to

$$\left.\begin{array}{c} v_t(x,t) - v\Delta v(x,t) = -grad\, p(x,t) + f(x,t) \\ div\, v(x,t) = 0 \\ v(x,t) = 0, \quad v(x,0) = a(x) \end{array}\right\}$$

1.3

Since we need to have T ∈ $\mathfrak{R}$ and Ω c $\mathfrak{R}$ for us to be able to have a 3-D surface plot for $v(x,t)$ and $f(x,t)$ in $\mathfrak{R}^3$, thus the problem becomes:

$$Min\ J(v,f) = Min \int_0^T \int_0^\lambda [v^2(x,t) + f^2(x,t)]dxdt \ 1.4$$

Subject to

$$\frac{\partial v(x,t)}{\partial t} - v\frac{\partial^2 v(x,t)}{\partial x^2} = \frac{\partial p(x,t)}{\partial x} + f(x,t)$$

$$div\, v(x,t) = 0 \ 1.4b$$

$$v(0,t) = v(\lambda,t) = 0, \qquad v(x,0) = a(x)$$

The above problem is transformed into an unconstrained control problem:

$$Min\left[\int_0^T \int_0^\lambda v^2(x,t) + f^2(x,t) + \mu\left\| \frac{\partial}{\partial t}v(x,t) - v\frac{\partial^2}{\partial x^2}v(x,t) + \frac{\partial}{\partial x}p(x,t) - f(x,t)\right\|^2\right]dxdt$$

where $\mu > 0$ is the penalty parameter and the viscosity v is a constant [6] been assumed. From (1.5) above, there is need to invoke Ladyzhenskaya's assumption [6] that $f(x,t)$ can be taken to be in $J(\Omega_T)$ since its gradient part can be incorporated in - $grad\ p(x,t)$.
Thus, we can absorb $\partial p(x,t)$ over ∂x in $f(x,t)$ in (1.4) to have

$$Min\, J(v,f,\mu) = Min\left[\int_0^T \int_0^\lambda [v^2(x,t) + f^2(x,t) +]\mu\left\|\frac{\partial v(x,t)}{\partial t} - v\frac{\partial^2 v(x,t)}{\partial x^2} - f(x,t)\right\|^2\right]dxdt \quad 1.6$$

2. A New Optimal Analytical Solution to the Navier-Stokes' Problem

Consider the constrained problem

$$Min \int_0^T \int_0^\lambda [v^2(x,t) + f^2(x,t)]dxdt \qquad 2.1$$

subject to the Navier-Stokes' problem

$$\left.\begin{array}{c} \frac{\partial v(x,t)}{\partial t} - v\frac{\partial^2 v(x,t)}{\partial x^2} = f(x,t) \\ div\, v(x,t) = 0 \\ v(0,t) = v(\lambda,t) = 0 \\ v(x,o) = a(x) \end{array}\right\} \qquad 2.2$$

while the usual $\frac{\partial v(x,t)}{\partial x}$ term that ought to appear in equation (2.2) has been absorbed over ∂x in $f(x,t)$ due to Ladyzhenskaya's assumption[6] explained earlier. The Hamiltonian associated with (1.7) and (1.8) is formed as follows:

$$H(x,t) = v^2(x,t) + f^2(x,t) + \lambda^T\left[v\frac{\partial^2 v(x,t)}{\partial x^2} + f(x,t)\right]$$

where $\lambda^T = \lambda^T(t)$

setting $h(v,f) = v\frac{\partial^2 v}{\partial x^2} + f(x,t)$

and $g(v,f) = v^2 + f^2$

We have the necessary optimality conditions as:

$$\frac{\partial v}{\partial t} = \frac{\partial H}{\partial \lambda} = v\frac{\partial^2 v}{\partial x^2} + f = h(v,f)$$

$$\frac{\partial \lambda}{\partial t} = \frac{\partial H}{\partial v} = -\left[\frac{\partial h}{\partial v}\right]^T \lambda = -\frac{\partial g}{\partial v} = -2v$$

$$\frac{\partial H}{\partial f} = 0 \quad \text{or} \quad \left[\frac{\partial h}{\partial f}\right]^T + \frac{\partial g}{\partial f} = 0$$

where

$$H = g(v,f) - \lambda^T(t)h(v,f)$$

Equation (2.6) gives

$$\lambda + 2f = 0 \quad \Rightarrow \quad \lambda = -2f \qquad 2.7$$

Thus, we have from (2.5) and (2.7)

$$\frac{\partial \lambda}{\partial t} = -2\frac{\partial f}{\partial t} = -2v$$

$$\Rightarrow \quad v(x,t) = \frac{\partial f(x,t)}{\partial t}$$

Admitting the Fourier series solution proposed by Duchateau and Zachmann [3] we have that

$$v(x,t) = \sum_{i=1}^{\infty} \alpha_i(t)\sin n\pi x$$

$$f(x,t) = \sum_{i=1}^{\infty} f_i(t)\sin n\pi x$$

Thus, our new velocity fluid, $v(x,t)$ is given by

$$v(x,t) = \sum_{i=1}^{\infty} f_{it}(t)\sin n\pi x$$

and

$$\alpha_i(t) = f_{it}(t)$$

with

$$v_t(x,t) = \sum_{i=1}^{\infty} f_{itt}(t)\sin n\pi x$$

$$vv_{xx}(x,t) = v\sum_{i=1}^{\infty}(i^2)(-\pi^2)f_{it}(t)\sin n\pi x \qquad 2.4$$

$$u(x) = \sum_{i=1}^{\infty} f_{it}(t)\sin n\pi x \qquad 2.5$$

2.6

So, the optimization problem becomes

$$Min\left[\int_0^T [f_1^2 + f_2^2 + \cdots + f_n^2]dt + \int_0^T [f_{1t}^2 + f_{2t}^2 + \cdots + f_{nt}^2]dt\right] \qquad 2.16$$

Subject to the following set of equations

$$\left.\begin{array}{l} f_{1tt} = -2v\pi^2 1^2 f_{1t} + f_1 \\ f_{2tt} = -2v\pi^2 2^2 f_{2t} + f_2 \\ f_{3tt} = -2v\pi^2 3^2 f_{3t} + f_3 \\ \vdots \\ f_{ntt} = -2v\pi^2 n^2 f_{nt} + f_n \end{array}\right\} \qquad \begin{array}{l} 2.8 \\ 2.9 \\ 2.17 \end{array}$$

while the unconstrained problem is

$$Min\left[\int_0^T [f_1^2 + f_2^2 + \cdots \ldots\ldots + f_n^2]dt + \int_0^T [f_{1t}^2 + f_{2t}^2 + \cdots \ldots\ldots + f_{nt}^2]dt + \mu\int_0^T [\|f_{itt} + \upsilon\pi^2 n^2 f_{it} - f_i\|^2 + \|f_{ntt} + \upsilon\pi^2 n^2 f_{nt} - f_n\|^2]dt\right] \quad 2.18$$

3. Finite Element Formulation

The stepwise solution procedure is as follows;

STEP 1: Solution Continuum Discretization

We discretize the domain $(0 \le t \le 1)$ with elements of equal length

STEP 2: Interpolation Model

Consider a one dimensional rod of length l. Let the nodes be denoted by i and j and the nodal values of the field variable f by f_i and f_j. Assuming the interpolation model for each element as:

$$f(t) = \propto_1 + \propto_2 (t)$$

and using the nodal conditions

$$f(t) = f_i \quad at \quad t = t_i$$

$$f(t) = f_j \quad at \quad t = t_j$$

From (3.1), we obtain

$$f_i = \propto_1 + \propto_2 t_i \quad 3.2$$

$$f_j = \propto_1 + \propto_2 t_j$$

The solution of equations (3.2) and (3.3) give

$$\propto_1 = \frac{f_i t_j - f_j t_i}{l}$$

$$\propto_2 = \frac{f_j - f_i}{l}$$

Substituting (3.4) and (3.5) in (3.1) we have

$$f(t) = \frac{f_i t_j - f_j t_i}{l} + \frac{(f_j - f_i)t}{l}$$

$$= \frac{(t_j - t)f_i}{l} + \frac{(t - t_i)f_j}{l}$$

$$= \emptyset_i(t)f_i + \emptyset_j(t)f_j$$

$$= [\emptyset(t)\vec{f}^e]$$

where $\emptyset_i(t) = \frac{t_j - t}{l}$, $\emptyset_j(t) = \frac{t - t_i}{l}$

and $\vec{f}^e = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$ = vector of nodal unknowns of element e

STEP 3: Element Characteristic Matrices and Vectors

The necessary conditions for optimality lead to the functional

$$J = \frac{1}{2}\int \left[-\left(\frac{\partial f_n}{\partial t}\right)^2 + 2\upsilon\pi^2 n^2 f_n - f_n^2\right] dt \quad 3.1$$

The element characteristics matrices and vectors are obtained by expressing the functional J in matrix form as follows:

Evaluating the functional J over the length of element e, we obtain

$$J^{(e)} = \frac{1}{2}\int_0^{(e)} \left[-\left(\frac{\partial f_n}{\partial t}\right)^2 + 2\upsilon\pi^2 n^2 - f_n^2\right] dt \quad 3.3$$

$$= \sum_{e=1}^{E} J^{(e)}$$

The functional J is expressed as the sum of E elemental quantities $J^{(e)}$ as

substituting (3.6) in (3.8) we obtain

$$J^{(e)} = \frac{1}{2}\int_{t_i}^{t_j}\left[-\vec{f}_n^{(e)^T}\left[\frac{\partial\emptyset}{\partial t}\right]^T\left[\frac{\partial\emptyset}{\partial t}\right]\vec{f}_n^{(e)} - \vec{f}_n^{(e)^T}[\emptyset]^T[\emptyset]\vec{f}_n^{(e)} + 2\upsilon\pi^2 n^2[\emptyset]\vec{f}_n^{(e)}\right] \quad 3.10$$

Using the equivalent minimization problem (2.16), the conditions for minimum are;

$$\frac{\partial J}{\partial f_i} = \frac{\partial}{\partial f_i}\sum_{e=1}^{E} J^{(e)} = \sum_{e=1}^{E}\frac{\partial J^{(e)}}{\partial f_i} = 0$$

$i = 1,2 \ldots\ldots\ldots m,$
where E is the number of nodal degrees of freedom.
Equation (3.11) can also be expressed as;

$$\sum_{e=1}^{E}\frac{\partial J^{(e)}}{\partial \vec{f}_n^{(e)}} = \vec{0} \quad 3.12$$

$$\text{i.e} \sum_{e=1}^{E}\int_{t_i}^{t_j}\left[-\left[\frac{\partial\emptyset}{\partial t}\right]^T\left[\frac{\partial\emptyset}{\partial t}\right]\vec{f}_n^{(e)} - [\emptyset]^T[\emptyset]\vec{f}_n^{(e)} + \upsilon\pi^2 n^2[\emptyset]^T\vec{f}_n^{(e)}\right]dt = \vec{0} \quad 3.13$$

$$\text{i.e} \sum_{e=1}^{E}[k^{(e)}]\vec{f}_n^{(e)} = \sum_{e=1}^{E}\vec{p}^{(e)}$$

J has been replaced by the sum of the element functional $J^{(e)}$,
e = 1, 2… m in the assembling.
Equation (3.11) is stated in a linear matrix form as;

$$\sum_{e=1}^{m}\frac{\partial J^{(e)}}{\partial f^{(e)}} = \sum_{e=1}^{m}\left[k^{(e)}\vec{f}_n^{(e)} - \vec{p}^{(e)}\right] = 0$$

This is necessary to form our governing finite element equations, where
$[k^{(e)}]$ = element characteristics matrix

$$= \int_{t_i}^{t_j}\left[\left[\frac{\partial\emptyset}{\partial t}\right]^T\left[\frac{\partial\emptyset}{\partial t}\right] + [\emptyset]^T[\emptyset]\right]dt \quad 3.11$$

$p^{(e)}$ = element characteristic vector

$$= \int_{t_i}^{t_j}\upsilon\pi^2 n^2[\emptyset]^T\,dt$$

$\vec{f}^{(e)}$ = the assembled external forces

$$= \begin{Bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f_m \end{Bmatrix} \quad 3.18$$

substituting $[\emptyset(t)] = [\emptyset_i(t) \quad \emptyset_j(t)]$

$$= \left[\frac{t_{j-t}}{l^{(e)}} \quad \frac{t-t_i}{l^{(e)}}\right] \quad 3.19$$

into (3.16) and (3.17) we have:3.14

$[k^{(e)}]$
$= \int_{t_i}^{t_j} \left[\begin{Bmatrix} \frac{-1}{l^{(e)}} \\ \frac{1}{l^{(e)}} \end{Bmatrix} \begin{Bmatrix} \frac{-1}{l^{(e)}} & \frac{1}{l^{(e)}} \end{Bmatrix} + \begin{Bmatrix} \frac{t_{j-t}}{l^{(e)}} \\ \frac{t-t_i}{l^{(e)}} \end{Bmatrix} \begin{Bmatrix} \frac{t_{j-t}}{l^{(e)}} & \frac{t-t_i}{l^{(e)}} \end{Bmatrix} \right]$

$= \frac{1}{l^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{l^{(e)}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\overline{p}^{(e)} = \upsilon\pi^2 n^2 \int_{t_i}^{t_j} \begin{Bmatrix} \frac{t_j - t}{l^{(e)}} \\ \frac{t-t_i}{l^{(e)}} \end{Bmatrix} dt$

$= \frac{\upsilon\pi^2 n^2}{2} \begin{Bmatrix} t_j - t_i \\ t_j - t_i \end{Bmatrix}$

STEP 4: Assembly of element characteristic matrices and vectors and derivation of governing equations.

The element characteristic matrices and vectors are assembled and the overall equations are obtained as:

$[k]\vec{f}$
$= \vec{p}$

STEP 5: Solution.

The system equation (3.20) is solved after incorporating the boundary value conditions.

4. Results and Conclusion

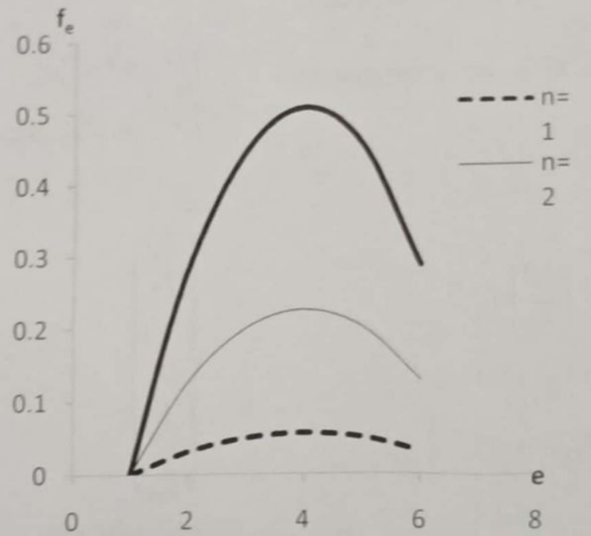

Fig.1 External forces profile f_e against the number of element e for various values of n when E= 6 and kinematic viscosity υ = 0.05

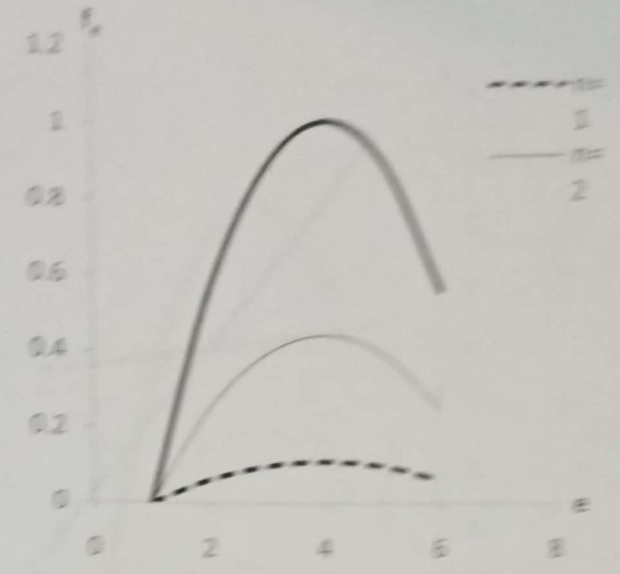

Fig. 2 External forces profile f_e against the number of element e for various values of n

when E = 6 and kinematic viscosity ν = 0.1

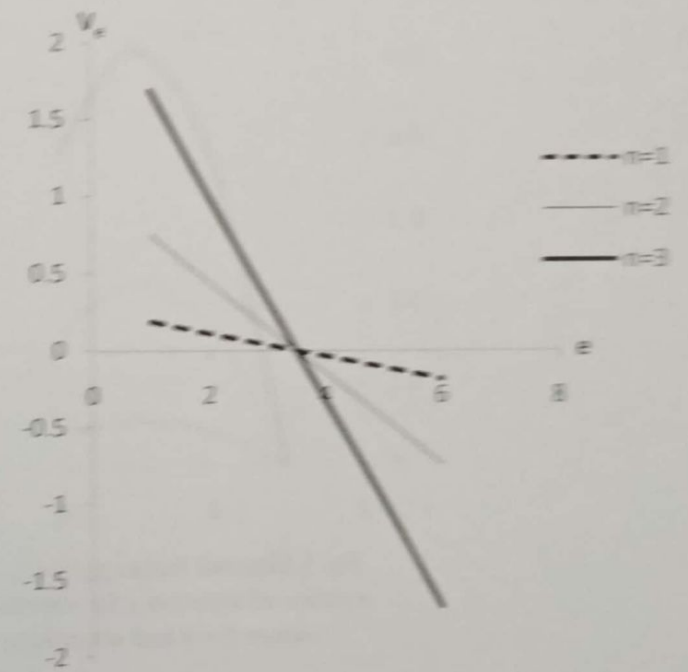

Fig. 3 Velocity profile V_e against the number of element e for various values of n when E = 6 and kinematic viscosity ν = 0.05

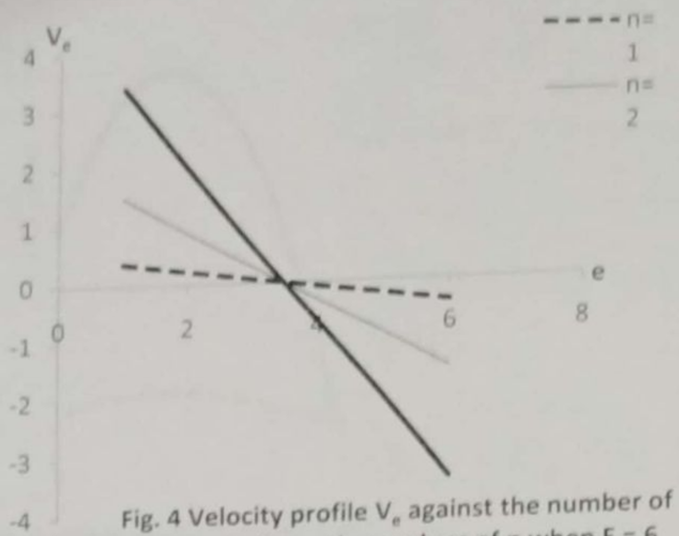

Fig. 4 Velocity profile V_e against the number of element e for various values of n when E = 6 and kinematic viscosity υ = 0.1

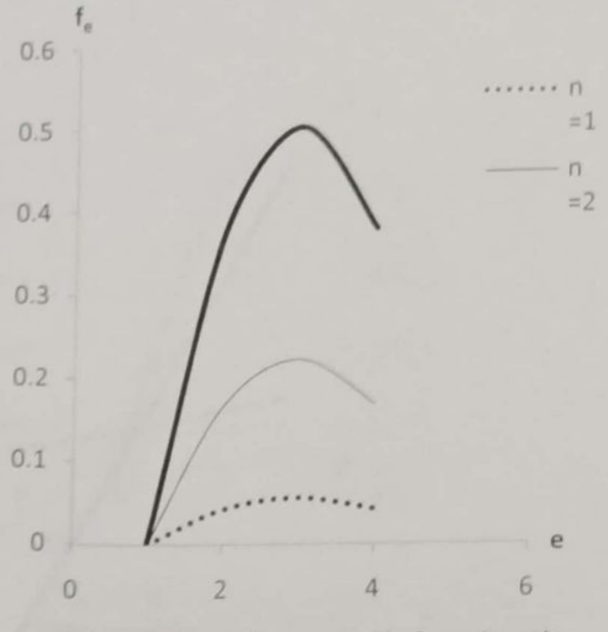

Fig. 5 External forces profile f_e against the number of element e for various values of n when E = 4 and viscosity υ = 0.05

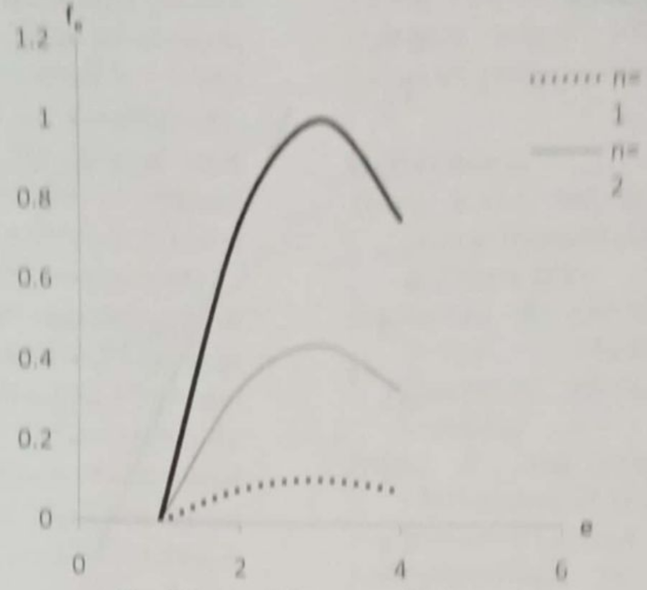

Fig. 6 External forces profile f_e against the number of element e for various values of n when E = 4 and viscosity υ = 0.1

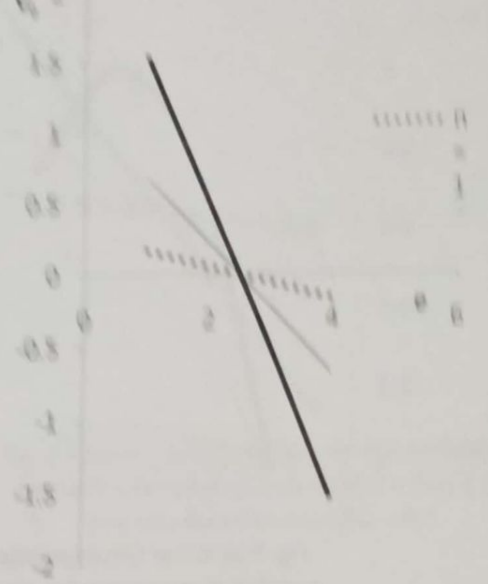

Fig.7 Velocity profile V_e against the number of element e for various values of n when E = 4 and viscosity υ =0.05

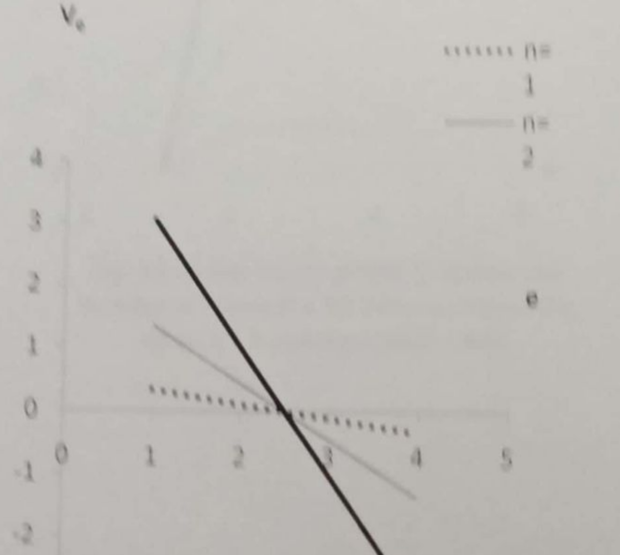

Fig. 8 Velocity profile V_e against the number of element e for various values of n when E = 4 and kinematic viscosity υ = 0.1

Figures 1and 2 show the optimal control profile f_e against the number of element e for various values of n when E = 6 and viscosity υ = 0.05 and 0.1 respectively. Figures 3 and 4 show the optimal state profile V_e against the number of element e for various values of n when E = 6 and viscosity υ = 0.05 and 0.1 respectively.

Figures 5 and 6 are the optimal control profiles f_e against the number of element e for various values of n when E = 4 and viscosity υ = 0.05 and 0.1 respectively while figures 7 and 8 show the optimal state profile V_e against the number of element e for various values of n when E = 4 and viscosity υ = 0.05 and 0.1 respectively.

It was discovered that the external forces increase with increase in viscosity and consequently increase in velocity, hence increase in motion. The motion is symmetrical about the co-ordinate axis .

5. Conclusion

Increase in number of elements yields better results with more computations giving rise to large systems of finite elements which can be solved by computer programming.

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