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An Alternative Approach to Solutions of Nonlinear Two-Point Boundary Value Problems

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ABSTRACT

We treat variational iterative method as an alternative technique for solving nonlinear two-point boundary value problems without the need of establishing a variational formulation for the boundary value problems. This method is based on Lagrange multipliers for the identification of optimal values of parameters in a functional and produces a sequence which provides an approximation to the solution. The numerical results of the solved problems compared with the analytical solutions confirm the ability of the method to produce good approximate results.

Keywords : *variational iteration method, nonlinear boundary value problems, lagrange multipler, correction functional, exact solution*

1. INTRODUCTION

Many problems in physical phenomena are stated as boundary value problems. Two-point boundary value problems occur in a wide variety of problems such as modeling of chemical reactions, the boundary layer theory in fluid mechanics and heat power transmission. The wide applicability of boundary value problems in engineering and sciences calls for faster and accurate numerical methods.

The variational iteration method has been widely used to handle nonlinear models and possesses the ability to solve nonlinear equations accurately and conveniently. The method is valid not only for weakly nonlinear equations but also for strongly nonlinear ones, and the solutions obtained are valid for the whole solution domain [1].
Wazwaz [2], [3] applied the method to solve nonlinear partial differential equations and nonlinear diffusion equations. Uremen and Yildirim [4] obtained exact solutions of the Poisson equations. He [5] used this method to have a series of nonlinear differential equations using an iterative formula.

Torvattanabun and Koonprasert [6] employ the method for solving eight-order boundary value problems; also Onur and Aysegül [7] used it in solving a class of nonlinear differential equations.

In the past and recent literature, several authors have studied boundary value problems and proposed some numerical solutions. These include Ha. [8] who employed shooting technique to explicitly construct the approximate solution of two-point boundary-value problem with fourth order Runge-Kutta method by reducing the second-order boundary-value problem to a system of first-order equations, Jafri et al. [9] considered solving directly two-point boundary-value problem for second-order ordinary differential equations using multistep method in terms of backward difference formula. Also, Attili and Syam [10] proposed an efficient shooting method for solving two-point boundary-value problem using the Adomian Decomposition

Method

In this paper, we use variational iteration method to find the approximate analytical solution of the nonlinear two-point boundary-value problems of the form:

$$u'' = f(t, u, u'), \quad t \in [a, b] \tag{1}$$

with boundary conditions:

$$u(a) = \alpha, \quad u(b) = \beta \tag{2}$$

where a, b, α, β are given constants and f is viewed as a nonlinear function of u and u'.

2. VARIATIONAL ITERATION METHOD

We consider the following general differential equation

$$Lu + Nu = g(t) \tag{3}$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is a continuous function. The basic of the method [11] is to construct a correction function for system (1), which reads

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(s) + Nu_n(s) - g(s))ds \tag{4}$$

where λ is the Lagrangian multiplier [11] which can be identified optimally via variational theory.

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n is the nth approximation, $\tilde{u}_n$ is considered as the restricted variation, i.e $\delta\tilde{u}_n = 0$

The Lagrangian multiplier, λ is first determined optimally followed by the successive approximation u_{n+1}, $n \geq 0$ of the solution u which will be readily obtained by using the determined Lagrange multiplier and any selective or trial function u_0 involving unknown parameter(s) which can be identified by the associated boundary conditions after first iteration. The accuracy of this method depends upon the choice of the trial function. The solution is given by $u = \lim_{n\to\infty} u_n$

3. NUMERICAL APPLICATION

3.1 Example 3.1

Now, we consider the following nonlinear problem with Neumann conditions [12]

$$u'' = u^3 - uu' \tag{5}$$

with boundary conditions

$$u'(0) = -1, \quad u'(1) = -\frac{1}{4} \tag{6}$$

The exact solution u_e, is

$$u_e = \frac{1}{t+1} \tag{7}$$

The correction functional is given as:

$$u_{n+1} = u_n + \int_0^t \lambda(s)\left(u_n''(s) - u_n^3(s) + u_n(s)u_n'(s)\right)ds \tag{8}$$

where the Lagrangian multiplier is identified as

$\lambda(s) = s - t$ after making the correction functional stationary.

This leads to the iteration formulation

$$u_{n+1} = u_n + \int_0^t (s-t)\left(u_n''(s) - u_n^3(s) + u_n(s)u_n'(s)\right)ds \tag{9}$$

Using the selective or trial function

$$u_0 = 1 - t + \frac{1}{2!}at^2 - \frac{1}{3!}bt^3 \tag{10}$$

we have

$$\begin{aligned}u_1 = {} & 1 - t + t^2 - \left(\frac{2}{3} + \frac{1}{6}a\right)t^3 + \left(\frac{1}{4} + \frac{1}{4}a + \frac{1}{24}b\right)t^4 - \\ & \left(\frac{1}{20} + \frac{3}{20}a + \frac{1}{40}a^2 + \frac{7}{120}b\right)t^5 + \\ & \left(\frac{1}{72}ab + \frac{1}{20}a + \frac{1}{40}a^2 + \frac{1}{30}b\right)t^6 - \\ & \left(\frac{1}{84}ab + \frac{1}{504}b^2 + \frac{1}{56}a^2 + \frac{1}{84}b\right)t^7 + \\ & \left(\frac{1}{112}ab + \frac{1}{672}b^2 + \frac{1}{448}a^3\right)t^8 - \\ & \left(\frac{1}{576}a^2b + \frac{1}{864}b^2\right)t^9 + \frac{1}{2160}ab^2t^{10} \\ & - \frac{1}{23760}b^3t^{11}\end{aligned} \tag{11}$$

Using the boundary conditions to solve, we have

$$a = 2.310123975, \quad b = 3.120247950$$

Consequently, we have the following approximants

$$u_0 = 1 - t + 1.155061988t^2 - 0.5200413250t^3$$

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$$u_1 = 1 - t + t^2 - 1.051687329t^3 + 0.9575413250t^4 - 0.7119498794t^5 + 0.4530446110t^6 - 0.2375723166t^7 + 0.1063652804t^8 - 0.04017773232t^9 + 0.01041261352t^{10} - 0.001278559323t^{11} \quad (12)$$

$$u_2 = 1 - t + t^2 - t^3 + 1.102921832t^4 - 1.009598830t^5 + 0.9596213535t^6 - 0.866580273t^7 + 0.7454316787t^8 - 0.6129524339t^9 + 0.4827825348t^{10} - 0.3644471225t^{11} + 0.2640729268t^{12} - 0.1840240094t^{13} + 0.1234968611t^{14} - 0.07985518916t^{15} + 0.04973468896t^{16} - 0.02978943536t^{17} + 0.01711539207t^{18} - 9.400336725t^{19} \times 10^{-3} + 4.915941660t^{20} \times 10^{-3} - 2.437576509t^{21} \times 10^{-3} + 1.141100006t^{22} \times 10^{-3} - 5.02014084t^{23} \times 10^{-4} + 2.065265004t^{24} \times 10^{-4} - 7.901129344t^{25} \times 10^{-5} + 2.792589394t^{26} \times 10^{-5} - 9.04022006t^{27} \times 10^{-6} + 2.655110390t^{28} \times 10^{-6} - 6.96321305t^{29} \times 10^{-7} + 1.597762388t^{30} \times 10^{-7} - 3.109858365t^{31} \times 10^{-8} + 4.899136974t^{32} \times 10^{-9} - 5.804083069t^{33} \times 10^{-10} + 4.551241808t^{34} \times 10^{-11} - 1.756368719t^{35} \times 10^{-12} \quad (13)$$

$$u_3 = 1 - t + t^2 - t^3 + t^4 - 1.002584366t^5 + 1.004045627t^6 - 0.9999804659t^7 + 0.9820420332t^8 - 0.9493566019t^9 + 0.9027569268t^{10} - 0.8445414212t^{11} + 0.7776289967t^{12} - 0.7051221832t^{13} + 0.6300058084t^{14} - 0.5549393907t^{15} + 0.4821453464t^{16} - 0.4133640929t^{17} + 0.3498538444t^{18} - 0.2924203823t^{19} + 0.2414665844t^{20} - 0.1970554435t^{21} + 0.1589806920t^{22} - 0.1268388503t^{23} + 0.100969597t^{24} - 0.07815157025t^{25} + 0.06037651448t^{26} - 0.04615883855t^{27} + 0.03492366594t^{28} - 0.02614957559t^{29} + 0.01937640860t^{30} - 0.01420738634t^{31} + 0.01030725137t^{32} - 0.007397830430t^{33} + 0.005252134178t^{34} - 0.003687810631t^{35} + 0.002560533312t^{36} - 0.001757708591t^{37} + 0.001192731542t^{38} - 7.999129422t^{39} \times 10^{-4} + 5.301177819t^{40} \times 10^{-4} - 3.471026283t^{41} \times 10^{-4} + 2.245051869t^{42} \times 10^{-4} - 1.434186021t^{43} \times 10^{-4} + 9.047385258t^{44} \times 10^{-5} - 5.635167535t^{45} \times 10^{-5} + 3.464842584t^{46} \times 10^{-5} - 2.102705725t^{47} \times 10^{-5} + 1.259259080t^{48} \times 10^{-5} - 7.440700293t^{49} \times 10^{-6} + 4.337023254t^{50} \times 10^{-6} - 2.493226659t^{51} \times 10^{-6} + 1.413291992t^{52} \times 10^{-6} - 7.897771536t^{53} \times 10^{-7} + 4.349866945t^{54} \times 10^{-7} - 2.360670578t^{55} \times 10^{-7} + 1.262015154t^{56} \times 10^{-7} - 6.644117007t^{57} \times 10^{-8} + 3.44364995t^{58} \times 10^{-8} - 1.756567369t^{59} \times 10^{-8} + 8.815005813t^{60} \times 10^{-9} - 4.350399491t^{61} \times 10^{-9} + 2.110626908t^{62} \times 10^{-9} - 1.006205235t^{63} \times 10^{-9} + 4.711518033t^{64} \times 10^{-10} - 2.165859212t^{65} \times 10^{-10} + 9.769663010t^{66} \times 10^{-11} - 4.321980719t^{67} \times 10^{-11} + 1.874140946t^{68} \times 10^{-11} - 7.961379080t^{69} \times 10^{-12} + 3.11150073t^{70} \times 10^{-12}$$

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$$-1.347406564t^{71}\times10^{-12}+5.361151250t^{72}\times10^{-13}$$
$$-2.084621794t^{73}\times10^{-13}+7.911514420t^{74}\times10^{-14}$$
$$-2.929813233t^{75}\times10^{-14}+1.057627072t^{76}\times10^{-15}$$
$$-3.718380932t^{77}\times10^{-15}+1.272021363t^{78}\times10^{-15}$$
$$-4.229758178t^{79}\times10^{-16}+1.365669956t^{80}\times10^{-16}$$
$$-4.276376658t^{81}\times10^{-17}+1.297042913t^{82}\times10^{-17}$$
$$-3.805225514t^{83}\times10^{-18}+1.078195914t^{84}\times10^{-19}$$
$$-2.945680084t^{85}\times10^{-19}+7.745504314t^{86}\times10^{-20}$$
$$-1.956186926t^{87}\times10^{-20}+4.734660132t^{88}$$
$$\times10^{-21}-1.095447254t^{89}\times10^{-21}+$$
$$2.415963099t^{90}\times10^{-22}-5.062897384t^{91}$$
$$\times10^{-23}+1.004489700t^{92}\times10^{-23}$$
$$-1.879035549t^{93}\times10^{-24}+3.298430435t^{94}$$
$$\times10^{-25}-5.403566781t^{95}\times10^{-26}+$$
$$8.208780565t^{96}\times10^{-27}-1.147721103t^{97}$$
$$\times10^{-27}+1.463757160t^{98}\times10^{-28}$$
$$-1.684574553t^{99}\times10^{-29}+1.726412601t^{100}$$
$$\times10^{-30}-1.549523511t^{101}\times10^{-31}+$$
$$1.191994662t^{102}\times10^{-32}-7.633053558t^{103}$$
$$\times10^{-34}+3.902051057t^{104}\times10^{-35}$$
$$-1.491365092t^{105}\times10^{-36}+3.784315963t^{106}$$
$$\times10^{-38}-4.777024158t^{107}\times10^{-40} \quad (14)$$

3.2 Example 3.2

Consider the nonlinear BVP [13]:

$$u''=\frac{3}{2}u^2 \quad (15)$$

with boundary conditions

$$u(0)=1, \quad u(1)=4 \quad (16)$$

The exact solution u_e, is

$$u_e=\frac{4}{(2-t)^2} \quad (17)$$

The correction functional is given as:

$$u_{n+1}=u_n+\int_0^t \lambda(s)\left(u_n''(s)-\frac{3}{2}u_n^2(s)\right)ds \quad (18)$$

where the Lagrangian multiplier is identified as $\lambda(s)=s-t$ after making the correction functional stationary.

The iteration formulation becomes:

$$u_{n+1}=u_n+\int_0^t (s-t)\left(u_n''(s)-\frac{3}{2}u_n^2(s)\right)ds \quad (19)$$

Using the selective or trial function

$$u_0=1+t+\frac{1}{2!}at^2+\frac{1}{3!}bt^3 \quad (20)$$

we have

$$u_1=1+t+\frac{3}{4}t^2+\frac{1}{2}t^3+\left(\frac{1}{8}+\frac{1}{8}a\right)t^4+$$
$$\left(\frac{1}{40}b+\frac{3}{40}a\right)t^5+\left(\frac{1}{80}a^2+\frac{1}{60}b\right)t^6+$$
$$\frac{1}{168}abt^7+\frac{1}{1344}b^2t^8 \quad (21)$$

Using the boundary conditions, we have

$$a=0.19177723, \quad b=11.42466831$$

Consequently, we have the following approximants

$$u_0=1+t+0.095888615t^2+1.904111385t^3 \quad (22)$$

$$u_1=1+t+0.75t^2+0.5t^3+0.1489721538t^4$$
$$+0.3t^5+0.1908708698t^6+0.0130416145t^7 \quad (23)$$
$$+0.0971153616t^8$$

$$u_1 = 1 + t + 0.75t^2 + 0.5t^3 + 0.3125t^4 + 0.1875t^5 + 9.302221543t^6 \times 10^{-2} + 3.888313383t^7 \times 10^{-2} + 3.897857049t^8 \times 10^{-2} + 2.097494007t^9 \times 10^{-2} + 1.381334931t^{10} \times 10^{-2} + 6.737008733t^{11} \times 10^{-3} + 3.472540298t^{12} \times 10^{-3} + 2.072341907t^{13} \times 10^{-3} + 6.022276935t^{14} \times 10^{-4} + 4.517696112t^{15} \times 10^{-4} + 2.327691926t^{16} \times 10^{-4} + 1.396920345t^{17} \times 10^{-4} + 4.62323207t^{18} \times 10^{-5} \quad (24)$$

$$u_2 = 1 + t + 0.75t^2 + 0.5t^3 + 0.3125t^4 + 0.1875t^5 + 0.109375t^6 + 0.0625t^7 + 3.428020797t^8 \times 10^{-2} + 1.869801538t^9 \times 10^{-2} + 1.033928370t^{10} \times 10^{-2} + 5.705447281t^{11} \times 10^{-3} + 3.184035488t^{12} \times 10^{-3} + 1.76163364t^{13} \times 10^{-3} + 9.659322847t^{14} \times 10^{-4} + 5.26300135t^{15} \times 10^{-4} + 2.781981520t^{16} \times 10^{-4} + 1.465379203t^{17} \times 10^{-4} + 7.647650261t^{18} \times 10^{-5} + 3.89750073t^{19} \times 10^{-5} + 1.986058047t^{20} \times 10^{-5} + 9.834750589t^{21} \times 10^{-6} + 4.865309614t^{22} \times 10^{-6} + 2.343845929t^{23} \times 10^{-6} + 1.102991587t^{24} \times 10^{-6} + 5.297530152t^{25} \times 10^{-7} + 2.399769810t^{26} \times 10^{-7} + 1.096094553t^{27} \times 10^{-7} + 4.998432758t^{28} \times 10^{-8} + 2.030394599t^{29} \times 10^{-8} + 9.169677305t^{30} \times 10^{-9} + 3.596365988t^{31} \times 10^{-9} + 1.306310318t^{32} \times 10^{-9} + 5.948688479t^{33} \times 10^{-10} + 1.638776093t^{34} \times 10^{-10} + 6.085199481t^{35} \times 10^{-11} + 2.585483154t^{36} \times 10^{-11} + 1.454569131t^{37} \times 10^{-12} + 2.280328043t^{38} \times 10^{-12} \quad (25)$$

$$u_3 = 1 + t + 0.75t^2 + 0.5t^3 + 0.3125t^4 + 0.1875t^5 + 0.109375t^6 + 0.0625t^7 + 3.515624998t^8 \times 10^{-2} + 1.953125001t^9 \times 10^{-2} + 1.071298610t^{10} \times 10^{-2} + 5.812758364t^{11} \times 10^{-3} + 3.130801535t^{12} \times 10^{-3} + 1.677834789t^{13} \times 10^{-3} + 8.96797455t^{14} \times 10^{-4} + 4.785857536t^{15} \times 10^{-4} + 2.550967871t^{16} \times 10^{-4} + 1.358235846t^{17} \times 10^{-4} + 7.217276471t^{18} \times 10^{-5} + 3.825587521t^{19} \times 10^{-5} + 2.021895957t^{20} \times 10^{-5} + 1.064735883t^{21} \times 10^{-5} + 5.585163180t^{22} \times 10^{-6} + 2.916828027t^{23} \times 10^{-6} + 1.516460940t^{24} \times 10^{-6} + 7.846663260t^{25} \times 10^{-7} + 4.040100222t^{26} \times 10^{-7} + 2.070522667t^{27} \times 10^{-7} + 1.055936877t^{28} \times 10^{-7} + 5.359311245t^{29} \times 10^{-8} + 2.707334255t^{30} \times 10^{-8} + 1.360752992t^{31} \times 10^{-8} + 6.806501998t^{32} \times 10^{-9} + 3.387411362t^{33}$$

$$
\begin{aligned}
&\times 10^{-9} + 1.677013062t^{34} \times 10^{-9} + \\
&8.261151907t^{35} \times 10^{-10} + 4.047597056t^{36} \\
&\times 10^{-10} + 1.972628343t^{37} \times 10^{-10} + \\
&9.562918179t^{38} \times 10^{-11} + 4.610051883t^{39} \\
&\times 10^{-11} + 2.210245849t^{40} \times 10^{-11} + \\
&1.053631415t^{41} \times 10^{-11} + 4.994142494t^{42} \\
&\times 10^{-12} + 2.352796316t^{43} \times 10^{-12} + \\
&1.101618154t^{44} \times 10^{-12} + 5.124773075t^{45} \\
&\times 10^{-12} + 2.367486727t^{46} \times 10^{-13} \\
&1.086107266t^{47} \times 10^{-13} + 4.944651301t^{48} \\
&\times 10^{-14} + 2.233401880t^{49} \times 10^{-14} + \\
&1.000631449t^{50} \times 10^{-14} + 4.442859188t^{51} \\
&\times 10^{-15} + 1.955352239t^{52} \times 10^{-15} + \\
&8.522438509t^{53} \times 10^{-16} + 3.676870733t^{54} \\
&\times 10^{-16} + 1.570551820t^{55} \times 10^{-16} + \\
&6.629693892t^{56} \times 10^{-17} + 2.768139617t^{57} \\
&\times 10^{-17} + 1.141363443t^{58} \times 10^{-17} + \\
&4.640592706t^{59} \times 10^{-18} + 1.864515888t^{60} \\
&\times 10^{-18} + 7.358625748t^{61} \times 10^{-19} + \\
&2.862487858t^{62} \times 10^{-19} + 1.096064189t^{63} \\
&\times 10^{-19} + 4.091823055t^{64} \times 10^{-20} + \\
&1.510494031t^{65} \times 10^{-20} + 5.413984098t^{66} \\
&\times 10^{-21} + 1.884136776t^{67} \times 10^{-21} + \\
&6.563755854t^{68} \times 10^{-22} + 2.110322014t^{69} \\
&\times 10^{-22} + 6.803873040t^{70} \times 10^{-23} + \\
&2.140051995t^{71} \times 10^{-23} + 5.828993169t^{72} \\
&\times 10^{24} + 1.808325040t^{73} \times 10^{-24} + \qquad (26) \\
&4.423052533t^{74} \times 10^{-25} + 9.533521616t^{75} \\
&\times 10^{-26} + 3.158704257t^{76} \times 10^{-26} + \\
&1.700390354t^{77} \times 10^{-27} + 1.298675346t^{78} \\
&\times 10^{-27}
\end{aligned}
$$

4. RESULTS AND DISCUSSION

The VIM is used to find the numerical solution of nonlinear two-point boundary value problems. The main advantage is its fast convergence to the solution,

Table 1 shows the approximate solutions for Example 3.1 and the error estimates at n=2. From the numerical results in Table 1 it is clear that the approximate solutions are in high agreement with the exact solutions.

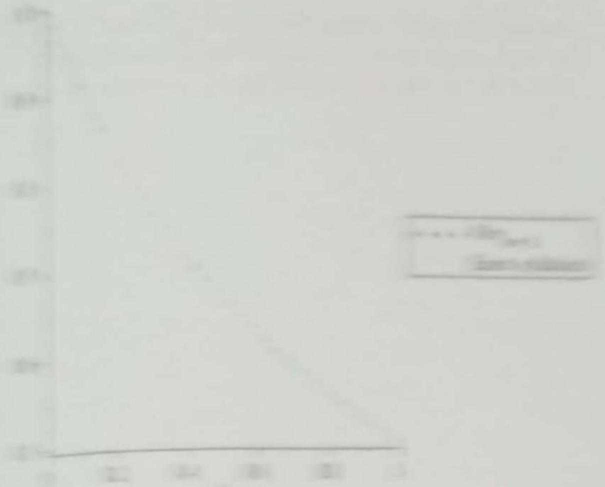
Table 2 indicates the approximate solution of example 3.2 and the error estimates for VIM at n=3, a good agreement is observed between the numerical solution and the exact solution.

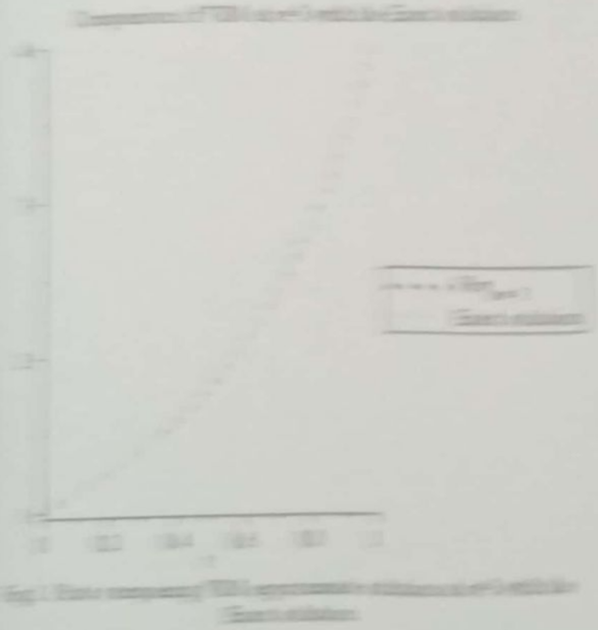
Table 1: Error estimates for vim at n=2

t	u_e	$u_{vim_{n=2}}$	Errors
0	1.0000000000	1.0000000000	0
0.1	0.9090909091	0.9090908882	2.09E-8
0.2	0.8333333333	0.8333328019	5.31 E-7
0.3	0.7692307692	0.7692276075	3.1617 E-6
0.4	0.7142857143	0.7142753601	1.03542 E-5
0.5	0.6666666667	0.6666422807	2.4386 E-5
0.6	0.6250000000	0.6249535039	4.64961 E-5
0.7	0.5882352941	0.5881589553	7.63388 E-5
0.8	0.5555555556	0.5554437911	1.117644 E-4
0.9	0.5263157895	0.5261669509	1.488386 E-4
1	0.5000000000	0.4998179728	1.820272 E-4

Table 2: Error estimates for vim at n=3

t	u_e	$u_{vim_{n=3}}$	Errors
0	1.000000000	1.000000000	0
0.1	1.108033241	1.108033241	0
0.2	1.234567901	1.234567901	0
0.3	1.384083045	1.384083044	1 E-9
0.4	1.562500000	1.562499994	6 E-9
0.5	1.777777778	1.777777711	6.7 E-8
0.6	2.040816326	2.040815825	5.01 E-7
0.7	2.366863905	2.366861069	2.869 E-6
0.8	2.777777778	2.777764648	1.313 E-5
0.9	3.305785124	3.305732941	5.2183 E-5
1	4.000000000	3.999816108	1.83892 E-4





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