A Homotopy-Perturbation analysis of the non-linear contaminant transport problem in one dimension with an initial continuous point source.

Aiyesimi, Y. M and Jimoh, O. R. Department of Mathematics/Statistics, School of Natural and Applied Sciences, Federal University of Technology, Minna, Nigeria.

Abstract

In this research work, a Homotopy-perturbation analysis of a non –linear contaminant flow equation with an initial continuous point source is provided. The equation is characterized by advection, diffusion and adsorption. We assume that the adsorption term is modeled by Freudlich Isotherm. We provide an approximation of this equation using homotopy-perturbation transformation and solve the resulting linear equations analytically by homotopy-perturbation method. Graphs are plotted using the solution obtained from the method and the results are presented, discussed and interpreted. The research findings show that the concentration increases with time and decreases as distance increases.

Keywords: Homotopy-perturbation, contaminant, advection, diffusion, adsorption

Email: jimohorazaq@yahoo.com. Phone: 08077808699, 08162934661

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Introduction

Human Society during the past several centuries has created a large number of chemical substances that often find their way into the groundwater system, either intentionally applied during agricultural practices or unintentionally released from leaking and community wastes disposal sites, or weapons production related activities. Consequently, as many of these chemicals represent a significant health risk when they enter the food chain, contamination of both surface and subsurface water supplies has become a critical issue. Several types of fertilizers, pesticides and fumigants are now routinely applied to agricultural lands making agricultural one of the most important sources for non-point source pollution. While many agricultural chemicals are generally beneficial in surface soils, their leaching into the groundwater system may pose serious threat. To predict the fate of such pollutants during their transport has become a tasking job for hydro-geologist and scientists. The problem involves defining the flow lines of groundwater in the aquifers, the travel time of the water along the flow lines and to predict the chemical reaction which alters the concentrations during transport.

Due to the presence of first partial derivative term (advective) and second partial derivative term (hydrodynamic dispersion) which both exist in the governing equation of the nonlinear contaminant flow problem, most of the numerical methods fail to analyse it sufficiently. Though researches have been carried out in the study of hydrodynamic dispersion with various initial and boundary conditions, none employed Homotopy perturbation method. Studies on non-linear contaminant transport equations using numerical methods in the analysis (Clint, 1993; Makinde and Chinyoka, 2010), perturbation methods (Aiyesimi, 2004, Gideon and Aiyesimi, 2005 and Gideon, 2011) have been employed. Other studies adopted the Laplace Transformation method (Singh and Singh 2010, Yadav, and Kumar 2011).

The Homotopy perturbation method was used to solve wide range of physical problems, eliminating the limitations of perturbation method (Rajabi, *et. al.* 2007, Jiya 2010 and Abdul-Sattar, *et. al*, 2011). This study, presents a homotopy-perturbation analysis for constant initial concentration for the non-linear contaminant transport problem.

Formulation of the problem

We consider an incompressible fluid flow through a homogeneous, saturated porous medium where the fluid is not solute-free, i.e. contaminated with solute of concentration $A > 0$. The following assumptions are made (i)The flow is steady (ii) the solute transport is described by advection, molecular diffusion and mechanical dispersion (iii) the flow is one dimensional and in x-direction.

Under these assumptions, mass conservation of the contaminant gives the equation:

$$C_t + UC_x + \frac{pb}{n} S_t - DC_{xx} = 0, 0 < x < \infty, t > 0 \qquad (1)$$

Where $S(x,t)$ is the mass of the contaminant absorbed on the solid matrix per unit mass of the solid, $pb > 0$ is the bulk density of the porous medium, $n > 0$ is the porosity $D > 0$ is the molecular diffusion and mechanical dispersion $[L^2T^{-1}]$, C is the concentration of the contaminant $[ML^{-3}]$ and U is the fluid velocity $[LT^{-1}]$.

Basic Idea of Homotopy-Perturbation Method (HPM)

In order to explain the method of homotopy-perturbation, we consider the function:

$$A(u) - f(r) = 0, r \in \Omega \quad (2)$$

having the boundary conditions:

$$B(u, \frac{\partial u}{\partial n}) = 0, r \in \Gamma, \quad (3)$$

where $A, B, f(r)$ and Γ are a general differential operator, a boundary operator, a known analytical function and a boundary of the domain respectively. The operator A can be divided into two parts L and N where L is linear and N is nonlinear. Equation (2) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, r \in \Omega \quad (4)$$

By homotopy-perturbation method, we form a homotopy:

$v(r, p) : \Omega \times [0,1] \rightarrow R$

which satisfies

$$H(v, p) = (1 - p)[L(u) - L(u_0)] + p[A(v) - f(r)] = 0, \quad (5)$$

$$p \in [0,1], r \in \Omega,$$

where $p \in [0,1]$ is an embedding parameter, while u_0 is an initial approximation of (2), which satisfies the boundary conditions (3). From equation (5), we have:

$$H(v,0) = L(v) - L(u_0) = 0 \quad (6)$$

$$H(v,1) = A(v) - f(r) = 0 \quad (7)$$

According to HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solutions of equation (4) can be written as a power series in p:

C and the best approximate solution is

$$u = \lim_{p \rightarrow 1} v = v^{(0)} + pv^{(1)} + p^2 v^{(2)} + ... \quad (8)$$

The convergence of the above solution is discussed in Abdul-Sattar, J. et al (2011).

Problem solution

We consider the non-linear contaminant flow equation (1):

$$C_t + UC_x + \frac{pb}{n} S_t - DC_{xx} = 0, 0 < x < \infty, t > 0$$

Equation (1) can be rewritten as

$$\frac{\partial C}{\partial t} + \frac{\partial \phi(C)}{\partial t} + U\frac{\partial C}{\partial x} - D\frac{\partial^2 C}{\partial x^2} = 0 \quad (9)$$

This could further be expressed as

$$\frac{\partial C}{\partial t} + \frac{\partial \phi}{\partial C}.\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} - D\frac{\partial^2 C}{\partial x^2} = 0 \quad (10)$$

Equation (3) takes the form

$$\frac{\partial C}{\partial t} + \varepsilon\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} - D\frac{\partial^2 C}{\partial x^2} = 0 \quad (11)$$

where $\frac{\partial \phi}{\partial C}$ is assumed as ε, the perturbation parameter.

(11) simplifies to

$$(1 + \varepsilon)\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} - D\frac{\partial^2 C}{\partial x^2} = 0, C(0, t) = A, C(\infty, t) = 0, C(x,0) = Ae^{-\lambda x} \quad (12)$$

By the Homotopy-perturbation transformation equation:

$$H(v, p) = [(1 - p)\frac{\partial C}{\partial t}] + p[(1 + \varepsilon)\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} - D\frac{\partial^2 C}{\partial x^2}] = 0 \quad (13)$$

$$C(x,t) = v^{(0)} + pv^{(1)} + p^2 v^{(2)} + p^3 V^{(3)} + ... \quad (14)$$

By substituting (14) in (13) and comparing the coefficients of like terms on both sides, we have the following set of equations:

$$p^0 : (1 + \varepsilon)\frac{\partial}{\partial t} v^{(0)}(x,t) = 0 \quad (15)$$

$$p^{(1)} : (1 + \varepsilon)\frac{\partial}{\partial t} v^{(1)}(x,t) + U\frac{\partial}{\partial x} v^{(0)}(x,t) - D\frac{\partial^2}{\partial x^2} v^{(0)}(x,t) = 0 \quad (16)$$

$$p^{(2)}: (1+\varepsilon)\frac{\partial}{\partial t}v^{(2)}(x,t) + U\frac{\partial}{\partial x}v^{(1)}(x,t) - D\frac{\partial^2}{\partial x^2}v^{(1)}(x,t) = 0 \quad (17)$$

$$p^{(3)}: (1+\varepsilon)\frac{\partial}{\partial t}v^{(3)}(x,t) + U\frac{\partial}{\partial x}v^{(2)}(x,t) - D\frac{\partial^2}{\partial x^2}v^{(2)}(x,t) = 0 \quad (18)$$

$$p^{(4)}: (1+\varepsilon)\frac{\partial}{\partial t}v^{(4)}(x,t) + U\frac{\partial}{\partial x}v^{(3)}(x,t) - D\frac{\partial^2}{\partial x^2}v^{(3)}(x,t) = 0 \quad (19)$$

$$p^{(n)}: (1+\varepsilon)\frac{\partial}{\partial t}v^{(n)}(x,t) + U\frac{\partial}{\partial x}v^{(n-1)}(x,t) - D\frac{\partial^2}{\partial x^2}v^{(n-1)}(x,t) = 0 \quad (20)$$

Equation (8) admits a solution of the form:

$$v^{(0)}(x,t) = Ae^{-\lambda x} \text{ and } v^{(i)}(x,0) = 0, i = 1,2,3,... \quad (21)$$

Such that $v^{(0)}(x,0) = Ae^{-\lambda x}, A > 0, v^{(0)}(0,t) = A$ and $v^{(0)}(\infty,t) = 0$.

The order 1 problem in equation (15) has a solution:

$$v^{(1)}(x,t) = \frac{A\lambda e^{-\lambda x}(U + D\lambda)t}{1+\varepsilon} \quad (22)$$

Similarly, the solutions of order 2 and 3 problems are:

$$v^{(2)}(x,t) = \frac{1}{2}\frac{A\lambda^2 e^{-\lambda x}(U + D\lambda)^2 t^2}{(1+\varepsilon)^2} \quad (23)$$

$$v^{(3)}(x,t) = \frac{1}{6}\frac{A\lambda^3 e^{-\lambda x}(U + D\lambda)^3 t^3}{(1+\varepsilon)^3} \quad (24)$$

Therefore, the approximate solution of contaminant flow equation in one dimension is

$$C(x,t) = \lim_{p\to 1}(v^{(0)} + pv^{(1)} + p^2 v^{(2)} + p^3 v^{(3)} + ...) \quad (25)$$

$$C(x,t) = Ae^{-\lambda x} + \frac{A\lambda e^{-\lambda x}(U + D\lambda)t}{1+\varepsilon} + \frac{1}{2}\frac{A\lambda^2 e^{-\lambda x}(U + D\lambda)^2 t^2}{(1+\varepsilon)^2} + \frac{1}{6}\frac{A\lambda^3 e^{-\lambda x}(U + D\lambda)^3 t^3}{(1+\varepsilon)^3} + o(p^4)$$

$$C(x,t) = Ae^{-\lambda x + (\frac{U+D\lambda}{1+\varepsilon})\lambda t} \quad (26)$$

Results and discussions

The analytical solution (26) is used with the help of input data to understand the concentration behavior in some cases. The graph (Fig. 1) of the concentration against distance is plotted when U=0.1, D=0.01, ε=0.001, λ=1 and time t varying as t=0.1, 0.4, 0.7, and 1. Similarly, the concentration is plotted against time varying x as x=0.1, 0.4, 0.7,1 and keeping U=0.1, D=0.01, ε=0.001, λ=1 t = 1 fixed (Fig. 2). Thirdly, the concentration is plotted against distance when U=0.1, ε=0.001, λ=1 t = 1 are fixed and the dispersion varied (Fig. 3). Finally, a three-dimensional graph of the concentration against distance (x) and time (t) is plotted when U=0.1, ε=0.001, λ=1 to show the effect of increase in distance and time on the concentration (Fig. 4).

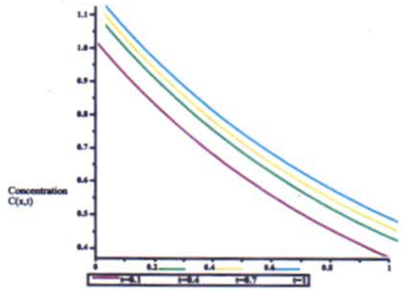

Figure 1: Graph of concentration against distance.

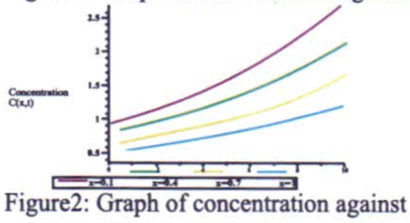

Figure2: Graph of concentration against time.

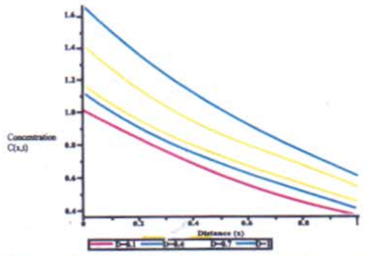

Figure 3: Graph of concentration against distance.

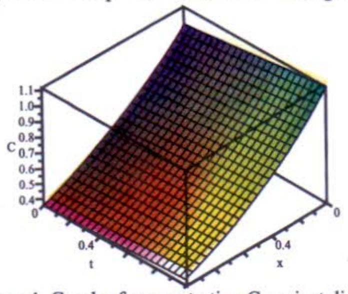

Figure 4: Graph of concentration C against distance (x) and time (t).

The research findings show that the concentration increases with time. Similarly, we observed that when the distance increases, the concentration decreases. This study could be useful to hydrogeologists and geophysicists in locating their wells.

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