Preconditioned SOR Iterative Methods for $L - $ matrices

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Abstract A preconditioner of the type $I + S$ for speeding up convergence of the successive overrelaxation (SOR) iterative method for solving linear system $Ax = b$ is proposed. Two forms of the preconditioned SOR iteration are obtained and implemented, under limited conditions imposed on the coefficient matrix of the original linear system. Convergence properties are analyzed and established in conformity with standard procedures. The rates of convergence of the preconditioned iterations are shown to supersede that of the SOR method. Numerical experiments confirmed the established theoretical results.

Keywords Spectral radius, Preconditioner, Iterative matrix, Linear system, L − matrix

1. Introduction

Suppose the linear system of equations
$$Ax = b \qquad (1)$$
is such that $A \in R^{(n \times n)}$ is an $L - $ matrix, $b \in R^n$ is the right-hand side vector, and $x \in R^n$ is the vector of unknown. All the numerical methods for solution of a linear system can be categorised into two thus; direct and iterative methods.

A direct method is one that produces the exact solution of a linear system by performing a prescribed, finite number of operations (steps), if there were no round-off error. Examples of direct methods for solving linear systems include, graphical method, matrix inversion method, Cramer's rule, elimination of unknowns, Gaussian elimination and LU decomposition. However, because direct methods are mostly restricted to small systems of linear equations, iterative methods remain the dominating and preferred techniques for solving very large linear systems.

An iterative method for solving a linear system consists of a process whereby the system $Ax = b$ is converted into an equivalent system of the form
$$x = Gx + k \qquad (2)$$
What follows next is application of the general linear iteration formula
$$x^{(n)} = G_n x^{(n-1)} + k_n \qquad (3)$$
where G_n , referred to as the Iteration matrix, is a matrix depending upon A and x, and k_n is a column vector. At each step of the iteration a solution vector, $x^{(n)}$, that more accurately approximates the solution to the linear system than its predecessor, is produced. This sequence $x^{(n)}$ should be so defined that $\lim_{n \to \infty} x^{(n)} = A^{-1}b$. For any iteration to be convergent, the spectral radius of the iteration matrix must be less than 1[1].

Iterative methods are further classified into two main types, namely, stationary and nonstationary iterative methods. Stationary iterative methods are those methods that can be expressed in the form
$$x^{(n)} = Gx^{(n-1)} + k \qquad (4)$$
where neither G nor k depend upon the iteration count n [2]. Stationary methods are simpler to implement and understood compared to nonstationary methods. However, they are usually not effective when used alone; hence the need to implement them along with preconditioners. If we assume that $A = I - L - U$, where I is the identity matrix, $-L$ the strictly lower triangular part of A, and $-U$ the strictly upper triangular part of A, then the successive overrelaxation method as invented by[3] has the following matrix from
$$x^{(n)} = G_{SOR} x^{(n-1)} + (I - \omega L)^{-1} \omega b \qquad (5)$$
where $G_{SOR} = (I - \omega L)^{-1}\{(1 - \omega)I + \omega U\}$ is the SOR iteration matrix. Various authors have introduced preconditioners for accelerating the convergence of the SOR method[4 -9].

2. Materials and Methods

Consider the preconditioner $P = I + S$, where

$$S = \begin{pmatrix} 0 & -\alpha_{12} & 0 & . & . & 0 \\ -a_{21} & 0 & -a_{23} & & & 0 \\ -a_{31} & . & . & & & . \\ . & . & . & & & . \\ . & . & . & & & -a_{n-1,n} \\ -a_{n1} & 0 & 0 & . & . & 0 \end{pmatrix} \qquad (6)$$

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The above preconditioning matrix is applied to the coefficient matrix A of system (1) thus

$$PAx = Pb \qquad (7)$$

which results into the equivalent preconditioned system

$$\bar{A}x = \bar{b} \qquad (8)$$

where $\bar{A} = PA$ and $\bar{b} = Pb$. The preconditioned coefficient matrix $\bar{A}$ is further analyzed as follows.

$$\bar{A} = PA = (I + S)A = A + SA \qquad (9)$$
$$= I - L - U + S - SL - SU \qquad (10)$$
$$= I - L - U + (S) - (SL + SU) \qquad (11)$$

Further splittings of the bracket terms in (11) are obtained as follows.

$$S = -L_S - U_S \qquad (12)$$
$$-SL - SU = D_1 - L_1 - U_1 \qquad (13)$$

Thus

$$\bar{A} = (I + D_1) - (L + L_S + L_1) - (U + U_S + U_1) \qquad (14)$$

or equivalently

$$\bar{A} = \bar{D} - \bar{L} - \bar{U} \qquad (15)$$

where $\bar{D} = I + D_1$ is the diagonal, $-\bar{L} = -(L + L_S + L_1)$ and $-\bar{U} = -(U + U_S + U_1)$ are strictly lower and strictly upper triangular parts of $\bar{A}$ respectively.

The application of overrelaxation parameter ω to the preconditioned linear system (8) yields

$$\omega \bar{A}x = \omega \bar{b} \qquad (16)$$

where,

$$\omega \bar{A} = \omega(\bar{D} - \bar{L} - \bar{U}) \qquad (17)$$
$$= (I - \omega \bar{L}) - (1 - \omega)I - \omega(\bar{U} - D_1) \qquad (18)$$

Thus

$$\omega \bar{A} = (I - \omega \bar{L}) - [(1 - \omega)I + \omega(\bar{U} - D_1)] \qquad (19)$$

is a regular splitting of $\omega \bar{A} = M - N$, where

$$M = (I - \omega \bar{L}) \text{ and } N = [(1 - \omega)I + \omega(\bar{U} - D_1)] \qquad (20)$$

Hence, the following preconditioned SOR iterative scheme is obtained.

$$x^{(n+1)} = G_1 x^{(n)} + M^{-1}\omega \bar{b} \qquad (21)$$

And the preconditioned SOR iterative matrix G_1 has the representation

$$G_1 = (I - \omega \bar{L})^{-1}[(1 - \omega)I + \omega(\bar{U} - D_1)] \qquad (22)$$

Similarly,

$$= (\bar{D} - \omega \bar{L}) - [(1 - \omega)\bar{D} + \omega \bar{U}] \qquad (23)$$

is another splitting of the the preconditioned coefficient matrix $\omega \bar{A} = M - N$, with

$$M = (\bar{D} - \omega \bar{L}) \text{ and } N = [(1 - \omega)\bar{D} + \omega \bar{U}] \qquad (24)$$

And this leads to the second preconditioned SOR iterative scheme

$$x^{(n+1)} = G_2 x^{(n)} + M^{-1}\omega \bar{b} \qquad (25)$$

with the second preconditioned iterative matrix G_2 having the representation

$$G_2 = (\bar{D} - \omega \bar{L})^{-1}[(1 - \omega)\bar{D} + \omega \bar{U}] \qquad (26)$$

3. Convergence Analysis

Convergence of the preconditioned iterative schemes (21) and (25) is established by showing that the spectral radii of the G_1 and G_2 are less than 1 in each case.

Lemma 3.1 (Varga[10]): Let $A \geq O$ be an irreducible $n \times n$ matrix. Then,

i. A has a positive real eigenvalue equal to its spectral radius.

ii. To $\rho(A)$ there corresponds an eigenvector $x > 0$.

iii. $\rho(A)$ increases when any entry of A increases.

iv. $\rho(A)$ is a simple eigenvalue of A.

Lemma 3.2 (Varga[10]): Let A be a nonnegative matrix. Then

i. If $\alpha x \leq Ax$ for some nonnegative vector $x, x \neq 0$, then $\alpha \leq \rho(A)$.

ii. If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector x, then $\alpha \leq \rho(A) \leq \beta$ and x is a positive vector.

Lemma 3.1 (Li and Sun[11]): Let $A = M - N$ be an $M -$ splitting of A. Then the splitting is convergent, i.e., $\rho(M^{-1}N < 1)$, if and only if A is a nonsingular $M -$matrix.

By employing the foregoing lemmas (lemmas 3.1 - 3.3), the next three theorems are proposed to establish convergence of the preconditioned iterative methods.

Theorem 3.1: Let $G_{SOR} = (I - \omega L)^{-1}[(1 - \omega)I + \omega U]$ be the SOR iteration matrix while $G_1 = (I - \omega \bar{L})^{-1}[(1 - \omega I + \omega(U - D1)]$ and $G2 = D - \omega L - 1[1 - \omega D + \omega U]$ be the preconditioned SOR iteration matrices. If A is an irreducible $L -$matrix with $0 \leq a_{1i}a_{i1} + a_{i,i+1}a_{i+1,i} < 1$, $i = 2(1)n$, and $0 < \omega < 1$, then G_{SOR}, G_1 and G_2 are nonnegative and irreducible matrices.

Proof

Since A is an $L -$matrix, $L \geq 0$ and $U \geq 0$. Now, $(I - \omega L)^{-1} = I + \omega L + \omega^2 L^2 + \cdots + \omega^{n-1}L^{n-1} \geq 0$, $(1 - \omega)I + \omega U \geq 0$, for $0 < \omega < 1$. Thus $G_{SOR} = (I - \omega L)^{-1}[(1 - \omega)I + \omega U] \geq 0$. Hence, G_{SOR} is a nonnegative matrix.

One can also obtain that for $0 < \omega < 1$,

$$G_{SOR} = [I + \omega L + \omega^2 L^2 + \cdots + \omega^{n-1}L^{n-1}][(1 - \omega)I + \omega U] \qquad (27)$$
$$= (1 - \omega)I + \omega(1 - \omega)L + \omega U + \omega^2 LU + \omega^2(1 - \omega)L^2 + \omega^3 L^2 U + \cdots \qquad (28)$$
$$= (1 - \omega)I + \omega(1 - \omega)L + \omega U + \text{nonnegative terms} \qquad (29)$$

Since $A = D - L - U$ is irreducible, so also is $D^{-1}A = I - L - U$, because it inherits the nonzero structure of the irreducible matrix A. The same thing applies to the matrix $(1 - \omega)I + \omega(1 - \omega)L + \omega U$ since the coefficients of I, L and U are different from zero and less than 1 in absolute value. Hence, G_{SOR} is an irreducible matrix.

$$G_1 = (I - \omega \bar{L})^{-1}[(1 - \omega)I + \omega(\bar{U} - D_1)] \qquad (30)$$

Since $\bar{L} \geq 0$, $\bar{U} \geq 0$, $-D_1 \geq 0$, for $0 < \omega < 1$, $(1 - \omega)I + \omega(\bar{U} - D_1) \geq 0$ and $(I - \omega \bar{L})^{-1} = I + \omega \bar{L} + \omega^2 \bar{L}^2 + \cdots + \omega^{n-1}\bar{L}^{n-1} \geq 0$. Consequently, one can find that $G_1 = (I - \omega \bar{L})^{-1}[(1 - \omega)I + \omega(\bar{U} - D_1)] \geq 0$. Hence G_1 is a nonnegative matrix.

Now,

$$\bar{A} = (I + S)A = (I - L_s - U_s)(I - L - U) \tag{31}$$
$$= I - L - U - L_s + L_sL + L_sU - U_s + U_sL + U_sU \tag{32}$$
$$= I - L - U - L_s + L_sL - (L_sU)_L - (L_sU)_U - U_s - (U_sL)_L - (U_sL)_U + U_sU \tag{33}$$
$$= I - L - L_s + L_sL - (L_sU)_L - (U_sL)_L - U - U_s + U_sU - (U_sL)_U - (L_sU)_U \tag{34}$$
$$= I - (L + L_s - L_sL + (L_sU)_L + (U_sL)_L) - (U + U_s - U_sU + (U_sL)_U + (L_sU)_U) \tag{35}$$
$$= I - \bar{L} - \bar{U} \tag{36}$$

where $\bar{L} = L + L_s - L_sL + (L_sU)_L + (U_sL)_L$, $\bar{U} = U + U_s - U_sU + (U_sL)_U + (L_sU)_U$ and $-(T)_L$ and $-(T)_U$ denote the strictly lower and strictly upper parts of the matrix T respectively. $\bar{A} = I - \bar{L} - \bar{U}$ is irreducible, since it inherits the nonzero structure of the irreducible matrix $\bar{A}$.

Now,

$$G_1 = (I - \omega\bar{L})^{-1}[(1 - \omega)I + \omega(\bar{U} - D_1)] \tag{37}$$
$$= (1 - \omega)I + \omega(1 - \omega)\bar{L} + \omega\bar{U} - \omega D_1 + \omega^2\bar{L}\bar{U} - \omega^2 D_1\bar{L} + \omega^2(1 - \omega)\bar{L}^2 + \cdots \tag{38}$$
$$= (1 - \omega)I + \omega(1 - \omega)\bar{L} + \omega\bar{U} + \text{nonnegative terms} \tag{39}$$

The matrix $(1 - \omega)I + \omega(1 - \omega)\bar{L} + \omega\bar{U}$ is irreducible, since the coefficients of I, $\bar{L}$ and $\bar{U}$ are different from zero and less than one in absolute value. Therefore, the matrix $G_1 = (I - \omega\bar{L})^{-1}[(1 - \omega)I + \omega(\bar{U} - D_1)]$ is irreducible. Hence G_1 is a nonnegative and irreducible matrix.

Similarly, consider

$$G_2 = (\bar{D} - \omega\bar{L})^{-1}[(1 - \omega)\bar{D} + \omega\bar{U}] \tag{40}$$
$$= [\bar{D}(I - \omega\bar{D}^{-1}\bar{L})]^{-1}[(1 - \omega)\bar{D} + \omega\bar{U}] \tag{41}$$
$$= (I - \omega\bar{D}^{-1}\bar{L})^{-1}[(1 - \omega)I + \omega\bar{D}^{-1}\bar{U}] \tag{42}$$
$$= [I + \omega\bar{D}^{-1}\bar{L} + \omega^2(\bar{D}^{-1}\bar{L})^2 + \cdots + \omega^{n-1}(\bar{D}^{-1}\bar{L})^{n-1}] \times [(1 - \omega)I + \omega\bar{D}^{-1}\bar{U}] \tag{43}$$
$$= (1 - \omega)I + \omega(1 - \omega)\bar{D}^{-1}\bar{L} + \omega\bar{D}^{-1}\bar{U} + \text{nonnegative terms} \tag{44}$$

Using similar arguments it is conclusive that $G_2 = (\bar{D} - \omega\bar{L})^{-1}[(1 - \omega)\bar{D} + \omega\bar{U}]$ is a nonnegative and irreducible matrix. The proof is completed.

Theorem 3.2: Let $G_{SOR} = (I - \omega L)^{-1}[(1 - \omega)I + \omega U]$ and $G_1 = (I - \omega\bar{L})^{-1}[(1 - \omega)I + \omega(\bar{U} - D_1)]$ be the SOR and preconditioned SOR iteration matrices respectively. If $0 < \omega < 1$ and if A is an irreducible $L-matrix$ with $0 \leq a_{1i}a_{i1} + a_{i,i+1}a_{i+1,i} < 1$, $i = 2(1)n$. Then,

(i) $\rho(G_1) < \rho(G_{SOR})$, if $\rho(G_{SOR}) < 1$
(ii) $\rho(G_1) = \rho(G_{SOR})$, if $\rho(G_{SOR}) = 1$
(iii) $\rho(G_1) > \rho(G_{SOR})$, if $\rho(G_{SOR}) > 1$

Proof

From Theorem 3.1, G_{SOR} and G_1 are nonnegative and irreducible matrices. Suppose $\rho(G_{SOR}) = \lambda$, then there exists a positive vector $x = (x_1, x_2, \cdots, x_n)^T$, such that

$$G_{SOR}x = \lambda x \tag{45}$$

That is,

$$(I - \omega L)^{-1}\{(1 - \omega)I + \omega U\}x = \lambda x \tag{46}$$
$$(1 - \omega)I + \omega U = \lambda(I - \omega L) \tag{47}$$

Therefore, for this $x > 0$,

$$= (I - \omega\bar{L})^{-1}\{(1 - \omega)I + \omega(\bar{U} - D_1)\}x - \lambda(I - \omega\bar{L})^{-1}(I - \omega\bar{L})x \tag{48}$$
$$= (I - \omega\bar{L})^{-1}\{(1 - \omega - \lambda)I + \omega U + \lambda\omega L + \lambda\omega L_s + \lambda\omega L_1 + \omega U_s - \omega D_1 + \omega U_1\}x \tag{49}$$

From equation (47)

$$\omega U + \lambda\omega L = -(1 - \omega - \lambda)I \tag{50}$$
$$G_1x - \lambda x = (I - \omega\bar{L})^{-1}\{(\lambda - 1)\omega L_1 + \lambda\omega L_s + \omega U_s - \omega(D_1 - L_1 - U_1)\}x \tag{51}$$
$$= (I - \omega\bar{L})^{-1}\{(\lambda - 1)\omega L_1 + \lambda\omega L_s - \omega L_s + \omega L_s + \omega U_s + \omega SL + \omega SU\}x \tag{52}$$
$$= (I - \omega\bar{L})^{-1}\{\omega(\lambda - 1)(L_1 + L_s) + (1 - \omega)S - S(I - \omega L) + \omega SU)\}x \tag{53}$$
$$= (I - \omega\bar{L})^{-1}\{\omega(\lambda - 1)(L_1 + L_s) + S[(1 - \omega)I + \omega U] - S(I - \omega L)\}x \tag{54}$$
$$= (I - \omega\bar{L})^{-1}\{\omega(\lambda - 1)(L_1 + L_s) + \lambda S(I - \omega L) - S(I - \omega L)\}x \tag{55}$$
$$= (\lambda - 1)(I - \omega\bar{L})^{-1}\{\omega(L_1 + L_s) + S(I - \omega L)\}x \tag{56}$$
$$G_1x - \lambda x = (\lambda - 1)/\lambda\, [(I - \omega\bar{L})^{-1}]\{\lambda\,\omega(L_1 + L_s) + (1 - \omega)S + \omega SU\}x \tag{57}$$

Let $R = Qx$, where $Q = (I - \omega\bar{L})^{-1}\{\lambda\,\omega(L_1 + L_s) + (1 - \omega)S + \omega SU\}$. Then $Q = (I - \omega\bar{L})^{-1}\{\lambda\,\omega(L_1 + L_s) + (1 - \omega)S + \omega SU\} \geq 0$, because $\lambda\,\omega(L_1 + L_s) \geq 0$, $(1 - \omega)S \geq 0$ and $\omega SU \geq 0$. Also, $(I - \omega\bar{L})^{-1} = I + \omega\bar{L} + \omega^2\bar{L}^2 + \cdots + \omega^{n-1}\bar{L}^{n-1} \geq 0$, since $\bar{L} \geq 0$. Therefore, $Q = (I - \omega\bar{L})^{-1}\{\lambda\,\omega(L_1 + L_s) + (1 - \omega)S + \omega SU\} \geq 0$.

So, $R = (I - \omega\bar{L})^{-1}\{\lambda\,\omega(L_1 + L_s) + (1 - \omega)S + \omega SU\}x \geq 0$, since $x > 0$.

i. If $\lambda < 1$, then $G_1x - \lambda x \leq 0$ but not equal to 0. Therefore,

$$G_1x \leq \lambda x \tag{58}$$

From Lemma 3.2, one obtains

$$\rho(G_1) < \lambda = \rho(G_{SOR}) \tag{59}$$

ii. If $\lambda = 1$, then $G_1 x - \lambda x = 0$

Therefore,

$$G_1 x = \lambda x \tag{60}$$

From Lemma 3.2, we have

$$\rho(G_1) = \lambda = \rho(G_{SOR}) \tag{61}$$

iii. If $\lambda > 1$, then $G_1 x - \lambda x \geq 0$ but not equal to 0.

Therefore,

$$G_1 x \geq \lambda x \tag{62}$$

From Lemma 3.2, we have

$$\rho(G_1) > \lambda = \rho(G_{SOR}) \tag{63}$$

The proof is completed.

Theorem 3.3: Let $G_{SOR} = (I - \omega L)^{-1}[(1 - \omega)I + \omega U]$ and $G_2 = (\overline{D} - \omega \overline{L})^{-1}[(1 - \omega)\overline{D} + \omega \overline{U}]$ be the SOR and the preconditioned SOR iteration matrices respectively. If

$0 < \omega < 1$ and if A is an irreducible $L-$matrix with $0 \leq a_{1i} a_{i1} + a_{i,i+1} a_{i+1,i} < 1$, $i = 2(1)n$. Then,

(i) $\rho(G_2) < \rho(G_{SOR})$, if $\rho(G_{SOR}) < 1$;
(ii) $\rho(G_2) = \rho(G_{SOR})$, if $\rho(G_{SOR}) = 1$;
(iii) $\rho(G_2) > \rho(G_{SOR})$, if $\rho(G_{SOR}) > 1$.

Proof

From Theorem 3.1, G_{SOR} and G_2 are nonnegative and irreducible matrices. Suppose $\rho(G_{SOR}) = \lambda$, then there exists a positive vector $x = (x_1, x_2, \cdots, x_n)^T$, such that

$$G_{SOR} x = \lambda x \tag{64}$$

That is,

$$(I - \omega L)^{-1}\{(1 - \omega)I + \omega U\}x = \lambda x \tag{65}$$

$$(1 - \omega)I + \omega U = \lambda(I - \omega L) \tag{66}$$

Therefore, for this $x > 0$,

$$G_2 x - \lambda x = (\overline{D} - \omega \overline{L})^{-1}\{(1 - \omega)\overline{D} + \omega \overline{U}\}x - \lambda x \tag{67}$$

$$= (\overline{D} - \omega \overline{L})^{-1}\{(1 - \omega)\overline{D} + \omega \overline{U} - \lambda(\overline{D} - \omega \overline{L})\}x \tag{68}$$

$$= (\overline{D} - \omega \overline{L})^{-1}\{(1 - \omega - \lambda)D_1 + \lambda \omega L_1 + \lambda \omega L_S + \lambda \omega L + \omega U_S + \omega U_1 + (1 - \omega - \lambda)I + \omega U)\}x \tag{69}$$

From equation (66)

$$(1 - \omega - \lambda)I + \omega U = -\lambda \omega L \tag{70}$$

$$G_2 x - \lambda x = (\overline{D} - \omega \overline{L})^{-1}\{(1 - \omega - \lambda)D_1 + \lambda \omega L_1 + \lambda \omega L_S + \omega U_S + \omega U_1\}x \tag{71}$$

$$= (\overline{D} - \omega \overline{L})^{-1}\{(\lambda - 1)(-D_1 + \omega L_1) + \omega SL + \omega SU + (\lambda - 1)\omega L_S + \omega(L_S + U_S)\}x \tag{72}$$

$$= (\overline{D} - \omega \overline{L})^{-1}\{(\lambda - 1)(-D_1 + \omega L_1 + \omega L_S) + S[(1 - \omega)I + \omega U] - S(I - \omega L)\}x \tag{73}$$

$$= (\lambda - 1)/\lambda \, [(\overline{D} - \omega \overline{L})^{-1}]\{-\lambda D_1 + \lambda \omega L_1 + \lambda \omega L_S + (1 - \omega)S + \omega SU\}x \tag{74}$$

Let $R = Qx$, with $Q = (\overline{D} - \omega \overline{L})^{-1}\{-\lambda D_1 + \lambda \omega L_1 + \lambda \omega L_S + (1 - \omega)S + \omega SU\}$. It is clear that $-\lambda D_1 + \lambda \omega L_1 + \lambda \omega L_S + (1 - \omega)S + \omega SU \geq 0$, since $(1 - \omega)S \geq 0$, $\omega SU \geq 0$, $-\lambda D_1 \geq 0$, $\lambda \omega L_1 \geq 0$ and $\lambda \omega L_S \geq 0$. Since $\overline{D}$ is a nonsingular matrix, we let $\overline{D} - \omega \overline{L}$ be a splitting of some matrix K, i.e., $K = \overline{D} - \omega \overline{L}$. Also, $\overline{D}$ is an $M-$matrix and $\omega \overline{L} \geq 0$. Thus, $K = \overline{D} - \omega \overline{L}$ is an $M-$splitting. Now, $\omega \overline{D}^{-1} \overline{L}$ is a strictly lower triangular matrix, and by implication its eigenvalues lie on its main diagonal; in this case they are all zeros. Therefore, $\rho(\omega \overline{D}^{-1} \overline{L}) = 0$. since $\rho(\omega \overline{D}^{-1} \overline{L}) < 1$, $K = \overline{D} - \omega \overline{L}$ is a convergent splitting. By the foregoing, $K = \overline{D} - \omega \overline{L}$ is an $M-$splitting and $\rho(\omega \overline{D}^{-1} \overline{L}) < 1$. Lemma 3.3 is invoked in order to establish that K is an $M-$matrix. Since K is an $M-$matrix, by definition, $K^{-1} = (\overline{D} - \omega \overline{L})^{-1} \geq 0$. Thus, $Q \geq 0$ and $R \geq 0$.

If $\lambda < 1$, then $G_2 x - \lambda x \leq 0$ but not equal to 0.

Therefore,

$$G_2 x \leq \lambda x \tag{75}$$

From Lemma 3.2, we have

$$\rho(G_2) < \lambda = \rho(G_{SOR}) \tag{76}$$

ii. If $\lambda = 1$, then $G_2 x - \lambda x = 0$

Therefore,

$$G_2 x = \lambda x \tag{77}$$

From Lemma 3.2, we have

$$\rho(G_2) = \lambda = \rho(G_{SOR}) \tag{78}$$

iii. If $\lambda > 1$, then $G_2 x - \lambda x \geq 0$ but not equal to 0.

Therefore,

$$G_2 x \geq \lambda x \tag{79}$$

From Lemma 3.2, we have

$$\rho(G_2) > \lambda = \rho(G_{SOR}) \tag{80}$$

The proof is completed.

If in G_{SOR}, G_1 and G_2 the relaxation parameter $\omega = 1$, the iteration matrices of the Gauss-Seidel method results in each case. Therefore, the following corollaries are direct implications of Theorem 3.1 and Theorem 3.2.

Corollary 3.1 Let $G_{GS} = (I - L)^{-1}U$ be the Gauss-Seidel iteration matrix and $G_{PGS} = (I - \overline{L})^{-1}(\overline{U} - D_1)$ be the preconditioned Gauss-Seidel iteration matrix. If A is an irreducible $L-$matrix with $0 < a_{1i} a_{i1} + a_{i,i+1} a_{i+1,i} < 1$, $i = 2(1)n$, then

(i) $\rho(G_{PGS}) < \rho(G_{GS})$, if $\rho(G_{GS}) < 1$;
(ii) $\rho(G_{PGS}) = \rho(G_{GS})$, if $\rho(G_{GS}) = 1$;
(iii) $\rho(G_{PGS}) > \rho(G_{GS})$, if $\rho(G_{GS}) > 1$.

Corollary 3.2 Let $G_{GS} = (I - L)^{-1}U$ be the Gauss-Seidel iteration matrix and $G_{PGS} = (\bar{D} - \bar{L})^{-1}\bar{U}$ be the preconditioned Gauss-Seidel iteration matrix. If A is an irreducible $L-$matrix with $0 < a_{1i}a_{i1} + a_{i,i+1}a_{i+1,i} < 1$, $i = 2(1)n$, then

(i) $\rho(G_{PGS}) < \rho(G_{GS})$, if $\rho(G_{GS}) < 1$;
(ii) $\rho(G_{PGS}) = \rho(G_{GS})$, if $\rho(G_{GS}) = 1$;
(iii) $\rho(G_{PGS}) > \rho(G_{GS})$, if $\rho(G_{GS}) > 1$.

4. Numerical Experiments

Example 4.1 Consider a 4×4 matrix of the form.

$$\begin{pmatrix} 1.000 & -0.279 & -0.233 & 0 \\ -0.306 & 1.000 & 0 & -0.204 \\ -0.265 & 0 & 1.000 & -0.245 \\ 0 & -0.236 & -0.273 & 1.000 \end{pmatrix}$$

Table 1 displays the results of comparing the spectral radiuses of G_{SOR}, G_1 and G_2 iterative matrices corresponding to the matrix in example 4.1.

Table 1 compares the spectral radiuses of G_{SOR}, G_1 and G_2 iteration matrices. It reveals that the two preconditioned SOR iteration matrices exhibit faster convergence than the SOR, because the spectral radiuses of G_1 and G_2 are less than the spectral radius of G_{SOR} $(\rho(G_2) < \rho(G_1) < \rho(G_{SOR}) < 1)$, for all values of relaxation parameter ω.

Example 4.2 Consider a 9×9 matrix of the form.

$$\begin{pmatrix} 1.000 & -0.274 & 0 & -0.233 & 0 & 0 & 0 & 0 & 0 \\ -0.296 & 1.000 & -0.296 & 0 & -0.209 & 0 & 0 & 0 & 0 \\ 0 & -0.315 & 1.000 & 0 & 0 & -0.191 & 0 & 0 & 0 \\ -0.253 & 0 & 0 & 1.000 & -0.253 & 0 & -0.253 & 0 & 0 \\ 0 & -0.230 & 0 & -0.276 & 1.000 & -0.276 & 0 & -0.229 & 0 \\ 0 & 0 & -0.211 & 0 & -0.295 & 1.000 & 0 & 0 & -0.211 \\ 0 & 0 & 0 & -0.281 & 0 & 0 & 1.000 & -0.225 & 0 \\ 0 & 0 & 0 & 0 & -0.258 & 0 & -0.247 & 1.000 & -0.247 \\ 0 & 0 & 0 & 0 & 0 & -0.238 & 0 & -0.267 & 1.000 \end{pmatrix}$$

The results of example 4.2 are displayed in Table 2 as follows.

Table 1. Result of Spectral Radiuses of G_{SOR}, G_1 and G_2 Iterative Matrices for Example 4.1

ω	$\rho(G_{SOR})$	$\rho(G_1)$	$\rho(G_2)$
0.1	0.9497367332	0.9343841025	0.9294898941
0.2	0.8966535877	0.8664813907	0.8563024920
0.3	0.8403551407	0.7960535419	0.7801233797
0.4	0.7803331920	0.7228051902	0.7005500711
0.5	0.7159112293	0.6463595589	0.6170462734
0.6	0.6461456054	0.5662177654	0.5288570679
0.7	0.5696315402	0.4816855287	0.4348316650
0.8	0.4840647590	0.3917272446	0.3329801041
0.9	0.3850038057	0.2946323649	0.2189565183

Table 2. Result of Spectral Radiuses of G_{SOR}, G_1 and G_2 Iterative Matrices for Example 4.2

ω	$\rho(G_{SOR})$	$\rho(G_1)$	$\rho(G_2)$
0.1	0.9704634607	0.9620727092	0.9587537313
0.2	0.9385935809	0.9213681941	0.9142487503
0.3	0.9040263188	0.8774831690	0.8659661402
0.4	0.8662980616	0.8299100531	0.8132363784
0.5	0.8248024823	0.7779928048	0.7551667064
0.6	0.7787177235	0.7208539881	0.6905110757
0.7	0.7268771927	0.6572651779	0.6174182078
0.8	0.6675138755	0.5853927211	0.5328622455
0.9	0.5976821338	0.5022225352	0.4310404963

Table 2 goes further to confirm the efficiency of the preconditioned iterations by revealing that the spectral radiuses of the preconditioned iterative matrices G_1 and G_2 are less than the spectral radius of the SOR iterative matrix G_{SOR}. That is, $(\rho(G_2) < \rho(G_1) < \rho(G_{SOR}) < 1)$, for all values of relaxation parameter ω.

5. Conclusions

In this research work, a preconditioning matrix is introduced. Two different forms of the preconditioned SOR-type iterations are formulated for the preconditioner. Some theorems are proposed and proven in order to establish the validity and efficiency of the preconditioned iterations. The preconditioned iterations are shown to satisfy standard convergence criteria under mild conditions imposed on the coefficient matrix of the linear system. Based on the results obtained, it is instructive to conclude that the preconditioned SOR iterative methods presented in this research work provide better and faster convergence properties than the SOR.

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