CHEBYSHEV COLLOCATION APPROACH FOR CONTINUOUS FOUR-STEP HYBRID BACKWARD DIFFERENCE FORMULA FOR STIFF SYSTEM

MOHAMMED, U.[*1], **AJINUHI, J. O.**[*2], **JIMOH, O. R.**[*3], **DAUDA, A. A.**[*4]; **AKINTUBUBO, B. G.**[*5]
[1, 2, 3, 5*]Department of Mathematics,
Federal University of Technology, Minna, Niger State, Nigeria
[4*]Department of Mathematics, Federal University Gashua, P.M.B 1005, Yobe State, Nigeria
E-mail: [1]umaru.mohd@futminna.edu.ng , [2]ajinuhijoel@gmail.com
[5]ben.gakintububo@yahoo.com

Abstract
In this paper, we developed an implicit continuous four-step hybrid backward difference formulae for the direct solution of stiff system. For this purpose, the Chebyshev polynomial was employed as the basis function for the development of schemes in a collocation and interpolation techniques. The schemes were analysed using appropriate existing theorem to investigate their stability, consistency, convergence and the investigation shows that the developed schemes are consistent, zero-stable and hence convergent. The methods were implemented on test problem from the literatures to show the accuracy and effectiveness of the scheme.

Introduction
Ordinary differential equation (ODE) has been an important tool in modelling real life situations in Engineering, Science and Technology. It is an equation of the form

$$y' = f(x,y) \quad y(a) = y_0 \quad x \in [a,b] \tag{1}$$

Where $f: \mathbb{R} \times \mathbb{R}^m \longrightarrow \mathbb{R}^m, f$ satisfies a Liptchitz conditions Henrici (1962), System (1) can be regarded as stiff if its exact solution contains very fast as well as very slow components (see, Dahlquist,1974).

Stiff Initial Value Problems (IVPs) occur in any field of Sciences and engineering especially in electric circuits, chemical reactions, vibration, automatic control and combustion kinetics, theory of fluid mechanics. The solution is characterized by the transient and steady state components which restrict the step size of many numerical methods excepts method that has the properties of A-stability Suleiman et al. (2013) and Suleiman et al. (2014). We discover that the nature of the problem made it difficult to develop suitable methods for the solution of stiff problems. However, effort have been made by researchers such as Ngwane and Jator (2012), Abasi et al. (2014), Musa et al. (2013), Ibrahim et al. (2007), Biala et al. (2015), Babangida et al. (2016) among others, to develop methods for stiff ordinary differential equations.

The Dahlquist barrier theorem was circumvented by several authors that proposed modified form of linear multi-steps methods (LMMs) known as hybrid methods by introducing the off-grid points in the process of derivation. Gear (1965), Gragg and Stetter (1964) working in conjunction with Butcher (1965) by introducing the off-grid step point. These methods were shown to be of order $2k + 2$ Gupta (1978) observes that the design of algorithm for hybrid methods is more difficult due to the occurrence of off-step function which increase the number of predictors involved to implement the method.

Moreover, Curtis and Hirschfelder (1952) developed the backward differentiation formulae (BDF). Since then, a great effort has been made in order to obtain a new numerical integration with stability properties that is strong which is desirable for solving stiff system.

In this paper, the modified four-step backward differentiation formulae (BDF) are obtained from the continuous scheme and assembled into a block matrix equation which is applied to provide the solution for (1). The block method was first introduced by Milne (1952) for use as a means of starting values for predictor –corrector algorithms and has since then been developed by researcher (see Majid (2014), WuXia (2001), Brumano and Trigiante (2000), for general purpose. The application is that it yields several results at a time which depends on the number of points on the structure of the block method. The research work is motivated by constructing a hybrid backward differentiation formula (HBDF) that satisfies the Dahlquist barrier theorem, whose stated that the maximum attainable order for even is k $k+2$ and k odd is $k+1$ if essential zero stability condition is to be achieved. In this research, we obtain the order of the method to be $k+1$ and the zero stability condition is achieved.

Derivation of the Method

The four-step HBDF type block method is of the form:

$$y_{n+k} = \sum_{j=0}^{k-1} \alpha_j y_{n+j} + h\beta_\mu f_{n+\mu} + h\beta_k f_{n+k} \tag{2}$$

Where α_j, β_μ and β_k are coefficients to be determined. We proceed by assuming that the exact solution $y(x)$ of the form

$$y(x) = \sum_{j=0}^{5} \phi_j T_j(t) \tag{3}$$

Where ϕ_j are unknown coefficients to be determined and $T_j(t)$ are Chebyshev polynomial basis functions. The method is constructed with the Chebyschev polynomial as $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$; $T_3(x) = 4x^3 - 3x$, $T_4(x) = 8x^4 - 8x^2 + 1$, $T_5(x) = 16x^5 - 20x^3 + 5x$ by imposing the following conditions

$$y(x_{n+j}) = y_{n+j} \tag{4}$$

$$y'(x_{n+j}) = f_{n+j} \tag{5}$$

It is observed that equations (4) and (5) is a system of $(k+1)$ equations which must be solved to obtain the coefficients of ϕ_j, $j = 0,1,2,\ldots,5$ which are substituted into (3) and after some algebraic computations, the continuous representation yields the form

$$y(x) = \sum_{j=0}^{3} \alpha_j(x) y_{n+j} + h\beta_\mu(x) f_{n+\mu} + h\beta_k(x) f_{n+k} \tag{6}$$

Where $\alpha_j(x)$, $\beta_j(x)$ and $\beta_\mu(x)$ are the continuous coefficients. The equation (6) is then used to obtain the main method by evaluating $x = x_{n+4}$ and additional method by evaluating $x = x_{n+\frac{15}{4}}$ and differentiating equation (6) once and evaluate at $x = x_{n+1}$, $x = x_{n+2}$, and $x = x_{n+3}$ to form the block.

The combination of these methods yields the block method as given below:

$$
\left.
\begin{aligned}
y_{n+4} &= -\frac{331}{24561}y_n + \frac{272}{2729}y_{n+1} - \frac{996}{2729}y_{n+2} + \frac{31408}{24561}y_{n+3} + \frac{8192}{8187}hf_{n+\frac{15}{4}} - \frac{564}{2729}hf_{n+4} \\
y_{n+\frac{15}{4}} &= -\frac{349811}{25150464}y_n + \frac{143325}{1397248}y_{n+1} - \frac{1043625}{2794496}y_{n+2} + \frac{16156525}{12575232}y_{n+3} + \frac{28105}{32748}hf_{n+\frac{15}{4}} - \frac{444675}{1397248}hf_{n+4} \\
f_{n+3} &= \frac{1}{147366}\frac{1}{h}\left(60318hf_{n+4} - 113664hf_{n+\frac{15}{4}} + 4601y_n - 35883y_{n+1} + 151983y_{n+2} - 120701y_{n+3}\right) \\
f_{n+2} &= \frac{1}{221049}\frac{1}{h}\left(56511hf_{n+4} - 95232hf_{n+\frac{15}{4}} + 10493y_n - 107730y_{n+1} - 75789y_{n+2} + 173026y_{n+3}\right) \\
f_{n+1} &= \frac{1}{147366}\frac{1}{h}\left(55242hf_{n+4} - 89088hf_{n+\frac{15}{4}} + 28831y_n + 163053y_{n+1} - 299079y_{n+2} + 107195y_{n+3}\right)
\end{aligned}
\right\} \quad (7)
$$

The above equation gives the main and additional methods.

Analysis and Implementation

In this section, we discuss the local truncation error and order, convergence, the stability and implementation of the methods.

From (7) the Four-step HBDF is written in the form:

$$A^{(0)}Y_r = A^{(1)}Y_{r-1} + h(B^{(1)}F_{r-1} + B^{(0)}F_r) \quad (8)$$

Where
$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$B^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B^{(1)} = \begin{bmatrix} \frac{11083}{3960} & -\frac{4073}{840} & \frac{2563}{360} & -\frac{32128}{3465} & \frac{1873}{360} \\ \frac{1552}{495} & -\frac{407}{105} & \frac{292}{45} & -\frac{29696}{3465} & \frac{217}{45} \\ \frac{1371}{440} & -\frac{963}{280} & \frac{291}{40} & -\frac{3456}{385} & \frac{201}{40} \\ \frac{70275}{22528} & -\frac{49725}{14336} & \frac{15675}{2048} & -\frac{2595}{308} & \frac{9975}{2048} \\ \frac{1544}{495} & -\frac{52}{15} & \frac{344}{45} & -\frac{4096}{495} & \frac{224}{45} \end{bmatrix}$$

where $Y_r = [y_{n+1}, y_{n+2}, y_{n+3}, y_{n+\frac{15}{4}}, y_{n+4}]^t$

$Y_{r-1} = [y_{n-4}, y_{n-3}, y_{n-2}, y_{n-1}, y_n]^t$

$F_r = [f_{n+1}, f_{n+2}, f_{n+3}, f_{n+\frac{15}{4}}, f_{n+4}]$

$F_{r-1} = [f_{n-4}, f_{n-3}, f_{n-2}, f_{n-1}, f_n]$

where these are vectors notations.

Local Truncation Error and Order

Following Fatunla (1991) and Lambert (1973), we define the Local Truncation Error (LTE) associated with (6) to be the linear difference operator

$$L[y(x);h] = A^{(0)}Y_r - A^{(1)}Y_{r-1} - h(B^{(1)}F_{r-1} + B^{(0)}F_r) \qquad (9)$$

where we have taken $y_{n+j} = y(x_n + jh)$ and $f_{n+j} = y'(x_n + jh)$

Assuming that $y(x)$ is sufficiently differentiable, we can expand the term in (9) as a Taylor's series about x_n to obtain the expression as

$$L[y(x);h] = C_0 y(x) + C_1 hy'(x) + ... + C_q h^q y^q(x) + ... \qquad (10)$$

Where C_j is a (5×1) matrix

These are vectors of numerical estimates from (7) $r = 0,\ r = 1$ which show that the method is zero-stable and consistent since the order $p = 5$. Hence it is convergent.

It is evident from the calculation that the block method is of order $(5,5,5,5,5)^T$ and the error constant $(\frac{263}{40935}, \frac{56749}{1964880}, -\frac{122703}{40}, -\frac{15663}{8}, \frac{3587045}{536543232})^T$

It is worth noting that zero stability is concerned with the stability of the difference system in the limit as h tends to zero. Thus, as $h \to 0$, the method (7) tends to the difference system

$A^{(0)}Y_r - A^{(1)}Y_{r-1} = 0$

The method (7) is zero stable if the roots $R_{ij} = 1,2,3,4,5$ of the first characteristics polynomial $\rho(R)$ specified by

$\rho(R) = \det\left[\sum_{i=0}^{1} A^{(i)}R^{1-i}\right] = 0$ Satisfies $|R_j| \le 1 = 1,2,3,4,5$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed 1, (See Fatunla (1991)).

Thus, the Four-step HBDF is zero stable since

$\rho(R) = R^4(R-1) \Rightarrow R_1 \Rightarrow R_2 \Rightarrow R_3 \Rightarrow R_4 = 0,\ R_5 = 1$

The HBDF is also consistent since each of it numerical integrations have order $p > 1$, according to Henrici (1962). We can conclude that, the method is convergent.

Region of Absolute Stability of HBDF methods

The absolute stability region of the HBDF methods is constructed by reformulating the integrators as a general linear method of Butcher (1967) using notations introduced by Burrage and Butcher (1967). General Linear Method (GLM) is represented by a partition and $(s+r)\times(s+r)$ characterised by the four matrices A, B, U, V expressed in the form.

$$\begin{bmatrix} Y'_{[n]} \\ \cdot \\ \cdot \\ \cdot \\ Y_{[n]}{}^{i} \\ Y'_{[n]} \\ \cdot \\ \cdot \\ \cdot \\ Y^{r}{}_{[n]} \end{bmatrix} = \begin{bmatrix} A & B \\ U & V \end{bmatrix} \begin{bmatrix} hf(Y'_{[n]}) \\ \cdot \\ \cdot \\ \cdot \\ hf(Y^{i}{}_{[n]}) \\ y'{}_{[n+1]} \\ \cdot \\ \cdot \\ \cdot \\ y^{r}{}_{[n]} \end{bmatrix}$$

The elements of these matrices A, B, U, V are substituted into the recurrence relation $y^{[i+1]} = M(z)y^{[i]}$, $i = 1,2,..., N-1$

Where $M(z) = U + zB(1 - zA)^{+1}V$ (11)

The stability polynomial

$\rho(\eta, z) = \det(\eta)$ (12)

Where
$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{5458}{6039} & 0 & 0 & 0 & -\frac{62}{183} \\ 0 & 0 & -\frac{8187}{2807} & 0 & -\frac{31744}{25263} & \frac{299}{401} \\ 0 & 0 & 0 & \frac{147366}{120701} & -\frac{113664}{120701} & \frac{60318}{120701} \\ 0 & 0 & 0 & 0 & \frac{28105}{32748} & -\frac{444675}{1397248} \\ 0 & 0 & 0 & 0 & \frac{8192}{8187} & -\frac{564}{2729} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{8192}{8187} & -\frac{564}{2729} \\ 0 & 0 & 0 & \frac{147366}{120701} & & \frac{60318}{120701} \\ 0 & 0 & -\frac{8187}{2807} & 0 & -\frac{31744}{25263} & \frac{299}{40} \\ 0 & -\frac{5458}{6039} & 0 & 0 & 0 & -\frac{62}{183} \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -\frac{9745}{14823} & \frac{1007}{549} & 0 & -\frac{2621}{14823} \\ \frac{24718}{10827} & 0 & -\frac{570}{401} & \frac{1499}{10827} \\ 0 & \frac{151983}{120701} & -\frac{35883}{120701} & \frac{107}{2807} \\ \frac{16156525}{12575232} & -\frac{1043625}{2794496} & \frac{143325}{1397248} & -\frac{349811}{25150464} \\ \frac{31408}{24561} & -\frac{996}{2729} & \frac{272}{2729} & -\frac{331}{24561} \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{31408}{24561} & -\frac{996}{2729} & \frac{272}{2729} & -\frac{331}{24561} \\ 0 & \frac{151983}{120701} & -\frac{35883}{120701} & \frac{107}{2807} \\ \frac{24718}{10827} & 0 & -\frac{570}{401} & \frac{1499}{10827} \\ -\frac{9745}{14823} & \frac{1007}{549} & 0 & -\frac{2621}{14823} \end{bmatrix}$$

The method is plotted in a MATLAB environment to produce the RAS of the HBDF. It is observed that, the method is A-stable as shown in figure 1.

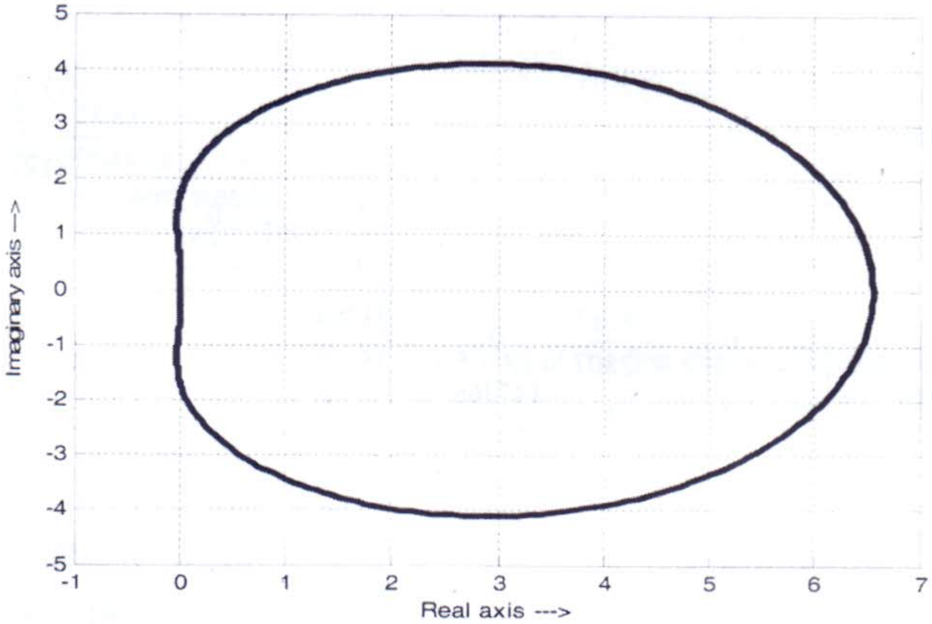

Figure 4.1. Showing Region of Absolute Stability of the HBDF

Numerical Solution

In this section, numerical examples are carried out for all computations with the written code in Maple 2015.

Example 4.1

We consider stiff systems (See Akinfenwa, 2011), in the range $0 \le t \le 10$

$$y_1'' = 998y_1 + 998y_2, y_1(0) = 1$$

$$y_2' = -999y_1 - 1999y_2, y_2(0) = 1$$

The exact solution is given by the sum of two decaying exponential components

$$y_1 = 4e^{-t} - 3e^{-1000t}, y_2 = -2e^{-t} + 3e^{-1000t}$$

The stiffness ratio is $1:1000$. The result of the proposed method is compared with Akinfenwa *et al.* (2011) at the end of $t = 10$ is presented in Table 1.

Table 1: Comparison of $BBDF_8$ for example 4.1 with h=0.1

t	Exact $y_1(t)\times 10^{-3}$ $y_2(t)\times 10^{-3}$	$BBDF_8$ $y_1\times 10^{-3}$ $y_2\times 10^{-3}$	Proposed method $(HBDF_4)$ $y_1\times 10^{-3}$ $y_2\times 10^{-3}$	$BBDF_8$ (Absolute error) $\lvert y_1(t)-y_1\rvert$ $\lvert y_2(t)-y_2\rvert$	Proposed method $\lvert y_1(t)-y_1\rvert$ $\lvert y_2(t)-y_2\rvert$
10	0.18159971904994	0.18159971946833	0.18159971904876	4.183×10^{-13}	1.180×10^{-15}
	−0.09079955952497	−0.09079985973416	−0.09079985952367	3.00209×10^{-10}	2.9999×10^{-10}

Example 4.2

We consider a system of equation which has been solved by Jackson and Kanue (1975) and Sahi *et al.* (2012)

$$y_1' = -y_1 + 95y_2,\ y_1(0) = 1$$
$$y'_2 = -y_1 - 97y_2,\ y_2(0) = 1$$

With exact solution of the system given by

$$y_1(t) = \frac{95}{47}e^{-2t} - \frac{48}{47}e^{-26t}$$
$$y_2(t) = \frac{48}{47}e^{-96t} - \frac{1}{47}e^{-2t}$$

The proposed method is compared with results obtained by Jackson and Kanue (1975) and Sahi *et al.* (2002) and the results are displayed in Table 2. As expected, the result show better accuracy than Jackson and Kanue (1975) and Sahi *et al.* (2002).

Table 2: Comparison between existing methods with the proposed method

h	Jackson and Kanue (1975) $\lvert y_1(t)-y_1\rvert$ $\lvert y_2(t)-y_2\rvert$	Sahi *et al.* (2012) $\lvert y_1(t)-y_1\rvert$ $\lvert y_2(t)-y_2\rvert$	Proposed Method $\lvert y_1(t)-y_1\rvert$ $\lvert y_2(t)-y_2\rvert$
0.0625	3×10^{-7}	9×10^{-11}	2×10^{-12}
	4×10^{-7}	1×10^{-8}	1×10^{-11}
0.03125	1×10^{-8}	4×10^{-12}	3×10^{-13}
	1×10^{-8}	4×10^{-12}	2×10^{-13}

Example 4.3

We consider the second order ordinary differential equation given by

$y'' + 1001y' + 1000y = 0$ and reduced to a system of first order differential equation

$$y' = z, y(0) = 1\ z' = -1000y - 1001z, z(0) = 0$$

The result of the problem has been compared with Abhulimen (2009), Abhulimen and Okunuga (2008).

The stiff system has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1000$ for the purpose of comparison, we solved the problem from the range of integration of $(0,1)$. Numerical results are shown below and it can clearly be seen that the results are more accurate than those presented by Abhulimen (2009), Abhulimen and Okunuga (2008) as displayed in Table 3.

Table 3: Comparison of proposed method for example 4.3

Method	t	$error = \lvert y_0 - y_1 \rvert$
Abhulimen (2009)	1	1.8×10^{-7}
Okunuga (1999)	1	5.26×10^{-8}
Abhulimen&Okunuga (2008)	1	5.29×10^{-9}
Proposed method	1	2.86×10^{-12}

Example 4.4. Consider the nonlinear problem

$$y_1' = -1002y_1 + 1000y_1^2, \quad y_1(0) = 1$$

$$y_2' = y_1 - y_2(1 + y_2), \quad y_2(0) = 1$$

This problem has been solved by Wu and Xia (2001) using two low accuracy explicit methods in vector form and also Akinfenwa *et al.*, (2013) solved the problem using a continuous block backward differentiation formula. Their results are reproduced in tables 4 and 5 and compared with our method in tables 6 and 7 with values of t the independent variable, h the step size, and N the number of computation step, the theoretical solutions; the proposed solution, and the absolute error.

Table 4: Absolute error for Akinfenwa, Jator, and Yao (2013)

t	h	N	Y	Theoretical	Akinfenwa, Jator, and Yao $k=4$	Absolute error
1	0.02	50	y_1	$1.353352832366127 \times 10^{-1}$	$1.35335286619327 \times 10^{-1}$	3.3827×10^{-9}
			y_2	$3.678794411714423 \times 10^{-1}$	$3.678794457979147 \times 10^{-1}$	4.6265×10^{-9}
10	0.02	500	y_1	$2.061153622416581 \times 10^{-9}$	$2.061154110095654 \times 10^{-9}$	4.8766×10^{-16}
			y_2	$4.539992976383902 \times 10^{-5}$	$4.539993515208483 \times 10^{-5}$	5.38966×10^{-12}

Table 5: Absolute error for Wu and Xia (2001)

t	h	N	Y	Theoretical	Wu and Xia (2001)	Absolute error
1	0.002	500	y_1	$1.353352832366127 \times 10^{-1}$	$1.353350271728111 \times 10^{-1}$	2.5606×10^{-7}
			y_2	$3.678794411714423 \times 10^{-1}$	$3.678795213211519 \times 10^{-1}$	8.0150×10^{-8}
10	0.001	10000	y_1	$2.061153622416581 \times 10^{-9}$	$2.061154177118385 \times 10^{-9}$	5.5468×10^{-16}
			y_2	$4.539992976383902 \times 10^{-5}$	$4.539993585613384 \times 10^{-5}$	6.0936×10^{-12}

Table 6: Absolute error for the proposed $HBDF_4$

t	h	N	Y	**Theoretical**	**Proposed Method**	**Absolute error**
1	0.02	50	y_1	$1.353352832366127\times10^{-1}$	$1.3533528814908014\times10^{-1}$	5.1537×10^{-11}
			y_2	$3.678794411714423\times10^{-1}$	$3.6787944124186063308\times10^{-1}$	7.0418×10^{-11}
10	0.02	500	y_1	$2.061153622416581\times10^{-9}$	$2.0611536298007748961\times10^{-9}$	7.38419×10^{-18}
			y_2	$4.539992976383902\times10^{-5}$	$4.5399929843545673782\times10^{-5}$	7.97066×10^{-14}

Table 7: Absolute error for the proposed $HBDF_4$

t	h	N	Y	**Theoretical**	**Proposed Method**	**Absolute error**
1	0.002	500	y_1	$1.353352832366127\times10^{-1}$	$1.3533528323661321371\times10^{-1}$	9.99201×10^{-16}
			y_2	$3.678794411714423\times10^{-1}$	$3.6787944117144303093\times10^{-1}$	9.99201×10^{-16}
10	0.001	10000	y_1	$2.061153622416583\times10^{-9}$	$2.061154177118387\times10^{-9}$	5.54702×10^{-16}
			y_2	$4.539992976383905\times10^{-5}$	$4.539993585613387\times10^{-5}$	6.09229×10^{-12}

Conclusion

A 4-step HBDF with continuous coefficients has been implemented as a self-starting method for solution of stiff systems of ODEs. The method avoids complicated subroutines needed for existing methods requiring starting values or predictors. The stability and consistency property of our method makes it attractive for numerical solution of stiff problems. We have demonstrated the accuracy of the methods for both linear and non linear problems. It is recommended that future research be focused on the implementation of the method to parabolic partial differential equations since it is *L*0- stable.

References

Abasi, N., Suleiman, M. B., Abbasi, N., & Musa, H. (2014). 2-Point Block BDF method with off-step points for solving stiff ODEs. *Journal of Soft Computing and Applications*, 1-15.

Abhulimen, C. E., & Otunta, F. O. (2009). A class of exponential fitting Numerical Integrators for initial value problems in ODEs. *Journal of Nigerian Mathematical Society*, 28, 13–28.

Abhulimen, C. E., & Okunuga, S. A. (2008). Exponentially fitted second derivative multistep method for stiff initial value problem for ODEs. *Journal of Engineering Science and Applications*, 5, 36–49.

Akinfenwa, O. A., Jator, S. N., & Yao, N. M. (2013). Continuous block backward differentiation formula for solving stiff ordinary differential equations. *Journal of Computers and Mathematics with Applications,* 65, 996-1005. http://doi:10.1016/j.camwa.2012.03.111.

Akinfenwa, O. A., Jator, S. N., & Yao, N. M. (2011). Implicit two step continuous hybrid block methods with four off-steps points for solving stiff ordinary differential equation. *International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering,* 5(3), 213-216.

Babangida, B., Musa, H., & Ibrahim, L. K. (2016). A new numerical method for solving stiff initial value problems. *Fluid Mech. Open Acc.,* 3(2), 136. http://*doi:10.4172/2476-2296.1000136.*

Biala, T. A., & Jator, S. N. (2015). A boundary value approach for solving three-dimensional elliptic and hyperbolic partial differential equations. *Springer plus Journals,* 4(1), 588.

Butcher, J. C. (1965). A modified multistep method for the numerical integration of ordinary differential equations. *J. Assoc. Comput. Mach.,* 12, 124-135.

Brugnano, L., & Trigiante, D. (2000). *Block implicit methods for ODEs, in recent trend in Numerical Analysis.* Nova Science Publisher Inc., Commack, New York.

Curtiss, C. F., & Hirschfelder, J. O. (1952). Integration of stiff equations. *Proc. Natl. Acad. Sci.* 38, 235-243.

Dahlquist, G. (1974). Problems related to the numerical treatment of stiff differential equations.

Fatunla, S. O. (1991). Block methods for second order IVPs. *Intern. J. Comput. Math.* 41, 55 - 63.

Gear, C. W. (1974). Hybrid methods for initial value problems in ordinary differential equations. *SIAM J. Numer. Anal.* 11, 321-331.

Gupta, G. K. (1978). Implementing second-derivative multistep methods using Nordsieck polynomial representation. *J. Math. Comp.* 32, 13-18.

Gragg, W., & Stetter, H. J. (1964). General multistep predictor-corrector methods. *J. Assoc. Comput. Mach.* 11, 188-209.

Henrici, P. (1962). *Discrete Variable Methods in ODEs.* John Wiley, New York.

Ibrahim, Z. B., Isk, K., & Othman, A. (2007). Variable step block backward differentiation formula for first order stiff ODEs. *Proceeding of the World Congress on Engineering,* 2, 2-6.

Ibrahim, Z. B., Othman, K. I., & Suleiman, M. B. (2007). Implicit r-point block backward differentiation formula for first order stiff ODEs. *Appl. Math. Comput.* 186, 558-565.

Jackson, L. W., & Kenue, S. K. (1974). A fourth-order exponentially fitted method. *SIAM J. Numer. Annal,* 11, 965–978.

Lambert, J. D. (1973). *Computational methods in ordinary differential equations.* John Wiley, New York, USA.

Majid, Z. (2004). *Parallel block methods for solving ordinary differential equations.* PhD Thesis, Universiti Putra Malaysia.

Milne, W. E. (1953). *Numerical solution of differential equations.* John Wiley and Sons, New York, USA.

Musa, H., Suleiman, M. B., Ismail, F., Senu, N., & Ibrahim, Z. B. (2013). An accurate block solver for stiff IVPs. *ISRN Applied Mathematics.*

Ngwane, F. F., & Jator, S. N. (2012). Block hybrid-second derivative method for stiff systems. *International Journal of Pure and Applied Mathematics,* 80(4), 543–559.

Okunuga, S. O. (2008). One-leg multistep method for numerical integration of periodic second order initial value problems. *Journal of Nigerian Association of Mathematical Physics,* 13, 63-68.

Suleiman, M. B., Musa, H., Ismail, F., Senu, N., & Ibrahim, Z. B. (2014). A new super class of block backward differentiation formulas for stiff ODEs. *Asian-European J. Math.*

Suleiman, M. B., Musa, H., Ismail, F., & Senu, N. (2013). A new variable step size block backward differentiation formula for solving stiff IVPs. *Int. J. Comput. Math.* 90, 2391-2408.

Sahi, R. K., Jator, S. N., & Khan, N. A. (2012). A Simpson's-type second derivative method for stiff systems. *International Journal of Pure and Applied Mathematics,* 81(4), 619–633.

Wu, X. Y., & Xia, J. L. (2001). Two low accuracy methods for stiff systems. *Appl. Math. Comput.* 123, 141-153.