Optimization
Iranian Journal of Optimization
Volume 13, Issue 2, 2021, 81-92
Research Paper

Online version is available on: www.ijo.rasht.iau.ir

Islamic Azad University
Rasht Branch
ISSN: 2588-5723
E-ISSN: 2008-5427

A Backward Diffrention Formula For Third-Order Inttial or Boundary Values Problems Using Collocation Method

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Revise Date: 19 June 2021
Accept Date: 19 September 2021

Abstract

We propose a new self-starting sixth-order hybrid block linear multistep method using backward differentiation formula for direct solution of third-order differential equations with either initial conditions or boundary conditions. The method used collocation and interpolation techniques with three off-step points and five-step points, choosing power series as the basis function. The convergence of the method is established, and three numerical experiments of initial and boundary value problems are used to demonstrate the efficiency of the proposed method. The numerical results in Tables and Figures show the efficiency of the method. Furthermore, the numerical method outperformed the results from existing literature in terms of accuracy as evident in the results of absolute errors produced.

Keywords:
Initial Value problems
Boundary Value Problems
Third Order ODE
Backward Differentiation Formula
Hybrid Linear Multistep

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INTRODUCTION

Initial and boundary value third-order ordinary differential equations (ODE) emerges in the modelling of real-life situations in areas of physical and applied sciences especially in mathematical representations of problems in electromagnetic waves, gravity-driven flows, problems related to thin films flow of viscous fluid, and quantum mechanics (Ahmed, 2017; Bernis & Peletier, 1996; Guo & Tsai, 2005; Morlando, 2017; Tuck & Schwartz, 1990). Solutions to this kind of problem is important in understanding the properties, and behaviours of the physical phenomenon under study (Cole & Tiamiyu, 2019). Most often, available analytical methods fail, in most cases, in finding exact solution to a general third-order ODE. Hence, numerical methods which find approximate solution to the equations is crucial in finding solution to the initial and boundary value problems of third-order ODE arising in computational fluid dynamics, sciences and engineering (Tiamiyu et al., 2021).

In this research, we propose a backward differentiation formula of hybrid block linear multistep method to the third-order initial or boundary value problems of the form;

$$D\{y(x)\} = f\left(x, y(x), \frac{dy(x)}{dx}, \frac{d^2y(x)}{dx^2}\right), \quad X \in (x_0, x_N) \qquad (1)$$

Couple with any of the initial or boundary conditions:

$$y(x_0) = y_0, \quad \frac{dy(x_0)}{dx} = z_0, \quad \frac{d^2y(x_0)}{dx^2} = w_0 \qquad (2)$$

$$y(x_0) = y_0, \quad \frac{dy(x_0)}{dx} = z_0, \quad y(x_N) = y_N \qquad (3)$$

$$y(x_0) = y_0, \quad \frac{dy(x_0)}{dx} = z_0, \quad \frac{dy(x_N)}{dx} = z_N \qquad (4)$$

where $D = \frac{d^3}{dx^3}, x_0, y_0, z_0, w_0, x_N, y_N,$ and $z_N \in \mathbb{R}$, $y(x) \in \mathbb{R}^n$ and f is a continuous-valued function. The Eq. 1 with 2 is a third-order initial value problem, while Eq. 1 coupled with either 3 or 4 is a third-order boundary value problem. Eq. 1 and 3 will be termed boundary value problem of type 1 while Eq. 1 and 4 will be termed boundary value problem of type 2.

In this research, we will focus on numerical solution of 1 at three different cases of 2, 3 and 4. Various researchers have addressed from the three cases of 1 and 2, 3 or 4. The theory of third order-ordinary differential equations was presented in Padhi & Pati (2014), by analysing the existence and uniqueness of solutions at different cases. Semi-analytic and numerical solutions of the initial value problem of type 1 with 2 have been studied by a number of authors including Adeyeye & Zurni (2019), Agboola et al. (2015), Allogmany & Ismail (2020), Saqlain et al. (2018), and Sunday (2018). Likewise, various authors like Abdulsalam (2019), Ahmed (2017), Jator (2009), Khan & Aziz (2002) and Taha & Khlefha (2015) have also worked on the boundary value problems coupled with different types of boundary conditions. Methods of solution utilized by these authors include finite difference method, variational iterative method, differential transformation method, Runge-Kutta method, and linear multistep method. Mohammed et al. (2019) developed a three-step hybrid linear multistep with one-off step collocation point for direct solution of boundary value problems using collocation approach. However, the limitations of these methods are the low degree of accuracy and low step number. The present study is triggered to address the limitations in the methods in the literature by increasing the step number in both grid and off-step points.

In this research, we develop a continuous five-step hybrid linear multistep method (HLMM), with three off-step points, for direct solution of 1 coupled with one of the conditions 2, 3 or 4. The new method will be self-starting and with higher order of accuracy to obtain a more efficient method than the available methods in the literature. Basic properties of the method will be analysed for stability and convergence. The method will handle efficiently both initial and boundary value problems of 1 - 4.

A brief introduction is presented in section 1, formulation and development of a continuous HLMM is provided in section 2, analysis of basic properties of the method is presented in section 3, experimentation with some numerical problems is presented in section 4, and the research will be concluded in section 5.

METHODOLOGY/DEVELOPMENT OF THE METHOD

In this section, we formulate and develop a numerical approximation to 1 by using power of the variable x as the basis functions and applying collocation technique. Consider the finite power series

$$y(x) = \sum_{j=0}^{m+n-1} a_j x^j \qquad (5)$$

where m is the number of collocation points, n is the number of interpolating points, $x \in [x_0, x_N]$, and a_js are unknown coefficients. The third derivative of 5 is

$$y'''(x) = \sum_{j=3}^{m+n-1} j(j-1)(j-2)a_j x^{j-3} \qquad (6)$$

Comparing 1 and 6, then 1 becomes

$$f\left(x, y(x), \frac{dy(x)}{dx}, \frac{d^2y(x)}{dx^2}\right) = \sum_{j=3}^{m+n-1} j(j-1)(j-2)a_j x^{j-3} \qquad (7)$$

We specify the method with $m = 1$, and $n = 9$. Interpolating 5 at $x = x_{n+j}; j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, 4$ and collocating 7 at $x = x_{n+5}$ to obtain a 9×9 system of equations. The system of equations is further expressed in matrix form to obtain the solution of the unknown coefficients. The solution of the unknown coefficients $\alpha_{n+j}; j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, 4$ and β_5 is obtain through matrix inversion method.

$$\alpha_0 = y_n,$$

$$\alpha_{\frac{1}{2}} = -\frac{12206443}{2302020}\frac{y_n}{h} + \frac{189824}{12789}\frac{y_{n+\frac{1}{2}}}{h} - \frac{2713120}{115101}\frac{y_{n+1}}{h} + \frac{64768}{2349}\frac{y_{n+\frac{3}{2}}}{h} - \frac{274055}{12789}\frac{y_{n+2}}{h} + \frac{5867392}{575505}\frac{y_{n+\frac{5}{2}}}{h} - \frac{275216}{115101}\frac{y_{n+3}}{h} + \frac{5501}{65772}\frac{y_{n+4}}{h} - \frac{20}{38367}h^2 f_{n+5}$$

$$\alpha_1 = \frac{50794333}{4604040}\frac{y_n}{h^2} - \frac{3134048}{63945}\frac{y_{n+\frac{1}{2}}}{h^2} + \frac{11670548}{115101}\frac{y_{n+1}}{h^2} - \frac{1500736}{11745}\frac{y_{n+\frac{3}{2}}}{h^2} + \frac{5261863}{51156}\frac{y_{n+2}}{h^2} - \frac{28743776}{575505}\frac{y_{n+\frac{5}{2}}}{h^2} + \frac{6831032}{575505}\frac{y_{n+3}}{h^2} - \frac{277429}{657720}\frac{y_{n+4}}{h^2} + \frac{103}{38367}h f_{n+5}$$

$$\alpha_{\frac{3}{2}} = -\frac{1773601}{147987}\frac{y_n}{h^3} + \frac{1080256}{16443}\frac{y_{n+\frac{1}{2}}}{h^3} - \frac{23462435}{147987}\frac{y_{n+1}}{h^3} + \frac{4625216}{21141}\frac{y_{n+\frac{3}{2}}}{h^3} - \frac{3037225}{16443}\frac{y_{n+2}}{h^3} + \frac{13661056}{147987}\frac{y_{n+\frac{5}{2}}}{h^3} - \frac{3309169}{147987}\frac{y_{n+3}}{h^3} + \frac{17194}{21141}\frac{y_{n+4}}{h^3} - \frac{527}{98658}f_{n+5}$$

$$\alpha_2 = \frac{44513677}{5919480}\frac{y_n}{h^4} - \frac{3817064}{82215}\frac{y_{n+\frac{1}{2}}}{h^4} + \frac{36619231}{295974}\frac{y_{n+1}}{h^4} - \frac{19464304}{105705}\frac{y_{n+\frac{3}{2}}}{h^4} + \frac{10781395}{65772}\frac{y_{n+2}}{h^4} - \frac{62811464}{739935}\frac{y_{n+\frac{5}{2}}}{h^4} + \frac{31195147}{1479870}\frac{y_{n+3}}{h^4} - \frac{669013}{845640}\frac{y_{n+4}}{h^4} + \frac{268}{49329}\frac{f_{n+5}}{h}$$

$$\alpha_{\frac{5}{2}} = -\frac{8316629}{2959740}\frac{y_n}{h^5} + \frac{307360}{16443}\frac{y_{n+\frac{1}{2}}}{h^5} - \frac{15773329}{295974}\frac{y_{n+1}}{h^5} + \frac{1778240}{21141}\frac{y_{n+\frac{3}{2}}}{h^5} - \frac{1292869}{16443}\frac{y_{n+2}}{h^5} + \frac{31321376}{739935}\frac{y_{n+\frac{5}{2}}}{h^5} - \frac{3204983}{295974}\frac{y_{n+3}}{h^5} + \frac{35779}{84564}\frac{y_{n+4}}{h^5} - \frac{305}{98658}\frac{f_{n+5}}{h^2}$$

$$\alpha_3 = \frac{129553}{211410}\frac{y_n}{h^6} - \frac{50312}{11745}\frac{y_{n+\frac{1}{2}}}{h^6} + \frac{270731}{21141}\frac{y_{n+1}}{h^6} - \frac{2233936}{105705}\frac{y_{n+\frac{3}{2}}}{h^6} + \frac{48352}{2349}\frac{y_{n+2}}{h^6} - \frac{1215464}{105705}\frac{y_{n+\frac{5}{2}}}{h^6} + \frac{321113}{105705}\frac{y_{n+3}}{h^6} - \frac{26353}{211410}\frac{y_{n+4}}{h^6} + \frac{7}{7047}\frac{f_{n+5}}{h^3}$$

$$\alpha_4 = -\frac{2549}{35721}\frac{y_n}{h^7} + \frac{2048}{3969}\frac{y_{n+\frac{1}{2}}}{h^7} - \frac{56986}{35721}\frac{y_{n+1}}{h^7} + \frac{1984}{729}\frac{y_{n+\frac{3}{2}}}{h^7} - \frac{10868}{3969}\frac{y_{n+2}}{h^7} + \frac{56384}{35721}\frac{y_{n+\frac{5}{2}}}{h^7} - \frac{15350}{35721}\frac{y_{n+3}}{h^7} + \frac{95}{5103}\frac{y_{n+4}}{h^7} - \frac{2}{11907}\frac{f_{n+5}}{h^4}$$

$$\beta_5 = \frac{17552}{5179545}\frac{y_n}{h^8} - \frac{14432}{575505}\frac{y_{n+\frac{1}{2}}}{h^8} + \frac{82220}{1035909}\frac{y_{n+1}}{h^8} - \frac{14656}{105705}\frac{y_{n+\frac{3}{2}}}{h^8} + \frac{16444}{115101}\frac{y_{n+2}}{h^8} - \frac{436832}{5179545}\frac{y_{n+\frac{5}{2}}}{h^8} + \frac{121748}{5179545}\frac{y_{n+3}}{h^8} - \frac{788}{739935}\frac{y_{n+4}}{h^8} + \frac{4}{345303}\frac{f_{n+5}}{h^8}$$

The continuous HLMM of backward differentiation formula is obtained as;

$$y(x) = \sum_{j=0}^{k-1}\alpha_j(x)\, y_{n+j} + \alpha_{\frac{1}{2}}y_{n+\frac{1}{2}} + \alpha_{\frac{3}{2}}y_{n+\frac{3}{2}} + \alpha_{\frac{5}{2}}y_{n+\frac{5}{2}} + \beta_5 h^3 f_{n+5} \quad (8)$$

where $k = 5$ is the step number of the method. Substituting the results of the unknown coefficients $\alpha_{n+j}; j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, 4$ and β_5 and evaluate 8 at $x = x_{n+5}$ to obtain the discrete scheme as;

$$y_{n+5} = -\frac{27341}{5481}y_n + \frac{25280}{609}y_{n+\frac{1}{2}} - \frac{822025}{5481}y_{n+1} + \frac{238720}{783}y_{n+\frac{3}{2}} - \frac{228350}{609}y_{n+2} + \frac{1499072}{5481}y_{n+\frac{5}{2}} - \frac{542270}{5481}y_{n+3} + \frac{7805}{783}y_{n+4} + \frac{100}{1827}h^3 f_{n+5} \quad (9)$$

We obtain the first and second derivatives of 8 and evaluating at $x = x_{n+j}; j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, 4, 5$. To obtain the sufficient schemes required for the backward differentiation formula of hybrid block LMM, we further obtain the third derivative of 8 and evaluating at $x = x_{n+j}; j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, 3, 4$. Combining all the 24 schemes together to form block method to solve 1, and after simplifications to express $y_{n+j}, z_{n+j}, w_{n+j}; j = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, 4, 5$ as the subject of relations. The hybrid block method for $y_{n+j}; j = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, 4, 5$ is;

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2}hz_n + \frac{1}{8}h^2 w_n + \frac{1610153}{33868800}h^3 f_{n+\frac{1}{2}} - \frac{448501}{5644800}h^3 f_{n+\frac{3}{2}} + \frac{262033}{1612800}h^3 f_{n+\frac{5}{2}} - \frac{216973}{1612800}h^3 f_{n+3} + \frac{161461}{5644800}h^3 f_{n+4} - \frac{128573}{33868800}h^3 f_{n+5} \quad (10)$$

$$y_{n+1} = y_n + hz_n + \frac{1}{2}h^2 w_n + \frac{40807}{132300}h^3 f_{n+\frac{1}{2}} - \frac{9271}{22050}h^3 f_{n+\frac{3}{2}} + \frac{5399}{6300}h^3 f_{n+\frac{5}{2}} - \frac{17863}{25200}h^3 f_{n+3} + \frac{6637}{44100}h^3 f_{n+4} - \frac{10561}{529200}h^3 f_{n+5} \quad (11)$$

$$y_{n+\frac{3}{2}} = y_n + \frac{3}{2}hz_n + \frac{9}{8}h^2 w_n + \frac{1078299}{1254400}h^3 f_{n+\frac{1}{2}} - \frac{117657}{125440}h^3 f_{n+\frac{3}{2}} + \frac{355833}{179200}h^3 f_{n+\frac{5}{2}} - \frac{295677}{179200}h^3 f_{n+3} + \frac{8829}{25088}h^3 f_{n+4} - \frac{58671}{1254400}h^3 f_{n+5} \quad (12)$$

$$y_{n+2} = y_n + 2hz_n + 2h^2w_n + \frac{56738}{33075}h^3f_{n+\frac{1}{2}} - \frac{16364}{11025}h^3f_{n+\frac{3}{2}} + \frac{5434}{1575}h^3f_{n+\frac{5}{2}} - \frac{649}{225}h^3f_{n+3} + \frac{6824}{11025}h^3f_{n+4} - \frac{2729}{33075}h^3f_{n+5} \qquad (13)$$

$$y_{n+\frac{5}{2}} = y_n + \frac{5}{2}hz_n + \frac{25}{8}h^2w_n + \frac{556375}{193536}h^3f_{n+\frac{1}{2}} - \frac{446125}{225792}h^3f_{n+\frac{3}{2}} + \frac{343625}{64512}h^3f_{n+\frac{5}{2}} - \frac{287125}{64512}h^3f_{n+3} + \frac{30875}{32256}h^3f_{n+4} - \frac{173125}{1354752}h^3f_{n+5} \qquad (14)$$

$$y_{n+3} = y_n + 3hz_n + \frac{9}{2}h^2w_n + \frac{21249}{4900}h^3f_{n+\frac{1}{2}} - \frac{837}{350}h^3f_{n+\frac{3}{2}} + \frac{5427}{700}h^3f_{n+\frac{5}{2}} - \frac{17883}{2800}h^3f_{n+3} + \frac{6723}{4900}h^3f_{n+4} - \frac{513}{2800}h^3f_{n+5} \qquad (15)$$

$$y_{n+4} = y_n + 4hz_n + 8h^2w_n + \frac{270112}{33075}h^3f_{n+\frac{1}{2}} - \frac{6592}{2205}h^3f_{n+\frac{3}{2}} + \frac{22688}{1575}h^3f_{n+\frac{5}{2}} - \frac{17392}{1575}h^3f_{n+3} + \frac{5408}{2205}h^3f_{n+4} - \frac{10768}{33075}h^3f_{n+5} \qquad (16)$$

$$y_{n+5} = y_n + 5hz_n + \frac{25}{2}h^2w_n + \frac{69875}{5292}h^3f_{n+\frac{1}{2}} - \frac{2875}{882}h^3f_{n+\frac{3}{2}} + \frac{5875}{252}h^3f_{n+\frac{5}{2}} - \frac{16375}{1008}h^3f_{n+3} + \frac{7625}{1764}h^3f_{n+4} - \frac{10625}{21168}h^3f_{n+5} \qquad (17)$$

where z_{n+j} and w_{n+j} are the first and second derivatives of y_{n+j}.

ANALYSIS

In this section, we present the analyses for consistency and zero stability of the proposed method.

Consistency

The proposed method in section 2 is commonly written as;

$$\sum_{j=0}^{5}\alpha_j y_{n+j} + \alpha_{\frac{1}{2}}y_{n+\frac{1}{2}} + \alpha_{\frac{3}{2}}y_{n+\frac{3}{2}} + \alpha_{\frac{5}{2}}y_{n+\frac{5}{2}} - h^3\beta_5 f_{n+5} = 0 \qquad (18)$$

Following Jator (2009), Mohammed et al (2019) and Tiamiyu et al. (2021), the local truncation error in a linear difference operator is defined as;

$$L[y(x);h] = \sum_{j=0}^{5}\alpha_j y(x+jh) + \alpha_{\frac{1}{2}}y\left(x+\frac{1}{2}h\right) + \alpha_{\frac{3}{2}}y\left(x+\frac{3}{2}h\right) + \alpha_{\frac{5}{2}}y\left(x+\frac{5}{2}h\right) - h^3\beta_5 y'''(x+5h) \qquad (19)$$

We assume that $y(x)$ is adequately differentiable. In Taylor's expansion about the point x, (19) can be expressed as;

$$L[y(x);h] = C_0y(x) + C_1hy'(x) + C_2h^2y''(x) + \cdots + C_ph^py^p(x) + C_{p+1}h^{p+1}y^{p+1}(x) + \cdots \qquad (20)$$

The discrete scheme in (9) is said to be consistent if $p \geq 1$ for $C_0 = C_1 = C_2 = \cdots = C_p = C_{p+1} = C_{p+2} = 0$, $C_{p+3} \neq 0$ is the error constant, and p is the order of the method (Jator, 2009). Computing for the order p and error constant C_{p+3} for (9) gives $(p, C_{p+3}) = \left(6, -\frac{796595}{44198784}\right)$.

Furthermore, we verify the following conditions that guarantee consistency of a linear multistep method (Tiamiyu et al., 2021);

i.) $\sum_{j=0}^{5}\alpha_j + \alpha_{\frac{1}{2}} + \alpha_{\frac{3}{2}} + \alpha_{\frac{5}{2}} = 0$

ii.) $\rho(r) = \rho'(r) = \rho''(r) = \cdots = \rho^{(n-1)}(r) = 0$

iii.) $\rho^n(r) = n!\,\sigma(r)$

where $r = 1$ is the principal root and $n = 3$ is the order of the differential equation, $\rho(r)$

is the first characteristic polynomial, and $\sigma(r)$ is the second characteristic polynomial of the method. From 9 $\alpha_0 = \frac{27341}{5481}$, $\alpha_{\frac{1}{2}} = -\frac{25280}{609}$, $\alpha_1 = \frac{822025}{5481}$, $\alpha_{\frac{3}{2}} = -\frac{238720}{783}$, $\alpha_2 = \frac{228350}{609}$, $\alpha_{\frac{5}{2}} = -\frac{1499072}{5481}$, $\alpha_3 = \frac{542270}{5481}$, $\alpha_4 = -\frac{7805}{783}$, $\alpha_5 = 1, \beta_5 = \frac{100}{1827}$

$$\rho(r) = \frac{27341}{5481} - \frac{25280}{609} r^{\frac{1}{2}} + \frac{822025}{5481} r - \frac{238720}{783} r^{\frac{3}{2}} + \frac{228350}{609} r^2 - \frac{1499072}{5481} r^{\frac{5}{2}} + \frac{542270}{5481} r^3 - \frac{7805}{783} r^4 + r^5$$

$$\sigma(r) = \frac{100}{1827} r^5$$

Eq. 9 satisfied the above conditions. Therefore, the method is consistent.

Zero Stability

We start by normalizing the first characteristic polynomials $\rho(r)$ of the proposed discrete schemes in (10) – (17) as;

$$\rho(r) = \det(rA^0 - A^1) \quad (21)$$

where $|r| \leq 1$ and the roots $|r| = 1$ has a multiplicity not exceeding the order 3 of the differential equation, A^0 is an identity matrix of 8×8 dimension, and A^1 from 10 to 17)is;

$$A^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore,

$$\rho(r) = \det(rA^0 - A^1) = r^7(r-1) \quad (22)$$

Since $|r| \leq 1$, then the proposed block method in 10 to 17 is zero stable.

Convergence

The fundamental theorem of Dahlquist states that, "the necessary and sufficient conditions for a linear multi-step method to be convergent are that it be consistent and zero-stable" (Lambert, 1973). By Adeyeye & Zurni (2019), Allogmany & Ismail (2020) and Tiamiyu et al. (2021), since the proposed hybrid block method is consistent and zero stable, the condition for convergence is satisfied.

NUMERICAL EXPERIMENTS

In this section, some numerical experiments will be performed to validate the efficiency of the proposed method.

Numerical Problems

Problem 1: Initial Value Problem

Consider the third-order initial value problem

$$y'''(x) = -6y''(x) - 11y'(x) - 6y(x)$$
$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0 \quad (23)$$

with the exact solution $y(x) = 3e^{-x} - 3e^{-2x} + e^{-3x}$

Problem 2: Boundary Value Problem of Type 1

Consider the third-order boundary value problem in Abdulsalam & Majid (2019) of type 1

$$-\epsilon y'''(x) + y(x) = 81\epsilon^2 \cos(3x) + 3\epsilon \sin(3x)$$
$$y(0) = 0, \quad y'(0) = 9\epsilon, \quad y(1) = 3\epsilon \sin(3) \quad (24)$$

with the exact solution $y(x) = 3\epsilon \sin(3x)$.

Problem 3: Boundary Value Problem of Type 2

Consider the third-order boundary value problem in Ahmed (2017) of type 2

$$y'''(x) = xy(x) + (x^3 - 2x^2 - 5x - 3)e^x$$
$$y(0) = 0, \quad y'(0) = 1, \quad y'(1) = -e$$

with the exact solution $y(x) = x(1-x)e^x$.

Numerical Results

The numerical results of the selected problems are presented in some Tables and Figures in this section. Exact and numerical solutions are compared in Tables 1, 3 and 6.

Table 1: Exact and Computed Results for Problem 1 at $h = \frac{1}{100}$

n	x_n	Exact Solution	Computed Solution
1	0.1	0.99913821555536510096	0.99913821555366816898
2.	0.2	0.99404375722105410064	0.99404375722112793535
3.	0.3	0.98258941350367341 22	0.98258941350384221868
4	0.4	0.96416745766745552245	0.96416745766774742614
5	0.5	0.93908381577200031349	0.93908381577243467435
6	0.6	0.90815116076705954628	0.90815116076763451979
7	0.7	0.87242144780239102348	0.87242144780310378013
8	0.8	0.83301529165711105223	0.83301529165794875761
9	0.9	0.79101782729678748580	0.79101782729773260849
10	1.0	0.74741954217235283213	0.74741954217338528978

Table 2: Maximum Absolute Error at distinct h for Problem 1

Step Sizes	**Maximum Absolute Error**
$h = \frac{1}{16}$	$6.191973979 \times 10^{-8}$
$h = \frac{1}{32}$	$9.720599132 \times 10^{-10}$
$h = \frac{1}{64}$	$1.508947299 \times 10^{-11}$
$h = \frac{1}{128}$	$2.342843261 \times 10^{-13}$

Table 3: Exact and Computed Results for Problem 2 at $h = \frac{1}{100}$ and $\epsilon = \frac{1}{16}$

n	x_n	Exact Solution	Computed Solution
1	0.1	0.05541003874900117033 3	0.05541003874900896583
2.	0.2	0.10587046376156912948	0.10587046376159386594
3.	0.3	0.14687379555515313534	0.14687379555519484929
4	0.4	0.17475732861885494056	0.17475732861890705160

5	0.5	0.18703030998826020580	0.18703030998831259471
6	0.6	0.18259643078966159747	0.18259643078970191196
7	0.7	0.16185175624666383200	0.16185175624668883217
8	0.8	0.12664934635334079873	0.12664934635334713819
9	0.9	0.080133727543843112730	0.080133727543837784674
10	1.0	0.026460001511225104144	0.026460001511225104155

Table 4: Maximum Absolute Error at distinct h for Problem 2

N	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$
10	$2.172231625 \times 10^{-7}$	$9.332503559 \times 10^{-8}$	$3.679731874 \times 10^{-8}$
20	$1.259856162 \times 10^{-9}$	$5.643907071 \times 10^{-10}$	$2.367721200 \times 10^{-10}$
40	$6.720773823 \times 10^{-12}$	$3.461736523 \times 10^{-12}$	$1.794300141 \times 10^{-12}$
80	$1.871873102 \times 10^{-13}$	$8.103007583 \times 10^{-14}$	$3.260234521 \times 10^{-14}$
100	$5.360673354 \times 10^{-14}$	$2.307602360 \times 10^{-14}$	$9.204725212 \times 10^{-15}$

Table 5: Maximum Absolute Error at distinct h for Problem 2 in Abdulsalam & Majid (2019)

N	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$
10	1.9×10^{-5}	3.7×10^{-5}	7.0×10^{-5}
20	2.6×10^{-6}	5.0×10^{-6}	9.9×10^{-6}
40	3.4×10^{-7}	6.6×10^{-7}	1.3×10^{-6}

Table 6: Exact and Computed Results for Problem 3 at $h = \frac{1}{100}$

n	x_n	Exact Solution	Computed Solution
1	0.1	0.09946538262680828623	0.09946538262680753277 6
2.	0.2	0.19542444130562717342	0.19542444130562429872
3.	0.3	0.28347034959096065184	0.28347034959095453309
4	0.4	0.35803792743390487627	0.35803792743389466775

5	0.5	0.41218031767503203670	0.41218031767501720534
6	0.6	0.43730851209372215398	0.43730851209970251828
7	0.7	0.42288806856880006954	0.42288806856877584396
8	0.8	0.35608654855879481674	0.35608654855876666025
9	0.9	0.22136428000412546974	0.22136428000409454131
10	1.0	0.00000000000000000000	$-3.19782158 \times 10^{-16}$

Table 7: Maximum Absolute Error at Distinct h for Problem 3

Step Sizes	**Maximum Absolute Error**
$h = \frac{1}{16}$	$2.442839703 \times 10^{-9}$
$h = \frac{1}{32}$	$3.331349298 \times 10^{-11}$
$h = \frac{1}{64}$	$4.796951000 \times 10^{-13}$
$h = \frac{1}{128}$	$7.213700000 \times 10^{-15}$

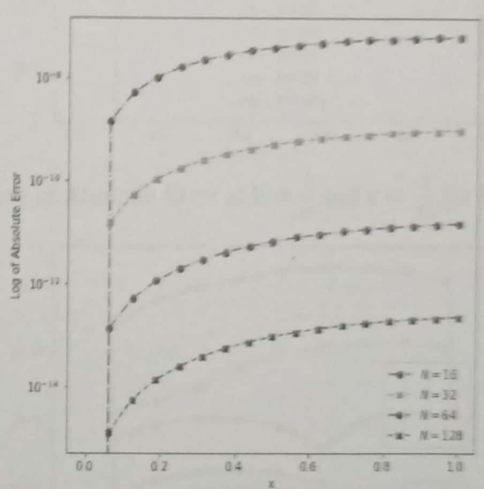

Fig. 1. Log of Absolute Error at $h = \frac{1}{N}$ for Problem 1

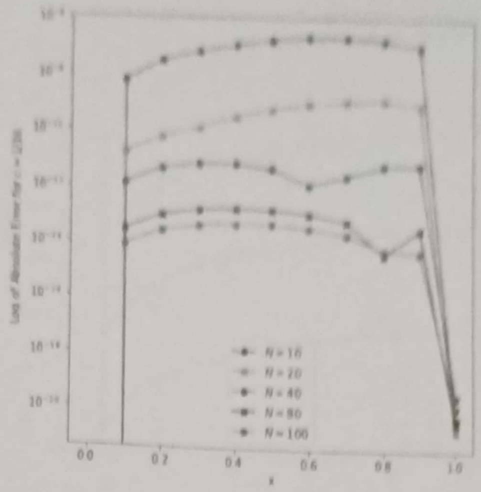

Fig. 2. Log of Absolute Error at $h = \frac{1}{N}$ and $\epsilon = \frac{1}{16}$ for Problem 2

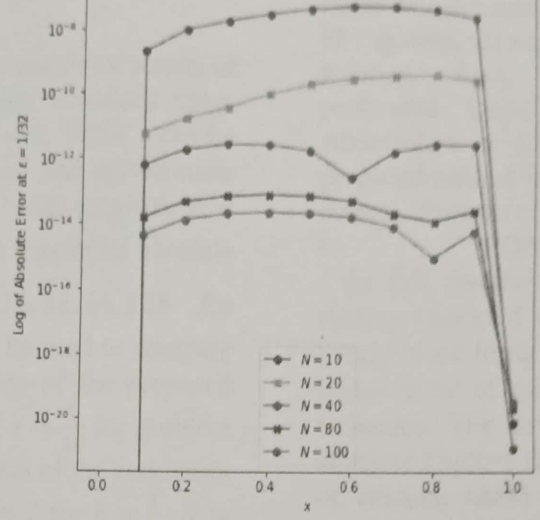

Fig. 3. Log of Absolute Error at $h = \frac{1}{N}$ and $\epsilon = \frac{1}{32}$ for Problem 2

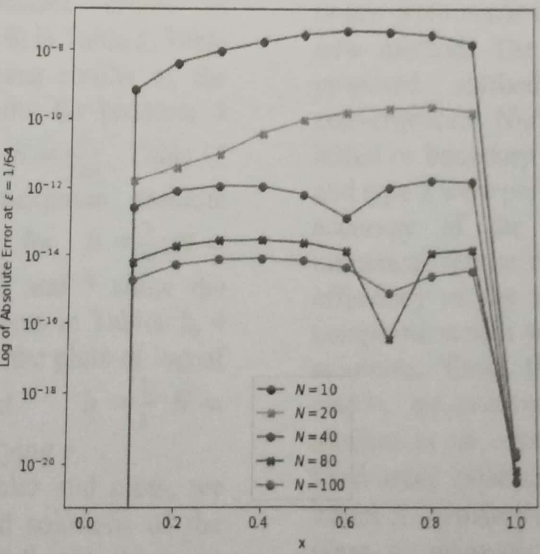

Fig. 4. Log of Absolute Error at $h = \frac{1}{N}$ and $\epsilon = \frac{1}{64}$ for Problem 2

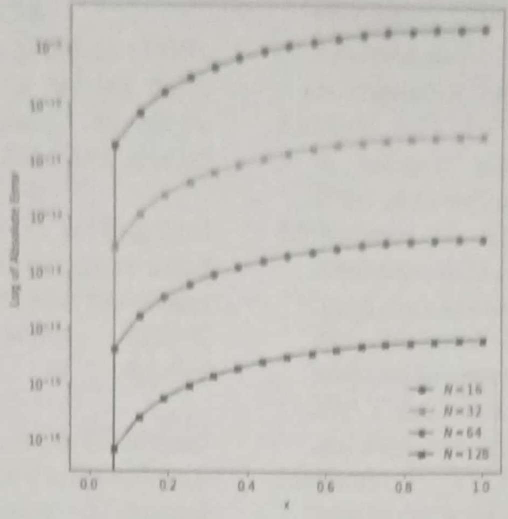

Fig. 5. Log of Absolute Error at $h = \frac{1}{N}$ for Problem 3

Discussion of Results

Tables 1 – 7 display the numerical results of the third-order initial and boundary value problems in examples 1 – 3. Table 1 can be used to compare the exact and approximate solutions for problem 1 with step-size $h = \frac{1}{100}$. Table 2 displays the maximum absolute errors at $h = \frac{1}{N}; N = 16, 32, 64, 128$ for problem 1. Table 3 shall be used to compare the exact and the solution of the proposed method with $h = \frac{1}{100}$ and $\epsilon = \frac{1}{16}$ for problem 2. Table 4 shows the results of the maximum absolute errors of problem 2 for $h = \frac{1}{N}; N = 10, 20, 40, 80, 100$, in contrast with the results of the maximum absolute errors of Abdulsalam & Majid (2019) in Table 5. Table 6 illustrates the comparison results of the exact and computed results for problem 3 from Ahmed (2017) at $h = \frac{1}{100}$. Table 7 displays the result of maximum absolute errors of problem 3 for $h = \frac{1}{N}; N = 16, 32, 64, 128$. Figures 1 and 5 show the plots of log of absolute errors in Tables 2, 4 and 7. Figures 2 – 4 show the plots of log of absolute errors at $h = \frac{1}{N}; N = 10, 20, 40, 80, 100$ and varying ϵ.

From the presented Tables and plots, we observe that the computed solutions of the proposed method agreed well with the exact solutions of the problems attempted in this paper. We also confirm that the smaller values of step-size, the more accurate approximated solution. Also, our proposed method performed better than the results in Abdulsalam & Majid (2019). The newly proposed method has therefore improved the numerical results of the problems under study.

CONCLUSION

In this research, we developed a self-starting backward differentiation formula of hybrid block linear multistep method with a higher-order of accuracy using collocation technique. The newly proposed method is aimed to improve the efficiency and accuracy of existing LMM by increasing the step-number both at grid and off-grid points. The choice of three off-step points and five step points were made in the development of the new method. The basic properties of the proposed method were analysed for convergence. Numerical experiments of initial or boundary value problems of type 1 and type 2 were performed to demonstrate the accuracy of the proposed method. The numerical results of the problems show the efficiency of the proposed methods as the computed results agreed well with the exact solutions. From the graphs and tabulated results, we can conclude that the proposed method is an excellent choice for handing third-order differential equations with either initial or boundary conditions appearing in all areas of sciences and engineering. Maple 2015 was used for all computations.

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