Development of Implicit Hybrid Adams Type Block Linear Multistep Method for the Solution of Stiff Ordinary Differential Equations

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Abstract
This paper proposes the derivation of a three-step eleventh order hybrid linear multi-step method (LMM) with nine off-step points for the solution of first order stiff differential equations. The obtained methods are then applied in block form as simultaneous numerical integrators over non-overlapping intervals. The numerical results show improved results over the existing methods in literatures considered, the schemes are consistent, zero-stable, and convergent.

Keywords: Hybrid, Collocation, Interpolation, Adams Type, Stiff differential equations, Zero-stable.

Introduction

Stiff differential equations arise frequently as singularly perturbed problems in chemical reaction systems and in electrical circuitry, and as space discretization of parabolic partial differential equations (Cole, 2019).

Consider the first order ODE of the form:

$$y' = f(x, y), y(x_0) = y_0 \quad (1)$$

where prime indicates derivative with respect to x and f satisfies the Lipschitz condition of the existence and uniqueness of solution. The collocation methods are widely considered as ways of generating numerical solution to (1). The second Dahlquist order barrier places a severe restriction on Linear Multistep Methods (LMMs), the condition that the LMM be implicit is a requirement for method suitable for integrating stiff initial value problems (IVPs). One of the ways the development of high order LMM which overcome Dahlquist order barrier has been achieved is by incorporating supplementary stages, extra division points, or future points, (Muka, 2016). Methods that fall into this class are block methods, hybrid methods, and extended methods. Block methods are used to compute previous k blocks to calculate the current block where each block contains r points (Cole, 2019). A subclass of LMMs is the Adams type given as.

$$y_{n+k} = y_{n+k-1} + h \sum_{j=0}^{k+1} \beta_j f_{n+j} \quad (2)$$

Cole, 2019 derived some hybrid Adams type block linear multistep methods using power series expansion, the idea of multistep collocation method (MCM) was adopted in the derivation of the schemes to obtain the continuous form which were evaluated at some off grid points and grid points to form block method. The schemes were consistent, zero-stable, and hence convergence. Muka, (2016) developed an extended block Adams-Moulton method for stiff IVPs which has superior stability regions with possible implementation on parallel computers like other block methods. The method is $A(a)$ stable. Nwachukwu and Okor, (2018) developed second derivative generalized backward differentiation formulae (SDGBDF) for solving stiff IVPs in ordinary differential equations (ODEs). The order, error constants, zero stability, and region of absolute stability were discussed. The methods are A stable. Ramos, (2017) developed a two-step block method of hybrid type for the direct solution of general first-order initial-value problems of the form $y' = f(x, y)$ where all the formulas in the method are obtained from a continuous approximation derived via interpolation and collocation at different points. The method is A-stable, which makes it appropriate for solving stiff problems.

Trapezoidal rule is the only Adams type family for integrating stiff IVPs, in this paper the derivation of implicit hybrid Adams type (IHAT) LMM for approximating stiff IVPs is presented.

Derivation of Method

The combination of a multi-step structure with the use of nine off-step points is emphasized here, andthe general multi-step method considered for the IVP (1) is given by.

$$\sum_{j=0}^{k} \varphi_j y_{n+j} + h\left(\sum_{j=0}^{k} \beta_j f_{n+j} + \beta_v f_{n+v}\right) \quad (3)$$

whereφ_j, s and $\beta_j's$ are coefficients and $v = \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, \frac{9}{4}, \frac{5}{2}, \frac{11}{4}\right\}$.

To obtain (3), we seek the approximation of the exact solution $y(x)$ by assuming a continuous solution of the form:

$$y(x) = \sum_{j=0}^{i+c-1} \varphi_j p_j(x) \quad (4)$$

such that$x \in [x_0, b]$, $\varphi_j's$ are unknown coefficients to be determined and $p_j(x)$are basis function of degree $i + c - 1$, furthermore, i and c which are the number of interpolation and collocation points respectively are carefully chosen to satisfy $1 \le c \le k$ and $c < 0$. The integer $k \ge 1$ is the step number of the method.

A k-step continuous multi-step method with $\varphi(x) = x^j, j = 0,1,2,..,13, i = 1, c = 13, k = 3$ is constructed, imposing this condition gives:

$$\sum_{j=0}^{13} \varphi_j x_{n+1}^{j-1} = f_{n+i}, i = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2, \frac{9}{4}, \frac{5}{2}, \frac{11}{4}, 3\right\} \quad (5)$$

$$\sum_{j=0}^{13} \varphi_j x_{n+1}^{j} = y_{n+i}, i = 2 \quad (6)$$

Equations (5) and (6) lead to system of $i + c$ equations which are solved to obtain the coefficients $\varphi_j's$. The values of $\varphi_j's$ are obtained using Maple 2015 software package. The scheme is obtained by substituting the values of the obtained $\varphi_j's$ into (4). Evaluating the scheme at points $x\left\{y_n, y_{n+\frac{1}{4}}, y_{n+\frac{1}{2}}, y_{n+\frac{3}{4}}, y_{n+1}, y_{n+\frac{5}{4}}, y_{n+\frac{3}{2}}, y_{n+\frac{7}{4}}, y_{n+2}, y_{n+\frac{9}{4}}, y_{n+\frac{5}{2}}, y_{n+\frac{11}{4}}, y_{n+3}\right\}$ gives the following twelve discrete equations:

$$\begin{aligned} y_n - y_{n+2} = &-\frac{42194069}{638512875}\Box f_n - \frac{97021984}{212837625}\Box f_{n+\frac{1}{4}} + \frac{19044664}{70945875}\Box f_{n+\frac{1}{2}} - \frac{173147168}{127702575}\Box f_{n+\frac{3}{4}} \\ &+ \frac{22373536}{14189175}\Box f_{n+1} - \frac{181696448}{70945875}\Box f_{n+\frac{5}{4}} + \frac{60377264}{30405375}\Box f_{n+\frac{3}{2}} - \frac{14992192}{7882875}\Box f_{n+\frac{7}{4}} \\ &+ \frac{10913087}{14189175}\Box f_{n+2} - \frac{43033184}{127702575}\Box f_{n+\frac{9}{4}} + \frac{19496392}{212837625}\Box f_{n+\frac{5}{2}} \\ &- \frac{358496}{23648625}\Box f_{n+\frac{11}{4}} \\ &+ \frac{739276}{638512875}\Box f_{n+3} \end{aligned} \quad (7)$$

$$y_{n+\frac{1}{4}} - y_{n+2} = \frac{250951589}{213497856000}\square f_n - \frac{2895487553}{35582976000}\square f_{n+\frac{1}{4}} - \frac{6476481263}{17791488000}\square f_{n+\frac{1}{2}} - \frac{293241529}{4269957120}\square f_{n+\frac{3}{4}} - \frac{809066881}{158145620}\square f_{n+1} + \frac{313854457}{5930496000}\square f_{n+\frac{5}{4}} - \frac{144310733}{277992000}\square f_{n+\frac{3}{2}} - \frac{475063673}{5930496000}\square f_{n+\frac{7}{4}} - \frac{138491}{63258624}\square f_{n+2} + \frac{1098060061}{21249785600}\square f_{n+\frac{9}{4}} - \frac{227568593}{17791488000}\square f_{n+\frac{5}{2}} + \frac{72298177}{3582976000}\square f_{n+\frac{11}{4}} - \frac{32315941}{213496780000}\square f_{n+3} \quad (8)$$

$$y_{n+\frac{1}{2}} - y_{n+2} = -\frac{44983}{336336000}\square f_n + \frac{81383}{28028000}\square f_{n+\frac{1}{4}} - \frac{642261}{7007000}\square f_{n+\frac{1}{2}} - \frac{5494261}{16816800}\square f_{n+\frac{3}{4}} - \frac{719601}{4484480}\square f_{n+1} - \frac{4912861}{14014000}\square f_{n+\frac{5}{4}} - \frac{18979}{125125}\square f_{n+\frac{3}{2}} - \frac{4742691}{14014000}\square f_{n+\frac{7}{4}} - \frac{1858267}{22422400}\square f_{n+2} - \frac{4559}{3363360}\square f_{n+\frac{9}{4}} + \frac{8389}{7007000}\square f_{n+\frac{5}{2}} - \frac{6887}{28028000}\square f_{n+\frac{11}{4}} + \frac{6887}{336336000}\square f_{n+3} \quad (9)$$

$$y_{n+\frac{3}{4}} - y_{n+2} = \frac{318845}{83691159552}\square f_n - \frac{8919685}{13948526592}\square f_{n+\frac{1}{4}} + \frac{13995335}{2324754432}\square f_{n+\frac{1}{2}} - \frac{4330920085}{41845579776}\square f_{n+\frac{3}{4}} - \frac{2749121675}{9299017728}\square f_{n+1} - \frac{512838115}{2324754432}\square f_{n+\frac{5}{4}} - \frac{4052105}{15567552}\square f_{n+\frac{3}{2}} - \frac{68674805}{258306048}\square f_{n+\frac{7}{4}} - \frac{1116220405}{9299017728}\square f_{n+2} + \frac{528744725}{41845579776}\square f_{n+\frac{9}{4}} - \frac{17042965}{6974263296}\square f_{n+\frac{5}{2}} + \frac{47735}{140894208}\square f_{n+\frac{11}{4}} - \frac{1935865}{83691159552}\square f_{n+3} \quad (10)$$

$$y_{n+1} - y_{n+2} = -\frac{28151}{5108103000}\square f_n + \frac{21128}{212837625}\square f_{n+\frac{1}{4}} - \frac{196739}{212837625}\square f_{n+\frac{1}{2}} + \frac{849752}{127702575}\square f_{n+\frac{3}{4}} - \frac{3920003}{37837800}\square f_{n+1} - \frac{21258704}{70945875}\square f_{n+\frac{5}{4}} - \frac{6237758}{30405375}\square f_{n+\frac{3}{2}} - \frac{21258704}{70945875}\square f_{n+\frac{7}{4}} - \frac{3920003}{37837800}\square f_{n+2} + \frac{849752}{127702575}\square f_{n+\frac{9}{4}} - \frac{196739}{212837625}\square f_{n+\frac{5}{2}} + \frac{21128}{212837625}\square f_{n+\frac{11}{4}} - \frac{28151}{5108103000}\square f_{n+3} \quad (11)$$

$$y_{n+\frac{5}{4}} - y_{n+2} = \frac{521303}{43051008000}\square f_n - \frac{1244171}{7175168000}\square f_{n+\frac{1}{4}} + \frac{4264699}{3587584000}\square f_{n+\frac{1}{2}} - \frac{22944767}{4305100800}\square f_{n+\frac{3}{4}} + \frac{54961629}{287067200}\square f_{n+1} - \frac{647419943}{3587584000}\square f_{n+\frac{5}{4}} - \frac{2033753}{8008000}\square f_{n+\frac{3}{2}} - \frac{985667673}{3587584000}\square f_{n+\frac{7}{4}} - \frac{329208157}{2870067200}\square f_{n+2} + \frac{45169279}{4305100800}\square f_{n+\frac{9}{4}} - \frac{6696251}{3587584000}\square f_{n+\frac{5}{2}} + \frac{152459}{7175168000}\square f_{n+\frac{11}{4}} - \frac{688087}{4305100800}\square f_{n+3} \quad (12)$$

$$\begin{aligned} y_{n+\frac{3}{2}} - y_{n+2} = {} & \frac{133787}{81729648000}\Box f_n - \frac{133787}{6810804000}\Box f_{n+\frac{1}{4}} + \frac{56333}{567567000}\Box f_{n+\frac{1}{2}} \\ & - \frac{181091}{817296480}\Box f_{n+\frac{3}{4}} - \frac{583669}{1816214400}\Box f_{n+1} + \frac{6742003}{1135134000}\Box f_{n+\frac{5}{4}} \\ & - \frac{3118879}{30405375}\Box f_{n+\frac{3}{2}} - \frac{38542363}{126126000}\Box f_{n+\frac{7}{4}} - \frac{7503059}{72648576}\Box f_{n+2} \\ & + \frac{28097519}{4068482400}\Box f_{n+\frac{9}{4}} - \frac{1742911}{1702701000}\Box f_{n+\frac{5}{2}} + \frac{89987}{756756000}\Box f_{n+\frac{11}{4}} \\ & - \frac{584203}{81729648000}\Box f_{n+3} \end{aligned} \quad (13)$$

$$\begin{aligned} y_{n+\frac{7}{4}} - y_{n+2} = {} & \frac{9959263}{951035904000}\Box f_n - \frac{252766961}{1743565824000}\Box f_{n+\frac{1}{4}} + \frac{821346049}{871782912000}\Box f_{n+\frac{1}{2}} \\ & - \frac{4014966413}{1046139494400}\Box f_{n+\frac{3}{4}} + \frac{172090819}{15498362887}\Box f_{n+1} - \frac{7236570071}{29059430400}\Box f_{n+\frac{5}{4}} \\ & + \frac{94985467}{1945944000}\Box f_{n+\frac{3}{2}} - \frac{49214636201}{290594304000}\Box f_{n+\frac{7}{4}} - \frac{54121130127}{7749184400}\Box f_{n+2} \\ & + \frac{2507349353}{209227898880}\Box f_{n+\frac{9}{4}} - \frac{184216480}{871782912000}\Box f_{n+\frac{5}{2}} + \frac{475414129}{1743565824000}\Box f_{n+\frac{11}{4}} \\ & - \frac{184329877}{1046139494400}\Box f_{n+3} \end{aligned} \quad (14)$$

$$\begin{aligned} y_{n+\frac{9}{4}} = y_{n+2} + {} & \frac{184329877}{10461394944000}\Box f_n - \frac{417640049}{1743565824000}\Box f_{n+\frac{1}{4}} + \frac{441509227}{290594304000}\Box f_{n+\frac{1}{2}} \\ & - \frac{6257449741}{1046139494400}\Box f_{n+\frac{3}{4}} + \frac{3821011693}{232475443200}\Box f_{n+1} - \frac{9816495959}{29059430400}\Box f_{n+\frac{5}{4}} \\ & + \frac{107296613}{1945944000}\Box f_{n+\frac{3}{2}} - \frac{2552320801}{32288256000}\Box f_{n+\frac{7}{4}} + \frac{44643543443}{232475443200}\Box f_{n+2} \\ & + \frac{115234170509}{10461394944000}\Box f_{n+\frac{9}{4}} - \frac{6054093569}{871782912000}\Box f_{n+\frac{5}{2}} + \frac{143115689}{193729536000}\Box f_{n+\frac{11}{4}} \\ & - \frac{456196373}{10461394944000}\Box f_{n+3} \end{aligned} \quad (15)$$

$$\begin{aligned} y_{n+\frac{5}{2}} - y_{n+2} = {} & -\frac{193087}{7429968000}\Box f_n + \frac{2349637}{6810804000}\Box f_{n+\frac{1}{4}} - \frac{3612439}{1702701000}\Box f_{n+\frac{1}{2}} \\ & + \frac{32731249}{4086482400}\Box f_{n+\frac{3}{4}} - \frac{12546839}{605404800}\Box f_{n+1} + \frac{44018707}{1135134000}\Box f_{n+\frac{5}{4}} \\ & - \frac{1625861}{30405375}\Box f_{n+\frac{3}{2}} + \frac{57802477}{1135134000}\Box f_{n+\frac{7}{4}} + \frac{34426087}{605404800}\Box f_{n+2} \\ & + \frac{1362297487}{4086482400}\Box f_{n+\frac{9}{4}} + \frac{154495511}{1702701000}\Box f_{n+\frac{5}{2}} - \frac{19099973}{6810804000}\Box f_{n+\frac{11}{4}} \\ & + \frac{10480453}{81729648000}\Box f_{n+3} \end{aligned} \quad (16)$$

$$\begin{aligned} y_{n+\frac{11}{4}} - y_{n+2} = {} & \frac{6279127}{43051008000}\Box f_n - \frac{13866379}{7175168000}\Box f_{n+\frac{1}{4}} + \frac{42570291}{3587584000}\Box f_{n+\frac{1}{2}} \\ & - \frac{38554547}{861020160}\Box f_{n+\frac{3}{4}} + \frac{333308253}{82700067200}\Box f_{n+1} - \frac{787623847}{3587584000}\Box f_{n+\frac{5}{4}} \\ & + \frac{2514233}{8008000}\Box f_{n+\frac{3}{2}} - \frac{1264870617}{3587584000}\Box f_{n+\frac{7}{4}} + \frac{46838683}{114802688}\Box f_{n+2} \\ & + \frac{324301823}{43051008000}\Box f_{n+\frac{9}{4}} + \frac{1302640901}{3587584000}\Box f_{n+\frac{5}{2}} + \frac{584576011}{7175168000}\Box f_{n+\frac{11}{4}} \\ & - \frac{50840663}{43051008000}\Box f_{n+3} \end{aligned} \quad (17)$$

$$\begin{aligned} y_{n+3} - y_{n+2} = &-\frac{5942359}{5108103000}\square f_n + \frac{3247592}{212837625}\square f_{n+\frac{1}{4}} - \frac{6564377}{7094875}\square f_{n+\frac{1}{2}} + \frac{43882936}{127702575}\square f_{n+\frac{3}{4}} \\ &- \frac{19812941}{22702680}\square f_{n+1} + \frac{113671024}{70945875}\square f_{n+\frac{5}{4}} - \frac{66615022}{30405375}\square f_{n+\frac{3}{2}} + \frac{17826416}{7882875}\square f_{n+\frac{7}{4}} \\ &- \frac{190748297}{113513400}\square f_{n+2} + \frac{34799384}{25540515}\square f_{n+\frac{9}{4}} - \frac{57330731}{212837625}\square f_{n+\frac{5}{2}} \\ &+ \frac{10782568}{23648625}\square f_{n+\frac{11}{4}} \\ &- \frac{337524401}{5108103000}\square f_{n+3} \end{aligned} \quad (18)$$

Basic Properties of the Developed Method

Order of the block

According to Fudziah et al. (2020), if y_{n+j} is the solution to y'_{n+j} and is sufficiently differentiable, then y_{n+j} and y'_{n+j} can be expanded into a Taylor's series about point x_n to obtain

$$T_n = \frac{1}{\square\sigma(1)}\left(C_0 y(x_n) + C_1 \square y'(x_n) + C_2 \square^2 y''(x_n) + \ldots\right) \quad (19)$$

where

$$\left.\begin{aligned} C_0 &= \sum_{j=0}^{k} \varphi_j \\ C_1 &= \sum_{j=0}^{k} j\varphi_j - \sum_{j=0}^{k} \beta_j \\ &\vdots \\ C_q &= \frac{1}{q!}\sum_{j=0}^{k} j^q \varphi_j - \frac{1}{(q-1)!}\sum_{j=0}^{k} j^{q-1}\beta_j \end{aligned}\right\} \quad (20)$$

Definition 1: A linear multistep method is said to be of order p if $C_0 = C_1 = \ldots = C_p = 0, C_{p+1} \neq 0$. Therefore, C_{p+1} is the error constant and $C_{p+1}\square^{p+1} y^{p+1}(x_n)$ is the principal truncation error at point (x_n). Thus, the local truncation error (LTE) of order p can be written as

$$\text{LTE} = C_{p+1}\square^{p+1} y^{p+1}(x_n) + O\left(h^{p+2}\right) \quad (21)$$

From our calculation, the implicit hybrid linear multistep method has high order and relatively small error constant shown in Table 1.

Consistency

Definition 2: A linear multistep method (3) is said to be consistent if (i) the order $p \geq 1$, (ii) $\sum_{j=0}^{k} \varphi_j = 0$, (iii) $\sum_{j=0}^{k} j\varphi_j = \sum_{j=0}^{k} \beta_j$.

For scheme (7).

Condition (i)

The method has order $p = 14 \geq 1$.

Condition (ii)

$$\sum_{j=0}^{k} \varphi_j = 1 - 1 = 0$$

Condition (iii)

$$\sum_{j=0}^{k} j\varphi_j = 0(1) - 2(1) = -2$$

$$\sum_{j=0}^{k} \beta_j = -\frac{42194069}{638512875} - \frac{97021984}{212837625} + \frac{19044664}{70945875} - \frac{173147168}{127702575} + \frac{22373536}{14189175} - \frac{181696448}{70945875} + \frac{60377264}{30405375} - \frac{14992192}{7882875} + \frac{10913087}{14189175} - \frac{43033184}{127702575} + \frac{19496392}{212837625} - \frac{358496}{23648625} + \frac{739276}{638512875} = -2$$

$$\Rightarrow \sum_{j=0}^{k} j\varphi_j = \sum_{j=0}^{k} \beta_j = -2$$

Hence the scheme is consistent.

For scheme (8).

Condition (i)
The method has order $p = 14 \geq 1$.

Condition (ii)

$$\sum_{j=0}^{k} \varphi_j = 1 - 1 = 0$$

Condition (iii)

$$\sum_{j=0}^{k} j\varphi_j = \frac{1}{4}(1) - 2(1) = -\frac{7}{4}$$

$$\sum_{j=0}^{k} \beta_j = \frac{250951589}{213497856000} - \frac{2895487553}{35582976000} - \frac{6476481263}{17791488000} - \frac{293241529}{4269957120} - \frac{809066881}{158145620} + \frac{313854457}{5930496000} - \frac{144310733}{277992000} - \frac{475063673}{5930496000} - \frac{138491}{63258624} + \frac{1098060061}{21249785600} - \frac{227568593}{17791488000} + \frac{72298177}{3582976000} - \frac{32315941}{213496780000} = -\frac{7}{4}$$

$$\Rightarrow \sum_{j=0}^{k} j\varphi_j = \sum_{j=0}^{k} \beta_j = -\frac{7}{4}$$

Hence the scheme is consistent.
Applying theseconditions to schemes (9-18) we have that schemes (9-18) are consistent as shown in Table 1.

Zero Stability
Definition 3:A linear multistep method of the form (3) is said to be zero stable ifno root of the first characteristic polynomial $\rho(r)$ has modulus greater than one and ifevery root with modulus one is simple.
Applying the above to schemes (7-18) they were found to be zero stable as shown in Table 1.

Consistency

According to Henrici (1962), we can establish the convergence ofthe three-step implicit hybrid Adams type block linear multistep method since convergence = consistency + zerostability.

Table 1: Order, Error Constant, Characteristics and Roots of Polynomials

Equation number	Order	Error Constant	Polynomial	Roots of polynomial
7	14	$\frac{7619}{486930382848000}$	$1-r^2$	$\rho(r)=-1,1$
8	14	$\frac{19061}{1072038370816000}$	$r^{\frac{1}{4}}-r^2$	$\rho(r)=0,1$
9	14	$\frac{6887}{22571126882304000}$	$r^{\frac{1}{2}}-r^2$	$\rho(r)=0,1$
10	14	$\frac{1935865}{11232837288754937856}$	$r^{\frac{3}{4}}-r^2$	$\rho(r)=0,1$
11	14	$\frac{1909}{514198484287488000}$	$r-r^2$	$\rho(r)=0,1$
12	14	$\frac{521303}{5778208481869824000}$	$r^{\frac{5}{4}}-r^2$	$\rho(r)=0,1$
13	14	$\frac{133787}{5484783832399872000}$	$r^{\frac{3}{2}}-r^2$	$\rho(r)=0,1$
14	14	$\frac{521303}{577208481869824000}$	$r^{\frac{7}{4}}-r^2$	$\rho(r)=0,1$
15	14	$\frac{1935865}{11232837288754937856}$	$r^{\frac{9}{4}}-r^2$	$\rho(r)=0,1$
16	14	$\frac{6887}{22571126882304000}$	$r^{\frac{5}{2}}-r^2$	$\rho(r)=0,1$
17	14	$\frac{19061}{10720238370816000}$	$r^{\frac{11}{4}}-r^2$	$\rho(r)=0,1$
18	14	$\frac{7619}{486930382848000}$	r^3-r^2	$\rho(r)=1,0,0$

Numerical experiments

In this section, effectiveness, and applicability of our new method (IHAT)is demonstrated on two stiff differential systems of ODEs, we consider both linear and nonlinear stiff system of IVPs in ordinary differential equation. Their performance is compared with the exact solution and with other methods in cited literature.

The notations used are:
IHAT: Implicit hybrid Adams type
Error: |Exact solution − Computed solution|
MaxErr: Maximum error = $\max_i \frac{|y_i - y(x_i)|}{|1+y(x_i)|}$

Example 1: This problem is a system of standard test problem which is a mildly stiff linear problem.

$$\begin{cases} y_1' = -8y_1 + 7y_2, y_1(0) = 1 \\ y_2' = 42y_1 - 43y_2, y_2(0) = 8 \end{cases}$$

The exact solution is

$$\begin{cases} y_1(x) = 2e^{-x} - e^{-50x} \\ y_2(x) = 2e^{-x} + 6e^{-50x} \end{cases}$$

Table 2: Comparison of absolute errors for problem 1

x	Error in NM y component	Mesh values [ref]	Error in GLM y component [ref]
0.1	2.85579712e-16	10.0	1.04e-2
0.2	3.46425700e-16	20.0	3.81e-3
0.3	3.15176715e-16	30.0	1.34e-5
0.4	2.54885792e-16	40.0	4.74e-6
0.5	1.93245058e-16	50.0	1.67e-8
0.6	1.40650865e-16	60.0	5.90e-9
0.7	9.9527242e-17	70.0	2.08e-11
0.8	6.8990087e-17	80.0	7,33e-12
0.9	4.7075183 e-17	90.0	2.58e-14
1.0	3.1725048e-17	100.0	9.12e-15

Table 3: Comparison of result for problem 1

X	Exact value	y_i	Approximate Value (IHAT)	Error
0.01	1.373569007785702683544012	y_1	1.373569007785702969123724	2.85579712e-16
	5.619283625774136648770609	y_2	5.619283625774134935292347	1.713478262e-15
0.02	1.592517905442068282846104	y_1	1.592517905442068629271804	3.46425700e-16
	4.167673993642164534014771	y_2	4.167673993642162455460567	2.078554204e-15
0.03	1.717760906948586524931776	y_1	1.717760906948586840108491	3.15176715e-16
	3.279672027987595327464740	y_2	3.279672027987593436404470	1.891060270e-15
0.04	1.786243595068033726984422	y_1	1.786243595068033981870214	2.54885792e-16
	2.733590577724322570242418	y_2	2.733590577724321040927705	1.529314713e-15
0.05	1.820373850377529223013322	y_1	1.820373850377529416258380	1.93245058e-16
	2.394968840744820789200023	y_2	2.394968840744819629729704	1.159470319e-15
0.06	1.833741998800633476094964	y_1	1.833741998800633616745829	1.40650865e-16
	2.182251477375681076950360	y_2	2.182251477375680233045213	8.43905147e-16
0.07	1.834590256389577956976159	y_1	1.834590256389578056503401	9.9527242e-17
	2.045971940345807462154663	y_2	2.045971940345806864991262	5.97163401e-16
0.08	1.827917053884537385527802	y_1	1.827917053884537454517889	6.8990087e-17
	1.956126526105676647583828	y_2	1.956126526105676233643359	4.13940469e-16
0.09	1.816753374004214066998564	y_1	1.816753374004214114073747	4.7075183e-17
	1.894516349771910212471566	y_2	1.894516349771909930020534	2.82451032e-16
0.1	1.802936889072833679231862	y_1	1.802936889072833710956910	3.1725048e-17
	1.850102518066431948908314	y_2	1.850102518066431758558087	1.90350227e-16

Example 2: Consider the stiff system of two dimensional Kaps problem with corresponding initial conditions.

$$\begin{cases} y_1' = -1002y_1 + 1000z, y_1(0) = 1 \\ y_2' = y_1 - y_2(1 + y_2), y_2(0) = 1 \end{cases}$$

The exact solution is

$$y_1(x) = e^{-2x}$$
$$y_2(x) = e^{-x}$$

Table 4: Comparison of result for problem 2

x	Exact	y_i	Approximate value (IHAT)	Error
0.1	0.9801986733067553022208135	y_1	0.9801986733067553022208141	6e-25
	0.9900498337491680535739060	y_2	0.9900498337491680535739064	4e-25
0.2	0.9607894391523232094392109	y_1	0.9607894391523232094392107	2e-25
	0.9801986733067553022208149	y_2	0.9801986733067553022208141	8e-25
0.3	0.9417645335842487095371528	y_1	0.9417645335842487095371537	9e-25
	0.9704455335485081769325284	y_2	0.9704455335485081769325296	1.2e-24
0.4	0.9231163463866357829107598	y_1	0.9231163463866357829107617	1.9e-24
	0.9607894391523232094392107	y_2	0.9607894391523232094392124	1.7e-24
0.5	0.9048374180359595731642491	y_1	0.9048374180359595731642518	2.7e-24
	0.9512294245007140090914253	y_2	0.9512294245007140090914275	2.2e-24
0.6	0.8869204367171575155275652	y_1	0.8869204367171575155275688	3.6e-24
	0.9417645335842487095371528	y_2	0.9417645335842487095371554	2.6e-24
0.7	0.8693582353988058196630844	y_1	0.8693582353988058196630887	4.3e-24
	0.9323938199059482288579726	y_2	0.9323938199059482288579756	3.0e-24
0.8	0.8521437889662113384563470	y_1	0.8521437889662113384563520	5.0e-24
	0.9231163463866357829107598	y_2	0.9231163463866357829107632	3.4e-24
0.9	0.8352702114112720213123850	y_1	0.8352702114112720213123905	5.5e-24
	0.9139311852712281867473535	y_2	0.9139311852712281867473573	3.8e-24
1.0	0.8187307530779818586699355	y_1	0.8187307530779818586699417	6.2e-24
	0.9048374180359595731642491	y_2	0.9048374180359595731642532	4.1e-24

Table 5: Comparison of Maximum error for Problem 2

Method	x	H	N	y_i	MAXE
IHAT	1	0.01	100	y_1	3.029999001e-25
				y_2	2.009999916e-25
	10	0.01	1000	y_1	3.408970783e-24
				y_2	2.152414669e-24
Wu and Xia As in Higinio (2017)	1	0.002	500	y_1	2.5606e-7
				y_2	8.0150e-8
	10	0.001	10000	y_1	5.5468e-16
				y_2	6.0936e-12
Akinfenwa et al.[ref] As in Higinio (2017)	1	0.02	50	y_1	9.1102e-13
				y_2	1.2527e-12
	10	0.02	500	y_1	2.1977e-20
				y_2	1.3542e-15
HBM [ref] As in Higinio (2017)	1	0.02	50	y_1	1.2258e-13
				y_2	2.4555e-15
	10	0.02	500	y_1	2.1200e-21
				y_2	3.0914e-18

Discussion of Result

Two numerical examples are used to test the efficiency of the newly developed scheme. Yakubu *et al.*, solved problem 1, a system of standard test problem which is a mildly stiff linear problem while Markus et al solved problem 2 in the interval [0, 15]. There is remarkable agreement with the exact values while the new IHAT performs better than Yakubu et al and Markus *et al*.

Conclusion

In this paper, it is shown that implicit linear multistep methods for stiff initial value problems can be formulated as implicit hybrid Adams type linear multistep methods for the direct solution of stiff systems of IVPs in ODEs. The three-step Adams type is of order eleven and gives very low error terms. The consistency and zero stability of the new method guarantee its convergence. Based on the result obtained in Tables 3 and 4there is improvement on the convergence rate of the three-step implicit scheme with nine off-grid points. The new method is highly accurate and performs better than the literatures cited.

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