ONE-STEP SECOND DERIVATIVE BLOCK INTRA-STEP POINTS FOR STIFF SYSTEMS OF INITIAL VALUE PROBLEMS

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ABSTRACT

Modified class of single-step numerical schemes were proposed to improve the order of accuracy by imposing intra-step points in the formulation process of the proposed algorithms. The behaviour of modified numerical algorithm is of great concern when varieties of countable off-grid points are imposed within the grid points in the derivation process. In this study, we present a one-step second derivative modified algorithms for numerical solution of first order initial value problems of ODEs. The consistency, convergence and order of accuracy of the algorithms are improved by interpolating and collocating the power series polynomials at carefully selected intra-points generated by the Bhaskara cosine formula. The proposed methods are self-starting and applied as simultaneous numerical integrators on non-overlapping intervals. In order to further illustrate the effectiveness of the proposed algorithms, stiff systems of IVPs are considered and results obtained are compared with those from related schemes and other methods in the literature.

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ABBREVIATIONS

- OSDBM2: The proposed one-step second derivative block method with 2 intra-points
- OSDBM3: The proposed one-step second derivative block method with 3 intra-points
- OSDBM4: The proposed one-step second derivative block method with 4 intra-points
- OSDBM5: The proposed one-step second derivative block method with 5 intra-points
- OSDBM6: The proposed one-step second derivative block method with 6 intra-points
- CEA: The method of Abhulimen (2014), of order p=5
- EOS: The method of Ehieie *et al.* (2013), (a 2-step method of order p=3).
- A&U: The method of Abhulimen and Ukpebor (2019), (a 4 step method of order p= 6)
- MOS: The method of Mohammed *et.al.* (2019), (a 2-step method of order p= 4)
- ABG: The method of Akintububo (2019), (a 4-step method of order p= 8).
- AAK: The method of Akinfenwa (2014), (a 2-step method of order p=8).
- NJ: The method of Ngwane and Jator (2012), (a one-step method of order p= 6).
- AJY: The method of Akinfenwa *et al.* (2013), (a 6-step method of order p=6).
- N.O. : The method of Nwachukwu and Okor (2018), (a 4-step method of order p=5).
- AAO: The method of Akinfenwa *et al.* (2017), (a 3-step method of order p= 8).
- HBSDBDF: The method Akinfenwa *et al.* (2020), (a 3-step method of order p=7)

CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

Numerical solutions for ordinary differential equations (ODEs) are very important in scientific computation, as they are widely used for solution of real life problems. Analytic methods have been used in some literature to solve mathematical problems; but some of the problems that occur in real life cannot be solved using analytic methods. The importance of numerical methods is to find approximate solution to problems. This is usually achieved with steps, with each step improving the accuracy of the later, until enough accuracy of approximation is obtained. It is important to note that numerical methods cannot give an exact solution, therefore the errors involved are of great concern of study.

Many applications are modeled by systems of ordinary differential equations; these systems exhibit a behaviour known as stiffness. Stiff systems are considered challenging because explicit numerical methods designed for non-stiff problems are used with very small step sizes or do not converge at all. The knowledge of stiffness, occurring in differential equations came as a result of some pioneering works done by the two chemists, Curtiss and Hirschfelder (1952). Shampine and Gear (1971) expounded the characteristics of numerical methods used for solving problems with stiffness and discussed the different realistic goals when solving stiff problems which involve methods with strong stability properties for solving stiff systems. Models associated to these problems are first order system of ordinary differential equation of the form described in Burden and Faires (2011) as

$$\left.\begin{array}{l} y_1' = f_1\left(x, y_{x1}, y_2, \ldots, y_n\right) \\ y_2' = f_1\left(x, y_1, y_2, \ldots, y_n\right) \\ \vdots \\ y_n' = f_1\left(x, y_1, y_2, \ldots, y_n\right) \\ with\ \ y_{n_0}\left(x_{n_0}\right) = y_{n_0}, x \in \left[x_0, x_n\right] \end{array}\right\} \quad (1.1)$$

where f satisfies the Lipschitz condition as given in Henrici (1962).

The common methods used to solve ODEs are categorized as single-step (multistage) methods such as Runge-Kutta methods and multistep (one-stage) methods such as Adams-Bashforth-Moulton methods (Avdelas and Simos, 1996). Stroller and Morrison (1958), Butcher (1965), Fang (2001), Watts and Shampine (1972) have fully studied implicit one-step methods.

However, problems frequently arise in which the magnitude of the derivative increases but the solution does not. In this situation, the error can grow so large that it dominates the calculations. Initial-value problems for which this is likely to occur are called stiff equations and are quite common, particularly in the study of vibrations, chemical reactions, and electrical circuits. This class of differential equations usually have a term of the form e^{-ct} in their exact solutions, where c is a large positive constant for which explicit methods are unsuitable (Baraff, 1997). For this class of differential equations, implicit methods are recommended. In this research work, a class of block hybrid second derivative implicit single-step methods for solving ODEs will be constructed through interpolation and collocation techniques – Gragg and Stetter (1964), Gear (1965), Kohfeld and Thompson (1967), Shampine and Watts (1969), Gladwell and Sayers (1976), Gupta (1978), Lambert (1991), Onumanyi *et al.* (1994), Akinfenwa *et al.* (2011), Mehdizadeh *et al.* (2012), Sahil *et al.* (2012) and Yakubu *et al.* (2017). The continuous representation generates a main discrete one-step second derivative block intra-step method (OSDBM) and additional methods which are combined and used as a block method to simultaneously produce approximations $\{y_{n+\eta}, y_{n+1}\}$ at a block points $\{x_{n+\eta}, x_{n+1}\}$, $h = x_{n+1} - x_n$, $n = 0,..., \mathrm{N}-1,$ on a partition $[a,b]$, where $[a,b]$ is the interval of integration, $\eta \in (0,1)$ are the intra-step points, h is the constant step-size, n is a grid index and $N > 0$ is the number of steps.

The method preserves the Rung-kutta traditional advantage of being self-starting and is more accurate since it is implemented as a block method. We note that block methods were first introduced by Milne (1953) for the purpose of obtaining starting values for predictor-corrector algorithms. However, Rosser (1967), developed Milne's idea into algorithms for general use. We emphasis that the continuous representation of our method is developed for general use, not only as a means of obtaining starting values for predictor-corrector algorithms; it generates a main discrete scheme and additional methods which are combined and implemented as a block method which simultaneously generates approximations $\{y_{n+\eta}, y_{n+j}\}$ to exact solutions $\{y(x_{n+\eta}), y(x_{n+1})\}$

1.2 Statement of the Research Problem

There exist certain classes of ordinary differential equations to which some numerical methods are not applicable. One of such classes is stiff system of ordinary differential equations which defies explicit methods owing to its property of containing components with different time scale which requires A-stability methods to be used. This places serious restriction on the choice of step-length to be used when applying some numerical methods in solving such problems. Explicit methods are said to be incapable for solving a stiff problem and as such, implicit methods (both single-step and multi-step) have been adopted over the years (Omar and Kuboye, 2015; Ndanusa and Tafida, 2016), most of which are used as predictor-corrector methods, requiring starting values. Implicit methods with large region of absolute stability are recommended and Improved Euler is one of such methods.

Several single-step methods and its modification have been formulated which have been very successful, but efforts are still being made towards having one which is of higher order.

In view of the above, some modified single-step methods incorporating second derivative with carefully selected off-step points are proposed, in which higher order is sought.

1.3 Aim and Objectives of the Study

This study aims at constructing a continuous formulation of one-step second derivative block intra-step methods for Stiff System of ordinary differential equations. The specific objectives are to;

1. derive families of one-step method by incorporating in the derivation process, off-step points generated from the Bhaskara cosine approximation formula;
2. compute order and error constant of the methods;
3. establish the convergence and stability analysis of the methods;
4. implement the methods for the solution of stiff systems of initial value problems of ODEs;
5. establish the efficacy of the methods by comparing the results of the proposed methods with some existing methods found in the literatures.

1.4 Significance of the Study

This research shall contribute to learning in the aspect of numerical methods by deriving a class of one-step numerical schemes that are effective and suitable for the solution of stiff systems of ordinary differential equations with the believe that the proposed methods will fulfill some required states of numerical conditions (convergence and stability).

1.5 Scope and Limitation of the Study

This research work focuses on the derivation and implementation of a class of one-step numerical schemes for the approximate solution of first order Stiff systems of initial value problems. Higher order ODEs are not considered in this project.

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Numerical Methods

Numerical analysis is the study of algorithms that use numerical approximation for the problems of mathematical analysis of real life problems arising from all fields of engineering, physical sciences, life sciences, social sciences, medicine and business. In most cases, some of these problems are dynamical in nature with respect to time, space and other physical quantities which can be transformed into ordinary differential equations (ODEs). However, some of the ODEs do not have analytical solution, therefore one of the possible ways to tackle this problem is to consider a discrete domain rather than a continuous one. Hence for practical purposes such as engineering, a numeric approximation to the solution is often sufficient. Numerical methods for ODEs are methods used to find numerical approximations to the solutions of ODEs. Conceptually, a numerical method starts from an initial point and then takes a short step forward in time to find the next solution point. The general numerical methods for approximating (1.1) for the value of $y(x)$ at discrete times t_i can be written as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\varphi_f\left(y_{n+k}, y_{n+k-1}, ..., y_n, x_n, h\right)$$

(2.1)

$y_n \approx y(x_n)$ where $x_n = x_0 + nh$ for

where h is the time step and n is an integer.

The common numerical methods used to solve ODEs are categorized as one-step ($k = 1$) methods and multistep ($k > 1$) methods, (Akinfenwa *et al.*, 2011).

2.1.1 One step method

The general form of a one-step method for solving (1.1) is given as

$$y_{n+1} = y_n + hf\left(x_n, y_n, h\right) \tag{2.2}$$

where y_n is the initial condition that is used to advance the solution of y_{n+1} and $f\left(x_n, y_n, h\right)$ is the incremental function per unit step.

Examples include

- The Euler method:

The Euler method is a first-order method which means that the local error is proportional to the square of the size and the global error is proportional to the step size. The Euler method often serves as the basis to construct more complex methods such as the predictor-corrector method.

The forward Euler, given as

$$y_{n+1} = y_n + hf\left(x_n, y_n\right) \tag{2.3}$$

The backward Euler, (which is an implicit method) given as:

$$y_{n+1} = y_n + hf\left(x_{n+1}, y_{n+1}\right) \tag{2.4}$$

Improved Euler method, given as:

$$y_{n+1} = y_n + \frac{h}{2}\left\{ f\left(x_n, y_n\right) + f\left(x_{n+1}, y_{n+1}\right)\right\} \tag{2.5}$$

The Heun's method also known as trapezoidal method:

$$y_{n+1} = y_n + \frac{h}{2}\left\{ f\left(x_n, y_n\right) + f\left(x_n + h, y_n + h\left(f\left(x_n, y_n\right)\right)\right)\right\} \tag{2.6}$$

The methods (2.3) to (2.6) are called Runge-Kutta methods. The methods (2.3) and (2.4) are of order 1 while (2.5) and (2.6) are of order 2. However, in order to obtain some higher order, zero stable one step methods, scholars have explored the introduction of off-step points and the introduction of higher derivatives into the various forms of one step methods mentioned above. Abdelrahim and Omar (2015) adopted the method of interpolation and collocation method in developing a new one step hybrid block method (with two off step points) for solving third order initial value problem of ordinary differential equations. Their method which is of order 4 and zero stable successfully solved third order IVPs. Omar and Abdelrahim (2016) developed an order 4, zero stable one step hybrid block method for solving second order IVPs using collocation and interpolation approach. In deriving their method, the power series used as basis function to approximate the solution is interpolated at the off step points while its second derivative is collocated at all points in the selected interval. Sunday *et al.* (2016) considered damping as an influence within or upon an oscillatory system that has the effect of reducing, restricting or preventing its oscillation, therefore proposed a one-step sixth-order computational method for the solution of second order free undamped and free damped motions in mass-spring systems through interpolation and collocation techniques of power series to generate a continuous computational hybrid block method. Similarly, Anake *et al.* (2012) developed an implicit on step method of order two for the numerical solution of second order initial value problems of ordinary differential equations by collocation and interpolation technique. In their method, the introduction of an off-step point guaranteed the zero stability and consistency of the method.

2.1.2 Linear multistep method (LMM)

The general k-step linear multistep method is given as

$$\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \ldots + \alpha_1 y_{n+1} + \alpha_0 y_n = h\left(\beta_k f_{n+k} + \beta_{k-1} f_{n+k-1} + \ldots + \beta_1 f_{n+1} + \beta_0 f_n\right) \quad (2.7)$$

or equivalently

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$$

It is always the case that $\alpha_k = 1$, also at least one of the α_0 and β_0 will be non-zero.

A linear multistep method is defined by a choice of quantities α_j's and β_j's. If $\beta_k = 0$, the method is called explicit (because the method can directly compute y_{n+k}). If $\beta_k \neq 0$, the method is called implicit (since the value of y_{n+k} depends on the value of $f(x_{n+k}, y_{n+k})$, and the equation must be solved for y_{n+k}. Sometimes an explicit multistep method is used to "predict" the value of y_{n+k}. That value is then used in an implicit formula to "correct" the value. The result is a predictor – corrector method.

2.1.3 Families of multistep methods

There are three families of linear multistep methods that are commonly used:

- The Adams – Bashforth methods are explicit methods and the coefficients are $\alpha_0 = \alpha_1 = ... = \alpha_{k-2} = 0$ and $\alpha_{k-1} = -1$, while β_j's are chosen such that the methods have an order k. The Adams – Bashforth methods with k=1,2,3,4,5 are: (Hairer *et al.,* 1993).

$$\left.\begin{aligned}
y_{n+1} &= y_n + hf(x_n, y_n) \\
y_{n+2} &= y_{n+1} + h\left(-\frac{1}{2} f(x_n, y_n) + \frac{3}{2} f(x_{n+1}, y_{n+1})\right) \\
y_{n+3} &= y_{n+2} + \frac{h}{12}\left(5f(x_n, y_n) - 16f(x_{n+1}, y_{n+1}) + 23f(x_{n+2}, y_{n+2})\right) \\
y_{n+4} &= y_{n+3} + \frac{h}{24}\left(-9f(x_n, y_n) + 37f(x_{n+1}, y_{n+1}) - 59f(x_{n+2}, y_{n+2}) + 55f(x_{n+3}, y_{n+3})\right) \\
y_{n+5} &= y_{n+4} + \frac{h}{720}\begin{pmatrix} 251f(x_n, y_n) - 1274f(x_{n+1}, y_{n+1}) + 2616f(x_{n+2}, y_{n+2}) - \\ 2774f(x_{n+3}, y_{n+3}) + 1901f(x_{n+4}, y_{n+4}) \end{pmatrix}
\end{aligned}\right\} \quad (2.8)$$

- The Adam-Moulton Methods – are implicit and the coefficients are $\alpha_0 = \alpha_1 = ... = \alpha_{k-2} = 0$ and $\alpha_{k-1} = -1$, while β_j's are chosen such that the methods have a highest order possible and $\beta_k \neq 0$. By removing the restriction that $\beta_k = 0$ a k-step Admas Moulton method can reach order k+1, while the Adams – Bashforth methods has only order k. The Adams – Moulton methods with k=1,2,3,4 are: (Hairer *et al.*, 1993).

$$\left.\begin{aligned}
y_{n+1} &= y_n + hf\left(x_{n+1}, y_{n+1}\right) \\
y_{n+1} &= y_n + \frac{h}{2}\left(f\left(x_n, y_n\right) + f\left(x_{n+1}, y_{n+1}\right)\right) \\
y_{n+2} &= y_{n+1} + \frac{h}{12}\left(-f\left(x_n, y_n\right) + 8f\left(x_{n+1}, y_{n+1}\right) + 5f\left(x_{n+2}, y_{n+2}\right)\right) \\
y_{n+3} &= y_{n+2} + \frac{h}{24}\left(f\left(x_n, y_n\right) - 5f\left(x_{n+1}, y_{n+1}\right) + 19f\left(x_{n+2}, y_{n+2}\right) + 9f\left(x_{n+3}, y_{n+3}\right)\right) \\
y_{n+4} &= y_{n+3} + \frac{h}{720}\begin{pmatrix} -19f\left(x_n, y_n\right) + 106f\left(x_{n+1}, y_{n+1}\right) - 264f\left(x_{n+2}, y_{n+2}\right) + 646f\left(x_{n+3}, y_{n+3}\right) + \\ 251f\left(x_{n+4}, y_{n+4}\right) \end{pmatrix}
\end{aligned}\right\}$$

(2.9)

- The backward differentiation formula (BDF) – is a family of implicit methods in which for a given function and time, approximate the derivative of a function using information from already computed times, thereby increasing the accuracy of the approximation. These methods are especially used for the solution of stiff differential equations (Curtiss and Hirschflder, 1952). The general formula for a BDF can be written as

  $$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta f\left(x_{n+k}, y_{n+k}\right)$$

  (2.10)

  The coefficients α_j 's and β are chosen so that the method achieves order k, which is the maximum possible.

Linear multistep methods (of different families as stated above) have been found to produce relatively higher order of accuracy to differential equations in the stiff systems of ordinary differential equations by many researchers – Awoyemi (2003), Ndanusa (2007), Ndanusa and Adeboye (2008), Ndanusa and Adeboye (2009), Yahaya and

Mohammed (2010), Mohammed and Yahaya (2010), Mohammed (2010), to mention but few. Some researchers have also attempted the approximate solution to higher order differential equations directly using linear multistep methods without reduction to system of first order ordinary differential equation (Mohammed *et al.,* 2010, Mohammed *et al.*, 2019).

2.2. Block Method

A block method is a method for computing approximations of IVPs simultaneously at points $x_{n+1}, x_{n+2}, ..., x_{n+N}$ when using linear multistep method. It is of the form:

$$A^{(1)}Y_w = A^{(0)}Y_{w-1} + h\left[B^{(0)}F_{w-1} + B^{(1)}F_w\right] \tag{2.11}$$

where

$$Y_w = \left(y_{n+1}, y_{n+2}, ..., y_{n+k}\right)^T, Y_{w-1} = \left(y_{n+k-1}, y_{n+k-2}, ..., y_n\right)^T,$$
$$F_w = \left(f_{n+1}, f_{n+2}, ..., f_{n+k}\right)^T, F_{w-1} = \left(f_{n+k-1}, f_{n+k-2}, ..., f_n\right)^T$$

The block method was first proposed by Milne (1953) who advocated their use only as a means of obtaining starting values for predictor – corrector algorithms (Sarafyan, 1965) and later the idea of proceeding in blocks for general use was developed by Rosser (1967). There is considerable literature on the methods of solution of ordinary differential equations (ODEs) by predictor-corrector methods (Lambert, 1973; 1991; Onumanyi *et al.,* 1994; Fatunla 1994; Awoyemi and Idowu, 2005; Areo and Adeniyi, 2013; Omar and Kuboye, 2015; Ndanusa and Tafida, 2016). The predictor-corrector method has a great setback of starting values which means more time and human efforts are needed to derive another method(s) to allow the main method to be implemented. Moreover, the predictors developed are of lower order to the corrector; therefore, corrupting the expected results generated by the corrector. However, in order to overcome the challenge in developing separate predictors, and other shortcomings in the predictor-corrector method, the development of the block method was proposed (Milne, 1953). This contains main and additional methods generated from the same continuous scheme, which are usually combined to simultaneously produce pieces of solutions to IVPs at non-overlapping points $(x_{n+1}, x_{n+2}, ..., x_{n+N})$; hence, it is self-starting. Block methods have been found to give better approximation as seen in the studies by (Badmus and Yahaya, 2009; Jator and Li 2012; Mohammed, 2011; Akinfenwa, *et al.*, 2013; Mohammed and Adeniyi, 2014; Badmus, *et al.*, 2015; Omar and Adeyeye, 2016; Akinfenwa *et al.*, 2017).

2.3 Continuous Multistep Collocation Method

Continuous multistep collocation involves using approximate function of the form

$$Y(x) = \sum_{j=0}^{r+s-1} a_j p_j(x) \quad (2.12)$$

for the solution of the differential equation (1.1). Where r is the number of points in which (2.12) is evaluated (interpolation points) and s is the number of points in which the derivative of (2.12) is evaluated (usually called the collocation points). $a_j,\ j = 0,1,2,...,r+s-1$ are unknown coefficients to be determined, $p_j(x)$ are terms of a choice of orthogonal polynomial which can be power series, Legendre polynomial, Laguerre polynomial, Hermit epolynomial or Chebyshev polynomial.

2.4. Stiff System

Many fields of application, notably spring and damping system, control system, chemical reaction, electrical circuits, diffusion and control theory, yield initial value problems involving systems of ordinary differential equations which exhibit a phenomenon known as ‘stiffness’ (Lambert, 1973). This behaviour is distressing because these systems are characterized by very high stability, which can turn into very high instability when approximated by standard numerical methods (Butcher, 2008). Attempts to use classical numerical methods to solve such systems of ODEs usually encounter very substantial difficulties. In mathematics, a stiff equation is a differential equation for which certain numerical methods for solving the equation are numerically unstable, unless the step size is taken to be extremely small. Although it has proven difficult to give a precise definition of stiffness; Lambert (1973) gives some features in which stiff systems usually exhibit:

- If a numerical method with a finite region of absolute stability, applied to a system with any initial conditions, is forced to use in a certain interval of integration a step size which is excessively small in relation to the smoothness of the exact solution in that interval, then the system is said to be stiff in that interval.
- A linear constant coefficient system is stiff if all of its eigenvalues have negative real part and the stiffness ratio is large.
- Stiffness occurs when stability requirements, rather than those of accuracy constrain the step length.

The behaviour of numerical methods on stiff problems can be analyzed by applying these methods to the test equation $y' = ky$ subject to the initial condition $y(0) = 1$ with $k \in \mathrm{Re}$. The solution of this equation is $y(t) = e^{kt}$. This solution approaches zero as $t \to \infty$ when $\mathrm{Re}(k) < 0$. If the numerical method also exhibits this behaviour (for a fixed step size), then the method is said to be A-stable. Therefore, a potentially good numerical method for solution of stiff systems of ODEs must have good accuracy and some reasonable wide range of absolute stability, (Dahlquist, 1963). Over the years, various notable researchers have made several attempts to develop effective numerical methods for stiff systems usually through higher derivatives of the solution or by inserting additional off step points. Enright (1974) proposed a second derivative method for the solution of stiff system of ODEs and the stability properties of the A-stable, high order method was improved by Cash (1981). Akinfenwa *et al.* (2014) developed a two-step Lo-stable second derivative hybrid block method of order eight which was applied directly to solve stiff initial value problems through interpolation and collocation technique. Their self-starting method effectively solved some stiff problems that they considered. Also Akinfenwa *et al.* (2017) derived a self-starting second derivative multistep block method which uses logic behind the Simpson's 3/8 rule for quadrature using collocation and interpolation techniques to obtain approximate solutions to stiff differential equations. Their method is of order eight and A-Stable, effective and reliable for stiff systems of ordinary differential equations. Chollom *et al.* (2014) in their quest for searching for a higher order A-stable linear multi-step method, constructed a block implicit multi-step methods of the hybrid form up to order twelve using multi-step collocation approach by inserting one or more off step points in the multi-step method which was found out to be an A-stable method. Their methods used as block integrators were tested on some stiff differential systems and the results reveal that the methods are efficient. Bakari *et al.* (2018), used second derivative hybrid block backward differentiation formula on stiff ordinary differential equations. In their method a continuous scheme of four and five steps with one off-grid point at collocation which provides the approximate solution of both linear and nonlinear stiff ordinary differential equations with constant step size. Their proposed methods which are of order 8 and $A(\alpha)-$stable give good approximate solution and reduced computational cost to the solution of both linear and nonlinear stiff systems. Abhulimen and Ukpebor (2019) derived a class of implicit four-step second derivative exponential fitting method of order six for the numerical integration of stiff initial value problems in ordinary differential equation which possess free parameters that allow it to be fitted automatically to exponential functions and their method was found to be A-stable. However, for the purpose of effective implementation of their method, they adopted the mechanism which splits the method into predictor-corrector schemes. In very recent times, some researchers have explored the advantage of including multi-derivatives (particularly second derivative) in their linear multistep methods in order to find and improve the numerical methods for systems of stiff problems. Ehigie *et al.* (2013), Yakubu and Markus (2016), Skwame, *et al.* (2017), Yakubu *et al.* (2017), Sabo *et al.* (2018), Skwame, (2018), Skwame, *et al.* (2018), Skwame, *et al.* (2017), Tumba *et al.* (2018), and Sabo *et al.* (2019).

Furthermore, the Backward differentiation formulae (BDF) are among the first most popular numerical methods to be proposed for stiff initial value problems (Curtis and Hirschfelder (1952) and Gear (1971). These methods are found to be A-stable up to order $p=2$ with order $p=k$ and $A(\alpha)-$stable for $k=3(1)6$, (Nwachukwu and Okor, 2018). In view of the above, many researchers have explored various methods of the backward differentiation formulae. Akinfenwa *et al.* (2013) considered a self-starting implicit continuous block backward differentiation formula for solving stiff ordinary differential equations. In their method, a block of p new values at each step which simultaneously provide approximate solutions for the ODEs was derived where p is the number of points and their method is found to be A-stable which makes it to effectively solve the stiff problem that they considered. Jator and Agyingi (2014), presented a generalized higher order block hybrid k-step backward differentiation formula for solving stiff systems, including large systems resulting from the semi-discretization parabolic partial differential equations. They examined a block scheme in which two off-grid points are specified by the zeros of the second degree Chebyshev polynomial of the first kind for convergence, L_o and A stability. They noted that stiff systems of ODEs are better handled by methods with larger stability intervals. In particular, A-stable methods are of great importance. However, for very large systems arising from the semi-discretization of parabolic PDEs, A-stable methods converge very slowly to the exact solution (Jator and Agyingi, 2014). Hence, in their paper, they sought methods which are at least L_0-stable for efficiently solving stiff ODEs when the system is very large. Mohammed and Adeniyi (2015), proposed a class of implicit six step hybrid backward differentiation formula of order 6 for the solution of second order initial value problems using also the interpolation and collocation of their assumed approximation solution and the continuous formulation is evaluated at grid points and its second derivative which leads to a system of equations from which the multiple finite difference methods are obtained and simultaneously applied to provide direct solution to IVPs. Many other notable researchers have extensively worked on backward differentiation formula, Zarina *et al.* (2005), Yayaha and Mohammed (2009), Mohammed and Yahaya (2010), Yayaha and Mohammed (2010).

CHAPTER THREE

3.0. MATERIALS AND METHODS

3.1. Derivation of the Method

The proposed one-step second derivative, intra-step block numerical method for the solution of stiff systems of first order ordinary differential equations is of the form:

$$y_{n+1} = y_n + h\left(\sum_{j=0}^{1} \beta_j f_{n+j}\right) + h^2\left(\sum_{j=0}^{1} \gamma_j g_{n+j}\right)$$

(3.1)

and the additional method

$$y_{n+\eta} = y_n + h\left(\sum_{j=0}^{1} \beta_j f_{n+j}\right) + h^2\left(\sum_{j=0}^{1} \gamma_j g_{n+j}\right)$$

(3.2)

where $\beta_1 \neq 0 \, and \, \gamma_1 \neq 0$ β_j, and γ_j are unknown coefficients, η is the intra-point derived from the Bhaskara cosine formula (Orakwelu, 2019).

Equations (3.1) and (3.2) are derived using interpolation and collocation techniques of a trial function of the form:

$$Y(x) = \sum_{j=0}^{r+2s-1} a_j x^j$$

(3.3)

where a_j are unknown coefficients to be determined, r and s are numbers of interpolation and collocation respectively. Interpolating (3.3) at x_n and collocating its first and second derivatives at $[x_n, x_{n+1}]$ with a countable number of intra-points defined as $x_{\eta+n} = x_n + h\eta$. Here, $\eta \in (0,1)$. These lead to a system of non-linear equations of the form

$$\left.\begin{aligned} &Y(x_n) = y_n \\ &Y'(x_{n+j}) = f_{n+j} \\ &Y''(x_{n+j}) = g_{n+j} \\ &j = 0,\ldots,1 \end{aligned}\right\}$$

(3.4)

which is solved using matrix inversion method to obtain a_j's and then substituted into (3.3) to get the continuous scheme of the form

$$y(x) = y_n + h\left(\beta_0(x) f_n + \beta_\eta(x) f_{n+\eta} + \beta_1(x) f_{n+1}\right) + h^2\left(\gamma_0(x) g_n + \gamma_\eta(x) g_{n+\eta} + \gamma_1(x) g_{n+1}\right)$$

(3.5)

The continuous scheme (3.5) generated, produces the main and additional algorithms which are merged to generate approximate solutions simultaneously.

The first-five intra-step points for one step method derived from the Bhaskara cosine approximation formula is provided in Table 3.1.

Table 3.1 List of Bhaskara Off-step Points Generated for the One-step Method (Source: Orakwelu, 2019)

m	Off-grid points
2	$\frac{1}{4}, \frac{3}{4}$
3	$\frac{5}{34}, \frac{1}{2}, \frac{29}{34}$
4	$\frac{5}{52}, \frac{10}{29}, \frac{19}{29}, \frac{47}{52}$
5	$\frac{5}{74}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{69}{74}$
6	$\frac{1}{20}, \frac{10}{53}, \frac{45}{116}, \frac{71}{116}, \frac{43}{53}, \frac{19}{20}$

3.2 Specification of the Methods

3.2.1 One-step second derivative block method with 2 intra-step points (OSDBM2)

The specification of one-step second derivative method with 2 intra-points is given as $k=1, m=2, \eta=\left\{\frac{1}{4},\frac{3}{4}\right\}, x\in\left[x_n, x_{n+1}\right]$ which results in system of equations

$$Y_\omega = D\Psi_{\omega-n}$$

(3.6)

where

$$Y_\omega=\left(y_n, f_n, f_{n+\frac{1}{4}}, f_{n+\frac{3}{4}}, f_{n+1,} g_n, g_{n+\frac{1}{4}}, g_{n+\frac{3}{4}}, g_{n+1,}\right)^T, \Psi_\omega=\left(\alpha_0,\beta_0,\beta_{\frac{1}{4}},\beta_{\frac{3}{4}},\beta_1,\gamma_0,\gamma_{\frac{1}{4}},\gamma_{\frac{3}{4}},\gamma_1\right)$$

and

$$D=\begin{pmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\
0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 \\
0 & 1 & 2(x_{n+\frac{1}{4}}) & 3(x_{n+\frac{1}{4}})^2 & 4(x_{n+\frac{1}{4}})^3 & 5(x_{n+\frac{1}{4}})^4 & 6(x_{n+\frac{1}{4}})^5 & 7(x_{n+\frac{1}{4}})^6 & 8(x_{n+\frac{1}{4}})^7 \\
0 & 1 & 2(x_{n+\frac{3}{4}}) & 3(x_{n+\frac{3}{4}})^2 & 4(x_{n+\frac{3}{4}})^3 & 5(x_{n+\frac{3}{4}})^4 & 6(x_{n+\frac{3}{4}})^5 & 7(x_{n+\frac{3}{4}})^6 & 8(x_{n+\frac{3}{4}})^7 \\
0 & 1 & 2(x_{n+1}) & 3(x_{n+1})^2 & 4(x_{n+1})^3 & 5(x_{n+1})^4 & 6(x_{n+1})^5 & 7(x_{n+1})^6 & 8(x_{n+1})^7 \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\
0 & 0 & 2 & 6x_{n+\frac{1}{4}} & 12(x_{n+\frac{1}{4}})^2 & 20(x_{n+\frac{1}{4}})^3 & 30(x_{n+\frac{1}{4}})^4 & 42(x_{n+\frac{1}{4}})^5 & 56(x_{n+\frac{1}{4}})^6 \\
0 & 0 & 2 & 6x_{n+\frac{3}{4}} & 12(x_{n+\frac{3}{4}})^2 & 20(x_{n+\frac{3}{4}})^3 & 30(x_{n+\frac{3}{4}})^4 & 42(x_{n+\frac{3}{4}})^5 & 56(x_{n+\frac{3}{4}})^6 \\
0 & 0 & 2 & 6x_{n+1} & 12(x_{n+1})^2 & 20(x_{n+1})^3 & 30(x_{n+1})^4 & 42(x_{n+1})^5 & 56(x_{n+1})^6
\end{pmatrix}$$

Equation (3.6) is solved by matrix inversion technique which yield the continuous coefficients $\alpha_0(x),\beta_0(x),\beta_{\frac{1}{4}}(x),\beta_{\frac{3}{4}}(x),\beta_1(x),\gamma_0(x),\gamma_{\frac{1}{4}}(x),\gamma_{\frac{3}{4}}(x),\gamma_1(x)$ as;

$$\alpha_0(x)=1$$

(3.7)

$$\beta_0(x) = x - \frac{33}{h^2}x^3 + \frac{8539}{54}\frac{x^4}{h^3} - \frac{8992}{27}\frac{x^5}{h^4} + \frac{9824}{27}\frac{x^6}{h^5} - \frac{38114}{189}\frac{x^7}{h^6} + \frac{1216}{27}\frac{x^8}{h^7}$$

(3.8)

$$\beta_{\frac{1}{4}}(x) = \frac{256}{9}\frac{x^3}{h^2} - \frac{1088}{9}\frac{x^4}{h^3} + \frac{5888}{27}\frac{x^5}{h^4} - \frac{5504}{54}\frac{x^6}{h^5} + \frac{2048}{21}\frac{x^7}{h^6} - \frac{512}{27}\frac{x^8}{h^7}$$

(3.9)

$$\beta_{\frac{3}{4}}(x) = \frac{64}{27}\frac{x^4}{h^3} - \frac{512}{27}\frac{x^5}{h^4} + \frac{1408}{27}\frac{x^6}{h^5} - \frac{10240}{189}\frac{x^7}{h^6} + \frac{512}{27}\frac{x^8}{h^7}$$

(3.10)

$$\beta_1(x) = \frac{41}{9}\frac{x^3}{h^2} - \frac{713}{18}\frac{x^4}{h^3} + \frac{3616}{27}\frac{x^5}{h^4} - \frac{5728}{27}\frac{x^6}{h^5} + \frac{3328}{21}\frac{x^7}{h^6} - \frac{1216}{27}\frac{x^8}{h^7}$$

(3.11)

$$\gamma_0(x) = \frac{1}{2}x^2 - \frac{38}{9}\frac{x^3}{h} + \frac{553}{36}\frac{x^4}{h^2} - \frac{1312}{45}\frac{x^5}{h^3} + \frac{272}{9}\frac{x^6}{h^4} - \frac{1024}{63}\frac{x^7}{h^5} + \frac{32}{9}\frac{x^8}{h^6}$$

(3.12)

$$\gamma_{\frac{1}{4}}(x) = -\frac{16}{3}\frac{x^3}{h} + \frac{104}{3}\frac{x^4}{h^2} - \frac{3856}{45}\frac{x^5}{h^3} + \frac{928}{9}\frac{x^6}{h^4} - \frac{1280}{21}\frac{x^7}{h^5} + \frac{128}{9}\frac{x^8}{h^6}$$

(3.13)

$$\gamma_{\frac{3}{4}}(x) = -\frac{16}{9}\frac{x^3}{h} + \frac{136}{9}\frac{x^4}{h^2} - \frac{2224}{45}\frac{x^5}{h^3} + \frac{224}{3}\frac{x^6}{h^4} - \frac{3328}{63}\frac{x^7}{h^5} + \frac{128}{9}\frac{x^8}{h^6}$$

(3.14)

$$\gamma_1(x) = -\frac{1}{3}\frac{x^3}{h} + \frac{35}{12}\frac{x^4}{h^2} - \frac{448}{45}\frac{x^5}{h^3} + 16\frac{x^6}{h^4} - \frac{256}{21}\frac{x^7}{h^5} + \frac{32}{9}\frac{x^8}{h^6}$$

(3.15)

Equations (3.7) to (3.15) are then substituted into (3.6) to get the continuous scheme of the form:

$$y(x) = y_n + h\left(\beta_0(x) f_n + \beta_{\frac{1}{4}}(x) f_{n+\frac{1}{4}} + \beta_{\frac{3}{4}}(x) f_{n+\frac{3}{4}} + \beta_1(x) f_{n+1}\right) +$$
$$h^2\left(\gamma_0(x) g_n + \gamma_{\frac{1}{4}}(x) g_{n+\frac{1}{4}} + \gamma_{\frac{3}{4}}(x) g_{n+\frac{3}{4}} + \gamma_1(x) g_{n+1}\right)$$

(3.16)

Evaluating (3.16) at $x = x_n + j,\ j = \frac{1}{4}, \frac{3}{4}, 1$ gives the one-step second derivative block method with 2 intra-points as follows:

$$y_{n+\frac{1}{4}} = y_n + h\left(\frac{20135}{193536} f_n + \frac{3413}{24192} f_{n+\frac{1}{4}} + \frac{11}{24192} f_{n+\frac{3}{4}} + \frac{857}{193536} f_{n+1}\right) +$$
$$h^2\left(\frac{233}{71680} g_n - \frac{1601}{161280} g_{n+\frac{1}{4}} - \frac{289}{161280} g_{n+\frac{3}{4}} - \frac{23}{71680} g_{n+1}\right)$$

(3.17)

$$y_{n+\frac{3}{4}} = y_n + h\left(\frac{1125}{7168} f_n + \frac{303}{896} f_{n+\frac{1}{4}} + \frac{177}{896} f_{n+\frac{3}{4}} + \frac{411}{7168} f_{n+1}\right) +$$
$$h^2\left(\frac{489}{71680} g_n + \frac{423}{17920} g_{n+\frac{1}{4}} - \frac{633}{17920} g_{n+\frac{3}{4}} - \frac{279}{71680} g_{n+1}\right)$$

(3.18)

$$y_{n+1} = y_n + h\left(\frac{61}{378} f_n + \frac{64}{189} f_{n+\frac{1}{4}} + \frac{64}{189} f_{n+\frac{3}{4}} + \frac{61}{378} f_{n+1}\right) +$$
$$h^2\left(\frac{1}{140} g_n + \frac{8}{315} g_{n+\frac{1}{4}} - \frac{8}{315} g_{n+\frac{3}{4}} - \frac{1}{140} g_{n+1}\right)$$

(3.19)

3.2.2 One-step second derivative block method with 3 intra-step points (OSDBM3)

The specification of one-step second derivative block method with intra-points is given as $k = 1, \eta = \left(\frac{5}{34}, \frac{1}{2}, \frac{29}{34}\right), x \in [x_n, x_{n+1}]$ which results in system of equations

$Y_{\omega} = D\Psi_{\omega-n}$

(3.20)

where $Y_{\omega} = \left(y_n, f_n, f_{n+\frac{5}{34}}, f_{n+\frac{1}{2}}, f_{n+\frac{29}{34}}, f_{n+1,} g_n, \text{g}_{n+\frac{5}{34}}, \text{g}_{n+\frac{1}{2}}, \text{g}_{n+\frac{29}{34}}, \text{g}_{n+1,} \right)^T$, and $\Psi_{\omega} = \left(\alpha_0, \beta_0, \beta_{\frac{5}{34}}, \beta_{\frac{1}{2}}, \beta_{\frac{29}{34}}, \beta_1, \gamma_0, \gamma_{\frac{5}{34}}, \gamma_{\frac{1}{2}}, \gamma_{\frac{29}{34}}, \gamma_1 \right)$

The D-matrix for this method is given as:

$$\begin{pmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10} \\
0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 & 9x_n^8 & 10x_n^9 \\
0 & 1 & 2(x_{n+\frac{5}{34}}) & 3(x_{n+\frac{5}{34}})^2 & 4(x_{n+\frac{5}{34}})^3 & 5(x_{n+\frac{5}{34}})^4 & 6(x_{n+\frac{5}{34}})^5 & 7(x_{n+\frac{5}{34}})^6 & 8(x_{n+\frac{5}{34}})^7 & 9(x_{n+\frac{5}{34}})^8 & 10(x_{n+\frac{5}{34}})^9 \\
0 & 1 & 2(x_{n+\frac{1}{2}}) & 3(x_{n+\frac{1}{2}})^2 & 4(x_{n+\frac{1}{2}})^3 & 5(x_{n+\frac{1}{2}})^4 & 6x_{n+\frac{1}{2}})^5 & 7(x_{n+\frac{1}{2}})^6 & 8(x_{n+\frac{1}{2}})^7 & 9(x_{n+\frac{1}{2}})^8 & 10(x_{n+\frac{1}{2}})^9 \\
0 & 1 & 2(x_{n+\frac{29}{34}}) & 3(x_{n+\frac{29}{34}})^2 & 4(x_{n+\frac{29}{34}})^3 & 5(x_{n+\frac{29}{34}})^4 & 6(x_{n+\frac{29}{34}})^5 & 7(x_{n+\frac{29}{34}})^6 & 8(x_{n+\frac{29}{34}})^7 & 9(x_{n+\frac{29}{34}})^8 & 10(x_{n+\frac{29}{34}})^9 \\
0 & 1 & 2(x_{n+1}) & 3(x_{n+1})^2 & 4(x_{n+1})^3 & 5(x_{n+1})^4 & 6(x_{n+1})^5 & 7(x_{n+1})^6 & 8(x_{n+1})^7 & 9(x_{n+1})^8 & 10(x_{n+1})^9 \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 & 72x_n^7 & 90x_n^8 \\
0 & 0 & 2 & 6(x_{n+\frac{5}{34}}) & 12(x_{n+\frac{5}{34}})^2 & 20(x_{n+\frac{5}{34}})^3 & 30(x_{n+\frac{5}{34}})^4 & 42(x_{n+\frac{5}{34}})^5 & 56(x_{n+\frac{5}{34}})^6 & 72(x_{n+\frac{5}{34}})^7 & 90(x_{n+\frac{5}{34}})^8 \\
0 & 0 & 2 & 6(x_{n+\frac{1}{2}}) & 12(x_{n+\frac{1}{2}})^2 & 20(x_{n+\frac{1}{2}})^3 & 30(x_{n+\frac{1}{2}})^4 & 42(x_{n+\frac{1}{2}})^5 & 56(x_{n+\frac{1}{2}})^6 & 72(x_{n+\frac{1}{2}})^7 & 90(x_{n+\frac{1}{2}})^8 \\
0 & 0 & 2 & 6(x_{n+\frac{29}{34}}) & 12(x_{n+\frac{29}{34}})^2 & 20(x_{n+\frac{29}{34}})^3 & 30(x_{n+\frac{29}{34}})^4 & 42(x_{n+\frac{29}{34}})^5 & 56(x_{n+\frac{29}{34}})^6 & 72(x_{n+\frac{29}{34}})^7 & 90(x_{n+\frac{29}{34}})^8 \\
0 & 0 & 2 & 6(x_{n+1}) & 12(x_{n+1})^2 & 20(x_{n+1})^3 & 30(x_{n+1})^4 & 42(x_{n+1})^5 & 56(x_{n+1})^6 & 72(x_{n+1})^7 & 90(x_{n+1})^8
\end{pmatrix}$$

(3.21)

The matrix D in equation (3.21) is solved by matrix inversion technique which yields the continuous coefficients

$\alpha_0(x), \beta_0(x), \beta_{\frac{5}{34}}(x), \beta_{\frac{1}{2}}(x), \beta_{\frac{29}{34}}(x), \beta_1(x), \gamma_0(x), \gamma_{\frac{5}{34}}(x), \gamma_{\frac{1}{2}}(x), \gamma_{\frac{29}{34}}(x), \gamma_1(x)$ as;

$\alpha_0(x) = 1$

(3.22)

$$\beta_0(x) = x - \frac{2056261}{21025}\frac{x^3}{h^2} + \frac{5039378801}{6097250}\frac{x^4}{h^3} - \frac{48823101516}{15243125}\frac{x^5}{h^4} - \frac{21365019612}{3048625}\frac{x^6}{h^5} - \frac{196026512848}{21340375}\frac{x^7}{h^6} + \frac{21841578444}{3048625}\frac{x^8}{h^7} - \frac{5617956544}{1829175}\frac{x^9}{h^8} + \frac{8504442304}{15243125}\frac{x^{10}}{h^9}$$

(3.23)

$$\beta_{\frac{5}{34}}(x) = \frac{3934423747}{45100800}\frac{x^3}{h^2} - \frac{1469808989117}{2179872000}\frac{x^4}{h^3} + \frac{247091297376217}{105360480000}\frac{x^5}{h^4} - \frac{21365019612}{3048625}\frac{x^6}{h^5} - \frac{103039975214047}{18438084000}\frac{x^7}{h^6} - \frac{42713383488589}{10536048000}\frac{x^8}{h^7} + \frac{11628587654147}{7111832400}\frac{x^9}{h^8} - \frac{5573630195359}{19755090000}\frac{x^{10}}{h^9}$$

$$\beta_{\frac{1}{2}}(x) = \frac{21025}{3888}\frac{x^3}{h^2} - \frac{188645}{2592}\frac{x^4}{h^3} + \frac{2363081}{6480}\frac{x^5}{h^4} - \frac{793883}{972}\frac{x^6}{h^5} + \frac{1044157}{1134}\frac{x^7}{h^6} - \frac{83521}{162}\frac{x^8}{h^7} + \frac{83521}{729}\frac{x^9}{h^8}$$

(3.24)

(3.25)

$$\beta_{\frac{29}{34}}(x) = -\frac{24137569}{9688320}\frac{x^3}{h^2} + \frac{511257848989}{12643257600}\frac{x^4}{h^3} - \frac{85446101169947}{316081440000}\frac{x^5}{h^4} - \frac{178053263898107}{189648864000}\frac{x^6}{h^5} - \frac{101577383358061}{55314252000}\frac{x^7}{h^6} + \frac{12859603673147}{6321628800}\frac{x^8}{h^7} + \frac{42182405245727}{35559162000}\frac{x^9}{h^8} + \frac{5573630195359}{19755090000}\frac{x^{10}}{h^9}$$

(3.26)

$$\beta_1(x) = \frac{1109}{145}\frac{x^3}{h^2} - \frac{5041537}{42050}\frac{x^4}{h^3} + \frac{11636847852}{15243125}\frac{x^5}{h^4} - \frac{7575363804}{3048625}\frac{x^6}{h^5} + \frac{96306438608}{21340375}\frac{x^7}{h^6} - \frac{2822442204}{609725}\frac{x^8}{h^7} - \frac{22936871104}{9145875}\frac{x^9}{h^8} - \frac{8504442304}{15243125}\frac{x^{10}}{h^9}$$

(3.27)

$$\gamma_0(x) = \frac{1}{2}x^2 - \frac{3182}{435}\frac{x^3}{h} + \frac{3956341}{84100}\frac{x^4}{h^2} - \frac{17312548}{105125}\frac{x^5}{h^3} + \frac{21604918}{63075}\frac{x^6}{h^4} - \frac{64162624}{147175}\frac{x^7}{h^5} + \frac{7016342}{21025}\frac{x^8}{h^6} - \frac{5345344}{37845}\frac{x^9}{h^7} + \frac{2672672}{105125}\frac{x^{10}}{h^8}$$

(3.28)

$$\gamma_{\frac{5}{34}}(x) = -\frac{1419857}{155520}\frac{x^3}{h} + \frac{86611277}{835200}\frac{x^4}{h^2} - \frac{508616914969}{1089936000}\frac{x^5}{h^3} + \frac{728836735669}{653961600}\frac{x^6}{h^4} - \frac{294772252199}{190738800}\frac{x^7}{h^5} - \frac{136546227833}{108993600}\frac{x^8}{h^6} - \frac{4513725403}{8174520}\frac{x^9}{h^7} + \frac{6975757441}{68121000}\frac{x^{10}}{h^8}$$

(3.29)

$$\gamma_{\frac{1}{2}}(x) = -\frac{21025}{7776}\frac{x^3}{h} + \frac{34945}{864}\frac{x^4}{h^2} - \frac{3117661}{12960}\frac{x^5}{h^3} + \frac{5538613}{7776}\frac{x^6}{h^4} - \frac{375989}{324}\frac{x^7}{h^5} + \frac{1378241}{1296}\frac{x^8}{h^6} - \frac{83521}{162}\frac{x^9}{h^7} + \frac{83521}{810}\frac{x^{10}}{h^8}$$

(3.30)

$$\gamma_{\frac{29}{34}}(x) = -\frac{1419857}{902016}\frac{x^3}{h} + \frac{356384107}{14532480}\frac{x^4}{h^2} - \frac{168634996033}{1089936000}\frac{x^5}{h^3} + \frac{324961251733}{653961600}\frac{x^6}{h^4} - \frac{169200099119}{190738800}\frac{x^7}{h^5} + \frac{3886148609}{4359744}\frac{x^8}{h^6} - \frac{19285917631}{40872600}\frac{x^9}{h^7} + \frac{6975757441}{68121000}\frac{x^{10}}{h^8}$$

(3.31)

$$\gamma_1(x) = -\frac{1}{3}\frac{x^3}{h} + \frac{3037}{580}\frac{x^4}{h^2} - \frac{3515976}{105125}\frac{x^5}{h^3} + \frac{6898286}{63075}\frac{x^6}{h^4} - \frac{29413264}{147175}\frac{x^7}{h^5} + \frac{868734}{4205}\frac{x^8}{h^6} - \frac{21381376}{189225}\frac{x^9}{h^7} + \frac{2672672}{105125}\frac{x^{10}}{h^8}$$

(3.32)

Equations (3.22) to (3.32) are then substituted into (3.20) to get the continuous scheme of the form:

$$y(x) = y_n + h\left(\beta_0(x) f_n + \beta_{\frac{5}{34}}(x) f_{n+\frac{5}{34}} + \beta_{\frac{1}{2}}(x) f_{n+\frac{1}{2}} + \beta_{\frac{29}{34}}(x) f_{n+\frac{29}{34}} + \beta_1(x) f_{n+1}\right) + h^2\left(\gamma_0(x) g_n + \gamma_{\frac{5}{34}}(x) g_{n+\frac{5}{34}} + \gamma_{\frac{1}{2}}(x) g_{n+\frac{1}{2}} + \gamma_{\frac{29}{34}}(x) g_{n+\frac{29}{34}} + \gamma_1(x) g_{n+1}\right)$$

(3.33)

Evaluating (3.32) at $x = x_{n+j}$, $j = \frac{5}{34}, \frac{1}{2}, \frac{29}{34}, 1$ gives the coefficient of the discrete one-step second derivative block method with 3 intra-step points.

$$\left.\begin{aligned} y_{n+\frac{5}{34}} = y_n &+ \frac{48236526812245}{791200932938304} hf_n + \frac{5815453137955}{69329555914752} hf_{n+\frac{5}{34}} + \frac{575288125}{463713937344} hf_{n+\frac{1}{2}} \\ &- \frac{29906050595}{69329555914752} hf_{n+\frac{29}{34}} + \frac{1109311642315}{791200932938304} hf_{n+1} + \frac{181522121275}{163696744745856} h^2 g_n \\ &- \frac{23029130875}{6773577302016} h^2 g_{n+\frac{5}{34}} - \frac{12819468125}{24024798277632} h^2 g_{n+\frac{1}{2}} - \frac{4299025}{14757249024} h^2 g_{n+\frac{29}{34}} \\ &- \frac{9971114725}{163696744745856} h^2 g_{n+1} \end{aligned}\right\}$$

(3.34)

$$\left.\begin{aligned} y_{n+\frac{1}{2}} = y_n &- \frac{2229002917}{20486760000} hf_n + \frac{531827571502403}{2548880732160000} hf_{n+\frac{5}{34}} + \frac{269281}{1632960} hf_{n+\frac{1}{2}} - \\ &\frac{2082565315751}{509776146432000} hf_{n+\frac{29}{34}} + \frac{89027711}{4097352000} hf_{n+1} + \frac{2540591}{847728000} h^2 g_n + \\ &\frac{37985434321}{2929747968000} h^2 g_{n+\frac{5}{34}} - \frac{82801}{4976640} h^2 g_{n+\frac{1}{2}} - \frac{103649561}{21701836800} h^2 g_{n+\frac{29}{34}} - \\ &\frac{157309}{169545600} h^2 g_{n+1} \end{aligned}\right\}$$

(3.35)

$$\left.\begin{aligned} y_{n+\frac{29}{34}} = y_n &+ \frac{2618145660737}{20275557960000} hf_n + \frac{364210994519}{1776660480000} hf_{n+\frac{5}{34}} + \frac{761804585009}{2318569686720} hf_{n+\frac{1}{2}} + \\ &\frac{214416025769}{1776660480000} hf_{n+\frac{29}{34}} + \frac{1410449179487}{20275557960000} hf_{n+1} + \frac{94010208599}{24330669552000} h^2 g_n + \\ &\frac{1033435097}{59222016000} h^2 g_{n+\frac{5}{34}} - \frac{12819468125}{24024798277632} h^2 g_{n+\frac{1}{2}} - \frac{788317237}{37287936000} h^2 g_{n+\frac{29}{34}} - \\ &\frac{68512139849}{24330669552000} h^2 g_{n+1} \end{aligned}\right\}$$

(3.36)

$$\left.\begin{aligned} y_{n+1} = y_n &+ \frac{83566921}{640211250} hf_n + \frac{1018388173679}{4978282680000} hf_{n+\frac{5}{34}} + \frac{269281}{816480} hf_{n+\frac{1}{2}} + \\ &\frac{1018388173679}{4978282680000} hf_{n+\frac{29}{34}} + \frac{83566921}{640211250} hf_{n+1} + \frac{103973}{26491500} h^2 g_n \\ &+ \frac{203039551}{11444328000} h^2 g_{n+\frac{5}{34}} - \frac{203039551}{11444328000} h^2 g_{n+\frac{29}{34}} - \frac{103973}{26491500} h^2 g_{n+1} \end{aligned}\right\}$$

(3.37)

3.2.3 One-step second derivative block method with 4 intra-step points (OSDBM4)

The specification of one-step second derivative block method with 4 intra-step points is given as $k=1, \eta = \left\{ \frac{5}{52}, \frac{10}{29}, \frac{19}{29}, \frac{47}{52} \right\}, x \in [x_n, x_{n+1}]$ which results in system of equations

$$Y_\omega = D\Psi_{\omega-n}$$

where

$$Y_\omega = \left(y_n, f_n, f_{n+\frac{5}{52}}, f_{n+\frac{10}{29}}, f_{n+\frac{19}{29}}, f_{n+\frac{47}{52}}, f_{n+1,} g_n, g_{n+\frac{5}{52}}, g_{n+\frac{10}{29}}, g_{n+\frac{19}{29}}, g_{n+\frac{47}{52}}, g_{n+1,} \right)^T,$$

and

$$\Psi_\omega = \left(\alpha_0, \beta_0, \beta_{\frac{5}{52}}, \beta_{\frac{10}{29}}, \beta_{\frac{19}{29}}, \beta_{\frac{47}{52}}, \beta_1, \gamma_0, \gamma_{\frac{5}{52}}, \gamma_{\frac{10}{29}}, \gamma_{\frac{19}{29}}, \gamma_{\frac{47}{52}},, \gamma_1 \right)$$

The D-matrix for this method is given as:

$$\begin{pmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10} & x_n^{11} & x_n^{12} \\
0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 & 9x_n^8 & 10x_n^9 & 11x_n^{10} & 12x_n^{11} \\
0 & 1 & 2(x_{n+\frac{5}{52}}) & 3(x_{n+\frac{5}{52}})^2 & 4(x_{n+\frac{5}{52}})^3 & 5(x_{n+\frac{5}{52}})^4 & 6(x_{n+\frac{5}{52}})^5 & 7(x_{n+\frac{5}{52}})^6 & 8(x_{n+\frac{5}{52}})^7 & 9(x_{n+\frac{5}{52}})^8 & 10(x_{n+\frac{5}{52}})^9 & 11(x_{n+\frac{5}{52}})^{10} & 12(x_{n+\frac{5}{52}})^{11} \\
0 & 1 & 2(x_{n+\frac{10}{29}}) & 3(x_{n+\frac{10}{29}})^2 & 4(x_{n+\frac{10}{29}})^3 & 5(x_{n+\frac{10}{29}})^4 & 6(x_{n+\frac{10}{29}})^5 & 7(x_{n+\frac{10}{29}})^6 & 8(x_{n+\frac{10}{29}})^7 & 9(x_{n+\frac{10}{29}})^8 & 10(x_{n+\frac{10}{29}})^9 & 11(x_{n+\frac{10}{29}})^{10} & 12(x_{n+\frac{10}{29}})^{11} \\
0 & 1 & 2(x_{n+\frac{19}{29}}) & 3(x_{n+\frac{19}{29}})^2 & 4(x_{n+\frac{19}{29}})^3 & 5(x_{n+\frac{19}{29}})^4 & 6(x_{n+\frac{19}{29}})^5 & 7(x_{n+\frac{19}{29}})^6 & 8(x_{n+\frac{19}{29}})^7 & 9(x_{n+\frac{19}{29}})^8 & 10(x_{n+\frac{19}{29}})^9 & 11(x_{n+\frac{19}{29}})^{10} & 12(x_{n+\frac{19}{29}})^{11} \\
0 & 1 & 2(x_{n+\frac{47}{52}}) & 3(x_{n+\frac{47}{52}})^2 & 4(x_{n+\frac{47}{52}})^3 & 5(x_{n+\frac{47}{52}})^4 & 6(x_{n+\frac{47}{52}})^5 & 7(x_{n+\frac{47}{52}})^6 & 8(x_{n+\frac{47}{52}})^7 & 9(x_{n+\frac{47}{52}})^8 & 10(x_{n+\frac{47}{52}})^9 & 11(x_{n+\frac{47}{52}})^{10} & 12(x_{n+\frac{47}{52}})^{11} \\
0 & 1 & 2(x_{n+1}) & 3(x_{n+1})^2 & 4(x_{n+1})^3 & 5(x_{n+1})^4 & 6(x_{n+1})^5 & 7(x_{n+1})^6 & 8(x_{n+1})^7 & 9(x_{n+1})^8 & 10(x_{n+1})^9 & 11(x_{n+1})^{10} & 12(x_{n+1})^{11} \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 & 72x_n^7 & 90x_n^8 & 110x_n^9 & 132x_n^{10} \\
0 & 0 & 2 & 6x_{n+\frac{5}{52}} & 12(x_{n+\frac{5}{52}})^2 & 20(x_{n+\frac{5}{52}})^3 & 30(x_{n+\frac{5}{52}})^4 & 42(x_{n+\frac{5}{52}})^5 & 56(x_{n+\frac{5}{52}})^6 & 72(x_{n+\frac{5}{52}})^7 & 90(x_{n+\frac{5}{52}})^8 & 110(x_{n+\frac{5}{52}})^9 & 132(x_{n+\frac{5}{52}})^{10} \\
0 & 0 & 2 & 6x_{n+\frac{10}{29}} & 2(x_{n+\frac{10}{52}})^2 & 20(x_{n+\frac{10}{52}})^3 & 30(x_{n+\frac{10}{52}})^4 & 42(x_{n+\frac{10}{52}})^5 & 56(x_{n+\frac{10}{52}})^6 & 72(x_{n+\frac{10}{52}})^7 & 90(x_{n+\frac{10}{52}})^8 & 110(x_{n+\frac{10}{52}})^9 & 132(x_{n+\frac{10}{52}})^{10} \\
0 & 0 & 2 & 6x_{n+\frac{19}{29}} & 2(x_{n+\frac{19}{29}})^2 & 20(x_{n+\frac{19}{29}})^3 & 30(x_{n+\frac{19}{29}})^4 & 42(x_{n+\frac{19}{29}})^5 & 56(x_{n+\frac{19}{29}})^6 & 72(x_{n+\frac{19}{29}})^7 & 90(x_{n+\frac{19}{29}})^8 & 110(x_{n+\frac{19}{29}})^9 & 132(x_{n+\frac{19}{29}})^{10} \\
0 & 0 & 2 & 6x_{n+\frac{47}{52}} & 2(x_{n+\frac{47}{52}})^2 & 20(x_{n+\frac{47}{52}})^3 & 30(x_{n+\frac{47}{52}})^4 & 42(x_{n+\frac{47}{52}})^5 & 56(x_{n+\frac{47}{52}})^6 & 72(x_{n+\frac{47}{52}})^7 & 90(x_{n+\frac{47}{52}})^8 & 110(x_{n+\frac{47}{52}})^9 & 132(x_{n+\frac{47}{52}})^{10} \\
0 & 0 & 2 & 6x_{n+1} & 2(x_{n+1})^2 & 20(x_{n+1})^3 & 30(x_{n+1})^4 & 42(x_{n+1})^5 & 56(x_{n+1})^6 & 72(x_{n+1})^7 & 90(x_{n+1})^8 & 110(x_{n+1})9 & 132(x_{n+1})^{10}
\end{pmatrix}$$

Following the same procedure as in the case of OSDBM2 and OSDBM3, the coefficients of the discrete one-step second derivative block method with 4 intra-points (OSDBM4) are given as.

$$
\left.
\begin{aligned}
y_{n+\frac{5}{52}} = y_n &+ \frac{3359041333385958632744 1775}{8442369160396436373157969 92} f_n + \frac{408266584242181971495700 27}{741331968767476413327000000} hf_{n+\frac{5}{52}} + \\
&\frac{4852939998194364415044286343471151 7}{5548825190585290619415713483980800000 0} hf_{n+\frac{10}{29}} \\
&+ \frac{1009161815202041989453836579161863}{11097650381170581238831426967961600000} hf_{n+\frac{19}{29}} \\
&- \frac{1219603262158196314174 9}{4942213125116509422180000 0} hf_{n+\frac{47}{52}} + \frac{485574234330527270749585}{844236916039643637315796992} hf_{n+1} + \\
&\frac{44511233781991957595 5}{945394082911135092178944} h^2 g_n - \frac{53693441819000827 3}{370647719291801952000} h^2 g_{n+\frac{5}{52}} \\
&- \frac{1966286260536702702388833611}{9238226368791976224287883264000} h^2 g_{n+\frac{10}{29}} - \frac{1932479923951651024725698 71}{18476452737583952448575766528 00} h^2 g_{n+\frac{19}{29}} - \\
&\frac{16492544961353959}{2223886315750811712 00} h^2 g_{n+\frac{47}{52}} - \frac{1557529062095031332 5}{945394082911135092178944} h^2 g_{n+1}
\end{aligned}
\right\}
$$

(3.38)

$$
\left.
\begin{aligned}
y_{n+\frac{10}{29}} = y_n &+ \frac{190403961745049632948 25}{246871863023952116881161} hf_n - \\
&\frac{617166404555170204454816279387176 96}{1392911126011291166041937972998974609375} hf_{n+\frac{5}{52}} + \\
&\frac{1132244250307753354541 29}{9406375541187138901406 25} hf_{n+\frac{10}{29}} + \frac{5771470012723070465 56}{188127510823742778028125} hf_{n+\frac{19}{29}} - \\
&\frac{1142991512617987869356705468907 52}{2971543735490754487556134342397812 5} hf_{n+\frac{47}{52}} + \\
&\frac{23818758412882105821 80}{246871863023952116881161} hf_{n+1} + \frac{39560759668814601 5}{2764522542261501868 77} h^2 g_n + \\
&\frac{5802891412214070579691 52}{8746263663049241380023637 5} h^2 g_{n+\frac{5}{52}} - \\
&\frac{36691979427479353}{4541589406163068875} h^2 g_{n+\frac{10}{29}} - \frac{1811250704143208}{9083178812326137 75} h^2 g_{n+\frac{19}{29}} - \\
&\frac{3257957222187540786774016}{25714015169364769657269494 25} h^2 g_{n+\frac{47}{52}} - \frac{7590468408889160 0}{2764522542261501868 77} h^2 g_{n+1}
\end{aligned}
\right\}
$$

(3.39)

$$
\left.\begin{aligned}
y_{n+\frac{19}{29}} = y_n &+ \frac{378857268686010446316101}{3599239874966498277900000} hf_n - \\
&\frac{1823344704705974128674771440127690506 24}{139291112601129116604193797299897460937 5} hf_{n+\frac{5}{52}} + \\
&\frac{349661403113425220810752 3}{137139168117613921875000 00} hf_{n+\frac{10}{29}} + \frac{188794381646135819700752 3}{137139168117613921875000 00} hf_{n+\frac{19}{29}} - \\
&\frac{529593455609669805408750006040924979 2}{464303708670430388680645990999658203125} hf_{n+\frac{47}{52}} + \\
&\frac{135986291382952380816101}{3599239874966498277900000} hf_{n+1} + \frac{339139055549533 74487}{153159143615595671400 00} h^2 g_n + \\
&\frac{14181527898424233819338182451 2}{133927162340441508631611949218 75} h^2 g_{n+\frac{5}{52}} + \\
&\frac{5316743354414436941}{125805800724738750000 0} h^2 g_{n+\frac{10}{29}} - \frac{17989384060285736941}{125805800724738750000 0} h^2 g_{n+\frac{19}{29}} - \\
&\frac{209781094300823472118910273536}{401781487021324525894835847656 25} h^2 g_{n+\frac{47}{52}} - \frac{16201832834773074487}{153159143615595671400 00} h^2 g_{n+1}
\end{aligned}\right\}
$$

(3.40)

$$
\left.\begin{aligned}
y_{n+\frac{47}{52}} = y_n &+ \frac{290532003058687665109909 3}{254109432652098895872000 00} hf_n + \frac{28405637831643391132138 81}{223135760130064031250000 00} hf_{n+\frac{5}{52}} + \\
&\frac{447279030876994239691077863288690177038 73}{173400787205790331856741046374400000000 0000} hf_{n+\frac{10}{29}} + \\
&\frac{445920168661183819872228685771849483226 23}{173400787205790331856741046374400000000 0000} hf_{n+\frac{19}{29}} \\
&\frac{535401138009409151910877}{74378586710021343750000 00} hf_{n+\frac{47}{52}} + \frac{190888742715576495569284 3}{254109432652098895872000 00} hf_{n+1} + \\
&\frac{6613222259939110796 83}{267483613317998837760 000} h^2 g_n + \frac{617771787635799690 7}{524343197277900000000 0} h^2 g_{n+\frac{5}{52}} + \\
&\frac{176554951287412122655019298334841}{288694574024749257008996352000000 00} h^2 g_{n+\frac{10}{29}} - \\
&\frac{185719095732763773326118307603591}{288694574024749257008996352000000 00} h^2 g_{n+\frac{19}{29}} - \\
&\frac{20928561589939759471}{157302959183370000000 0} h^2 g_{n+\frac{47}{52}} - \frac{539791825586778535933}{267483613317998837760 000} h^2 g_{n+1}
\end{aligned}\right\}
$$

(3.41)

$$
\left.\begin{aligned}
y_{n+1} = y_n &+ \frac{56707444490210519}{49350051620100000} hf_n + \frac{7219980490343762883362816}{56825510262911199755859375} hf_{n+\frac{5}{52}} \\
&\frac{30998530191808415604964879}{120132510104561160937500000} hf_{n+\frac{10}{29}} + \frac{30998530191808415604964879}{120132510104561160937500000} hf_{n+\frac{19}{29}} \\
&+ \frac{7219980490343762883362816}{56825510262911199755859375} hf_{n+\frac{47}{52}} + \frac{56707444490210519}{49350051620100000} hf_{n+1} \\
&+ \frac{27508470287}{11052643140000} h^2 g_n + \frac{952233551720728576}{803167152450222656 25} h^2 g_{n+\frac{5}{52}} \\
&+ \frac{1119686762986609487}{180007507180462500000} h^2 g_{n+\frac{10}{29}} - \frac{1119686762986609487}{180007507180462500000} h^2 g_{n+\frac{19}{29}} \\
&- \frac{952233551720728576}{80316715245022265625} h^2 g_{n+\frac{47}{52}} - \frac{27508470287}{11052643140000} h^2 g_{n+1}
\end{aligned}\right\}
$$

(3.42)

3.2.4 One-step second derivative block method with 5 intra-step points (OSDBM5)

The specification of one-step second derivative block method with 5 intra-step points is given as $k = 1, \eta = \left\{ \frac{5}{74}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{69}{74} \right\}, x \in [x_n, x_{n+1}]$ which results in system of equations

$$Y_\omega = D\Psi_{\omega - n}$$

(3.43)

Where;

$$Y_\omega = \left(y_n, f_n, f_{n+\frac{5}{74}}, f_{n+\frac{1}{4}}, f_{n+\frac{1}{2}}, f_{n+\frac{3}{4}}, f_{n+\frac{69}{74}}, f_{n+1}, g_n, g_{n+\frac{5}{74}}, g_{n+\frac{1}{4}}, g_{n+\frac{1}{2}}, g_{n+\frac{3}{4}}, g_{n+\frac{69}{74}} g_{n+1}, \right)^T,$$ and

$$\Psi_\omega = \left(\alpha_0, \beta_0, \beta_{\frac{5}{74}}, \beta_{\frac{1}{4}}, \beta_{\frac{1}{2}}, \beta_{\frac{3}{4}}, \beta_{\frac{69}{74}}, \beta_1, \gamma_0, \gamma_{\frac{5}{74}}, \gamma_{\frac{1}{4}}, \gamma_{\frac{1}{2}}, \gamma_{\frac{3}{4}}, \gamma_{\frac{69}{74}}, \gamma_1 \right)$$

Following the same procedure as in the case of OSDBM2 and OSDBM3, the coefficients of the discrete one-step second derivative block method with 5 intra-step points (OSDBM5) are given as.

$$
\left.\begin{aligned}
y_{n+\frac{5}{74}} = y_n &+ \frac{229000240671549836847060385}{8197163229070036195593562944} hf_n + \frac{30821966534203471711296035015}{794803209597695208727781572608} hf_{n+\frac{5}{74}} + \\
&\frac{4799066842733736181946180000}{7906636097053004139215836776327} hf_{n+\frac{1}{4}} + \frac{3450076071236439375}{34102759007157505753088} hf_{n+\frac{1}{2}} + \\
&\frac{6140574416512977257660000}{7906636097053004139215836776327} hf_{n+\frac{3}{4}} - \frac{107315656776943207525779655}{794803209597695208727781572608} hf_{n+\frac{69}{74}} + \\
&\frac{2277064615176808809084415}{8197163229070036195593562944} hf_{n+1} + \frac{55058925194280275534125}{237598934175943078133146752} h^2 g_n - \\
&\frac{20909102194147642656275}{29304099459426191709044736} h^2 g_{n+\frac{5}{74}} - \frac{295700279607826605965000}{2899389841236891873566496801} h^2 g_{n+\frac{1}{4}} - \\
&\frac{26786422976951818125}{282361305345975432249344} h^2 g_{n+\frac{1}{2}} - \frac{89947576742759078605000}{2899389841236891873566496801} h^2 g_{n+\frac{3}{4}} - \\
&\frac{102495359829189387925}{4186299922775170244149248} h^2 g_{n+\frac{69}{74}} - \frac{1334898531657665905075}{237598934175943078133146752} h^2 g_{n+1}
\end{aligned}\right\}
$$

(3.44)

$$
\left.\begin{aligned}
y_{n+\frac{1}{4}} = y_n &+ \frac{1943212527496001}{34093872933120000} hf_n + \frac{672766595746510020168338779492501}{6873973704628715318726759546880000} hf_{n+\frac{5}{74}} + \\
&\frac{47060462768187769}{526168068417351360} hf_{n+\frac{1}{4}} + \frac{1963790013}{671759728640} hf_{n+\frac{1}{2}} + \frac{19206102090155}{105233613683470272} hf_{n+\frac{3}{4}} - \\
&\frac{641769922504699857049741591517}{274958948185148612749070381875200} hf_{n+\frac{69}{74}} + \frac{6710626995623}{1363754917324800} hf_{n+1} + \\
&\frac{597715882969}{790582560768000} h^2 g_n + \frac{4963027059592093628627263}{136995059619632044834816000} h^2 g_{n+\frac{5}{74}} - \\
&\frac{669184831493}{154358069209344} h^2 g_{n+\frac{1}{4}} - \frac{299379159}{150323855360} h^2 g_{n+\frac{1}{2}} - \\
&\frac{445505665001}{771790346046720} h^2 g_{n+\frac{3}{4}} - \frac{119775813900021295336589}{273990119123926408966963200} h^2 g_{n+\frac{69}{74}} - \\
&\frac{2241217229}{22588073164800} h^2 g_{n+1}
\end{aligned}\right\}
$$

(3.45)

$$
\left.\begin{aligned}
y_{n+\frac{1}{2}} = y_n &+ \frac{17996522320261}{213086705832000} hf_n + \frac{239716174678396247167366105871}{2685145978370591921377640448000} hf_{n+\frac{5}{74}} + \\
&\frac{319896474521248}{1644275213804223} hf_{n+\frac{1}{4}} + \frac{153451983}{1312030720} hf_{n+\frac{1}{2}} + \frac{26394146969312}{8221376069021115} hf_{n+\frac{3}{4}} - \\
&\frac{124826096558323497238818907979}{13425729891852959606888202240000} hf_{n+\frac{69}{74}} + \frac{22213495592711}{1065433529160000} hf_{n+1} + \\
&\frac{1596937613}{1235285251200} h^2 g_n + \frac{3370173164797268362157}{535136951413918767513600} h^2 g_{n+\frac{5}{74}} + \\
&\frac{1551045736}{430686577035} h^2 g_{n+\frac{1}{4}} - \frac{5197001}{293601280} h^2 g_{n+\frac{1}{2}} - \frac{1671184136}{602961207849} h^2 g_{n+\frac{3}{4}} - \\
&\frac{5047051038806095992223}{2675684757069593837568000} h^2 g_{n+\frac{69}{74}} - \frac{2579379871}{6176426256000} h^2 g_{n+1}
\end{aligned}\right\}
$$

(3.46)

$$
\left.\begin{aligned}
y_{n+\frac{3}{4}} = y_n + \frac{1564933587251}{15589333760000}hf_n + \frac{862350650512803320512765 56917}{104770213452655316548190208 0000}hf_{n+\frac{5}{74}} + \\
\frac{15845155729673}{80196321965760}hf_{n+\frac{1}{4}} + \frac{155171040579}{671759728640}hf_{n+\frac{1}{2}} + \frac{26061104016757}{240588965897280}hf_{n+\frac{3}{4}} - \\
\frac{5625175217543541746966475 9647}{31431064035796594964457062 40000}hf_{n+\frac{69}{74}} + \frac{753115181853}{15589333760000}hf_{n+1} + \\
\frac{1747267617}{1084475392000}h^2g_n + \frac{231080940715692766313}{2982887181003836620 8000}h^2g_{n+\frac{5}{74}} + \\
\frac{681768631}{117633035520}h^2g_{n+\frac{1}{4}} - \frac{299379159}{150323855360}h^2g_{n+\frac{1}{2}} - \frac{3778927211}{352899106560}h^2g_{n+\frac{3}{4}} - \\
\frac{28572045606573546074 59}{62640630801080569036 8000}h^2g_{n+\frac{69}{74}} - \frac{1034958591}{1084475392000}h^2g_{n+1} -
\end{aligned}\right\}
$$

(3.47)

$$
\left.\begin{aligned}
y_{n+\frac{69}{74}} = y_n + \frac{20221603651679559569621}{1925357182716106739600 00}hf_n + \frac{49850707848181703965480 1}{62228034491160880742400 00}hf_{n+\frac{5}{74}} + \\
\frac{11916070295137996850198 41568}{60254809457803720006217 32035}hf_{n+\frac{1}{4}} + \frac{39868522991012976378507}{170513795035787528765440}hf_{n+\frac{1}{2}} + \\
\frac{35638633238676056778197 24704}{18076442837341116001865 196105}hf_{n+\frac{3}{4}} - \frac{76905143170319465132065 3}{18668410347348264222720 000}hf_{n+\frac{69}{74}} + \\
\frac{148963089941292358633 71}{19253571827161067396 0000}hf_{n+1} + \frac{131291407474186803729}{770142873086442695840 00}h^2g_n + \\
\frac{8611510805044157833}{1055389290508320768000}h^2g_{n+\frac{5}{74}} + \frac{140129175010732991280 88}{220956396984978804569 9205}h^2g_{n+\frac{1}{4}} - \\
\frac{26786422976951818125}{2823613053459754322493 44}h^2g_{n+\frac{1}{2}} - \frac{613149069216113684775 2}{946955987078480591013 945}h^2g_{n+\frac{3}{4}} - \\
\frac{28171179896528379749}{3166167871524962304000}h^2g_{n+\frac{69}{74}} - \frac{113877534642128422479}{770142873086442695840 00}h^2g_{n+1} -
\end{aligned}\right\}
$$

(3.48)

$$
\left.\begin{aligned}
y_{n+1} = y_n + \frac{3506128349813}{33294797786250}hf_n + \frac{262138373250404721337405181}{327776608687816396652544 0000}hf_{n+\frac{5}{74}} + \\
\frac{1625876519575552}{8221376069021115}hf_{n+\frac{1}{4}} + \frac{153451983}{656015360}hf_{n+\frac{1}{2}} + \frac{1625876519575552}{8221376069021115}hf_{n+\frac{3}{4}} + \\
\frac{262138373250404721337405181}{327776608687816396652544 0000}hf_{n+\frac{69}{74}} + \frac{3506128349813}{33294797786250}hf_{n+1} + \\
\frac{330127123}{193013320500}h^2g_n + \frac{10692342218160370021}{1306486697787887616000}h^2g_{n+\frac{5}{74}} + \frac{19213240832}{3014806039245}h^2g_{n+\frac{1}{4}} - \\
- \frac{19213240832}{3014806039245}h^2g_{n+\frac{3}{4}} - \frac{10692342218160370021}{1306486697787887616000}h^2g_{n+\frac{69}{74}} - \frac{330127123}{193013320500}h^2g_{n+1}
\end{aligned}\right\}
$$

(3.49)

3.2.5 One-step second derivative block method with 6 intra-step points (OSDBM6)

The specification of one-step second derivative block method with 6 intra-step points is given as $k=1, \eta = \left(\frac{1}{20}, \frac{10}{53}, \frac{45}{116}, \frac{71}{116}, \frac{43}{53}, \frac{19}{20} \right), x \in [x_n, x_{n+1}]$ which results in system of equations

$$Y_{\omega} = D\Psi_{\omega-n}$$

(3.50)

where

$$Y_{\omega} = \left(y_n, f_n, f_{n+\frac{1}{20}}, f_{n+\frac{10}{53}}, f_{n+\frac{45}{116}}, f_{n+\frac{71}{116}}, f_{n+\frac{53}{43}}, f_{n+\frac{19}{20}}, f_{n+1,} g_n, g_{n+\frac{1}{20}}, g_{n+\frac{10}{53}}, g_{n+\frac{45}{116}}, g_{n+\frac{71}{116}}, g_{n+\frac{43}{53}}, g_{n+\frac{19}{20}}, g_{n+1,} \right)^T,$$

$$\Psi_{\omega} = \left(\alpha_0, \beta_0, \beta_{\frac{1}{20}}, \beta_{\frac{10}{53}}, \beta_{\frac{45}{116}}, \beta_{\frac{71}{116}}, \beta_{\frac{43}{53}}, \beta_{\frac{19}{20}}, \beta_1, \gamma_0, \gamma_{\frac{1}{20}}, \gamma_{\frac{10}{53}}, \gamma_{\frac{45}{116}}, \gamma_{\frac{71}{116}}, \gamma_{\frac{43}{53}}, \gamma_{\frac{19}{20}}, \gamma_1 \right)$$

Similar to the above procedure, the coefficients of the discrete one-step second derivative block method with 6 intra-step points (OSDBM6) are given as

$$
\left.
\begin{aligned}
y_{n+\frac{1}{20}} = y_n &+ \frac{847643686742018502407616289619130484 09}{41019967251473976301536000000000000000000} hf_n + \\
&\frac{2272196741385440775659840323979459365 21}{7908560194612450895811646872477496934400} hf_{n+\frac{1}{20}} + \\
&\frac{830986430887875704073320014190249787010545394214081421 49}{188525992486786370170966240978621709005200000000000000000000} hf_{n+\frac{10}{53}} \\
&+ \frac{2237461594437433856156987578767262227872477611003180621}{2678438046346951234797094447834643212287187500000000000000000} hf_{n+\frac{45}{116}} \\
&- \frac{3473498118550410305713907331743566495312353584536559}{2142750437077560987837675558267714569829750000000000000000} hf_{n+\frac{71}{116}} - \\
&\frac{1198504877973930208756755775476817045672589525331568 9}{150820793989429096136772992782897367204160000000000000000000} hf_{n+\frac{43}{53}} - \\
&\frac{614592394296913367433882817337280649}{7908560194612450895811646872477496934400} hf_{n+\frac{19}{20}} \\
&+ \frac{123147359183157643972253430629221307}{820399345029479526030720000000000000000000} hf_{n+1} \\
&+ \frac{153111187368927437788484727 11}{1208812766377708800000000000000000000} h^2 g_n - \frac{763315543400393644060752017 3}{195691816483694575210274734080 00} h^2 g_{n+\frac{1}{20}} \\
&- \frac{110748444791834369897239845156563551467888701}{2013583555785763043650283828112000000000000000000} h^2 g_{n+\frac{10}{53}} - \\
&\frac{3067190167814926026545020639000467150569 59}{124070784806442028110615551992500000000000000000} h^2 g_{n+\frac{45}{116}} - \\
&\frac{76011322557910326185055128821149911969 51}{496283139225768112442462207970000000000000000000} h^2 g_{n+\frac{71}{116}} - \\
&\frac{8456574360853384607611027477964765178763 1}{732212202103913834054648664768000000000000000000000} h^2 g_{n+\frac{10}{53}} - \\
&\frac{6282612181877105303051431 9}{6523060549456485840342491136000} h^2 g_{n+\frac{19}{20}} - \frac{5447599228449762232070806 9}{24176255327554176000000000000000} h^2 g_{n+1}
\end{aligned}
\right\} \qquad (3.51)
$$

$$
\left.
\begin{aligned}
& y_{n+\frac{10}{53}} = y_n + \frac{27441347628779617907200170289513875065735}{62973200514393426369953789536622813693214 3} hf_n + \\
& \frac{344387101054959507313346483367059933210937500000000000}{474261549737997263103025019134580412945464573043625223 7} hf_{n+\frac{1}{20}} + \\
& \frac{19423288443435809543431369122273091340603}{28386015914203629599605712420076564139987 5} hf_{n+\frac{10}{53}} \\
& + \frac{12700879266066244166053769600833686880279397421287644025221 12}{52632261590717182697226904346845470686860887416288059577118487 5} hf_{n+\frac{45}{116}} \\
& + \frac{4383056087911429311698346426660032648572494073803317535334 4}{10526452318143436539445380869369094137372177483257611915423697 5} hf_{n+\frac{71}{116}} - \\
& \frac{6401488130432950938635540528269662764}{5677203182840725919921142484015312827997 5} hf_{n+\frac{43}{53}} - \\
& \frac{66972588075670136459558333053663269453125000000000000}{474261549737997263103025019134580412945464573043625223 7} hf_{n+\frac{19}{20}} + \\
& \frac{17364257464418460724626595551129149456 60}{62973200514393426369953789536622813693214 3} hf_{n+1} \\
& + \frac{401146602873847260974921674881 35}{92787505478477363798789667938135397} h^2 g_n + \frac{247584516201592008800497774296875000000000 00}{11735271891819841116835486056332769139833646 59} h^2 g_{n+\frac{1}{20}} \\
& - \frac{44333388816344454498375351731 11}{17675488150631901217542647728645 65} h^2 g_{n+\frac{10}{53}} - \frac{673595551025452878540224635071684214968648611 84}{121901755558856955671161411239783624694905226134438 5} h^2 g_{n+\frac{45}{116}} - \\
& \frac{735789851762799032458336835230215203861587251 2}{243803511177139113423228224795672493898104522688 77} h^2 g_{n+\frac{71}{116}} - \frac{76800148921142757046670911288}{35350976301263802435085295457291 3} h^2 g_{n+\frac{43}{53}} - \\
& \frac{6948878157241182354218341953125000000000 00}{39117572972732803722784953521109230466112155 3} h^2 g_{n+\frac{19}{20}} - \frac{38345299405690475221762866812 00}{92787505478477363798789667938135397} h^2 g_{n+1}
\end{aligned}
\right\}
$$

(3.52)

$$
\left.
\begin{aligned}
& y_{n+\frac{45}{116}} = y_n + \frac{19629673882951274012574136032547890390165}{28994202824905806506677886305649583928115 2} hf_n + \\
& \frac{1928964942259961599107980232876921332168579101562 5}{29928744395769861373052857253577244052404311755980 8} hf_{n+\frac{1}{20}} + \\
& \frac{2010956976395118025900804601709063533276661829569410570654289}{133256100143086679176277460458442341214870764984018419056640 00} hf_{n+\frac{10}{53}} \\
& + \frac{90778610913131338115693609320042225542903}{95198890606725558773795500466419325747200 0} hf_{n+\frac{45}{116}} + \frac{11206002562267901797021158493638430972614 3}{31034838337792532160257333152052700193587200} hf_{n+\frac{71}{116}} - \\
& \frac{357797004394792133928861436094727889800546724632975464157}{266512200286173358352554920916884682429741529968036838113280 0} hf_{n+\frac{43}{53}} - \\
& \frac{210750249165410056757521328435887943267822265625}{348862664735967709256444348354581372389988910039 04} hf_{n+\frac{19}{20}} + \frac{18427505788686264901356564361552435258 5}{152601067499504244771988875292892546990 08} hf_{n+1} \\
& + \frac{30030924169128727774687874314513 5}{38449209478571958167751486631785267 2} h^2 g_n + \frac{1703963523224048350500207886505126953125}{44433956713442481669676959330176194320793 6} h^2 g_{n+\frac{1}{20}} \\
& + \frac{1073575544828017516394158297228373459713673712837}{426979253762687427265982100773490277679901987307520} h^2 g_{n+\frac{10}{53}} - \frac{630458209502363115888736687137}{1150960233538852647686197578956 80} h^2 g_{n+\frac{45}{116}} - \\
& \frac{202687523088875461863864761427}{12938380556333309073989669335859 2} h^2 g_{n+\frac{71}{116}} - \frac{855219982195357993095610244414591575238104028 79}{85395850752537485453196420154698055535980397461504} h^2 g_{n+\frac{43}{53}} - \\
& \frac{40632023663651565694375353240966796875}{51794182672110868203917928053579613011968} h^2 g_{n+\frac{19}{20}} - \frac{365051350693965458247874352167 5}{20236426041353662193553414016729088} h^2 g_{n+1}
\end{aligned}
\right\}
$$

(3.53)

$$
\left.
\begin{aligned}
y_{n+\frac{71}{116}} = y_n &+ \frac{1446543698451751010537145316203794557835503}{166094857323300167927845201086449319936000000} hf_n + \\
&\frac{690685007521106660138413821911807664872888549072265625}{12436291158774250496344653774578952221095563663962701824} hf_{n+\frac{1}{20}} + \\
&\frac{41952599766483647136999039243680317995429741190956780315124116762341}{273216647824622406898611380646200062722114715331295264822067200000000} hf_{n+\frac{10}{53}} \\
&+ \frac{1727258621434778882502352699136629270254815231}{8889237405312389266441603272645720844800000000} hf_{n+\frac{45}{116}} + \\
&\frac{911706528436974152032094995508703695437396481}{8889237405312389266441603272645720844800000000} hf_{n+\frac{71}{116}} + \\
&\frac{685017726303832999846803958604526150888849943749006833360177656091}{273216647824622406898611380646200062722114715331295264822067200000000} hf_{n+\frac{43}{53}} - \\
&\frac{185986254508909725581903164937693226187177674072265625}{12436291158774250496344653774578952221095563663962701824} hf_{n+\frac{19}{20}} + \frac{522616853641476093788316461561168156116753}{166094857323300167927845201086449319936000000} hf_{n+1} \\
&+ \frac{5301114601090068992469941737851889}{4964553821144996298971794455920640000} h^2 g_n + \frac{284099998133653068936149754008138345849609375}{55390926099410263224602600731404342078358093824} h^2 g_{n+\frac{1}{20}} \\
&+ \frac{12955844981309264950450255335666588836767000457374100771}{2918136337434366885720946419973822616518580144504832000000} h^2 g_{n+\frac{10}{53}} - \frac{252747680396554380369930119800541}{292355169657774446877432210432000000} h^2 g_{n+\frac{45}{116}} - \\
&\frac{67052792164553341926105092925384439}{8478299920075458959445534102528000000} h^2 g_{n+\frac{71}{116}} - \frac{776461286366680238988666850466537374685938207916777911}{265285121584942444156449674543074783319870922227712000000} h^2 g_{n+\frac{43}{53}} - \\
&\frac{115139242574992946982034199980374979150390625}{55390926099410263224602600731404342078358093824} h^2 g_{n+\frac{19}{20}} - \frac{16233698035798595690847228641594473}{34751876748014974092802561191444480000} h^2 g_{n+1}
\end{aligned}
\right\}
$$

(3.54)

$$
\left.
\begin{aligned}
y_{n+\frac{43}{53}} = y_n &+ \frac{76360855369795597472770832504540887740227}{792045989842321133610295817181164094900000} hf_n + \frac{12707448606305104184617345508649607341242515625000000}{249611341967366980580539483755042322602876091075592223} hf_{n+\frac{1}{20}} + \\
&\frac{54813811143342682546814260938547002547337}{357025367756343838902306871345624462500000} hf_{n+\frac{10}{53}} \\
&+ \frac{324845287563140269653938715898992418488057555523458736127314970048}{1644758174709911959288340760838920958964402731759001861784952734375} hf_{n+\frac{45}{116}} + \\
&\frac{1644758174709911959288340760838920958964402731759001861784952734375}{1644758174709911959288340760838920958964402731759001861784952734375} hf_{n+\frac{71}{116}} + \\
&\frac{30343895432665667569360015620632303647337}{357025367756343838902306871345624462500000} hf_{n+\frac{43}{53}} - \\
&\frac{109642836342729541451572752008083720672642203125000000}{4742615497379972631030250191345804129454645730436252237} hf_{n+\frac{19}{20}} + \frac{2317392999922920646373963435525661328433}{41686631044332691242647148272692847100000} hf_{n+1} \\
&+ \frac{173037982728813253777716148495033}{143378668745232733985613332207580000} h^2 g_n + \frac{354272303334395430669870292244876562500000}{61764588904314953246502558191225100735966561} h^2 g_{n+\frac{1}{20}} \\
&+ \frac{499386898175078232238402489179443}{95594852085624127731436710268500000} h^2 g_{n+\frac{10}{53}} + \frac{811136823168597294719096277028731459962052621140384}{380942986214279864723794101243238271715788316701203125} h^2 g_{n+\frac{45}{116}} - \\
&\frac{6725459155041353065588944414161857412681232664736}{2254100510143667838602331960019161371099339152078125} h^2 g_{n+\frac{71}{116}} - \frac{759924465846260451549970349079443}{95594852085624127731436710268500000} h^2 g_{n+\frac{43}{53}} - \\
&\frac{1487931648684942855116369356092551562500000}{39117572972732803722849535211092304661121553} h^2 g_{n+\frac{19}{20}} - \frac{43096638396934797533194685429749}{52823720064033112521015438181740000} h^2 g_{n+1}
\end{aligned}
\right\}
$$

(3.55)

$$
\left.\begin{aligned}
y_{n+\frac{19}{20}} = y_n + \frac{1184333240849174609162419188939652 7}{1960917699802879808000000000000000 0} hf_n + \\
\frac{57160369035493076669172340402002811}{115301941895501543895781409425244160 0} hf_{n+\frac{1}{20}} \\
\frac{3373609093801931407066822345257220019600782457256937 1}{2198874383866877039463084892592176224000000000000000000 0} hf_{n+\frac{10}{53}} + \\
\frac{77255002779997246404835004983498411917031878501078828 1}{39049978806632909094577845864333623156250000000000000000 0} hf_{n+\frac{45}{116}} + \\
\frac{77255002779997246404835004983498411917031878501078828 1}{39049978806632909094577845864333623156250000000000000000 0} hf_{n+\frac{71}{116}} + \\
\frac{42046776780577422974443636297580870597819016353081486 89}{27485929798335962993288561157402202800000000000000000000 000} hf_{n+\frac{43}{53}} + \\
\frac{1468928510761518133806866689880 87}{7073738766595186742072479105843200} hf_{n+\frac{19}{20}} + \\
\frac{46948309239092924652478684799254549}{598045884990143990400000000000000000000} hf_{n+1} + \\
\frac{1192005784582358527015405 3}{9567176623488000000000000000 0} h^2 g_n + \frac{32003784426150384166456636 3}{542082594137658103075553280000} h^2 g_{n+\frac{1}{20}} \\
+ \frac{12114286962924793880800982901562695729372 19}{2231117513336025533130508396800000000000000000 0} h^2 g_{n+\frac{10}{53}} + \\
\frac{20374629629623758954952734892735669643}{843402170565348660745479000000000000000000 0} h^2 g_{n+\frac{45}{116}} - \\
\frac{64925113159120050336190569717204446441 63}{2643741419272150609644482250000000000000000 0} h^2 g_{n+\frac{71}{116}} - \\
\frac{30656919690647666634324258280834624796309941}{557779378334006383282627099200000000000000000000} h^2 g_{n+\frac{43}{53}} - \\
\frac{69878204475107849951343 7}{110855336224469959729152000} h^2 g_{n+\frac{19}{20}} - \frac{37554346843908793413061364 9}{3348511818220800000000000000 00} h^2 g_{n+1}
\end{aligned}\right\}
$$

(3.56)

$$
\left.
\begin{aligned}
y_{n+1} = y_n + & \frac{181599473993366169835764 7}{183124853801223108489000000} hf_n + \\
& \frac{75847977666974768164531250000000}{153238161012921063054386138867139} hf_{n+\frac{1}{20}} \\
& \frac{106711123217557710996820477191299233034 7739}{69556520250437710364140437196953109875 00000} hf_{n+\frac{10}{53}} + \\
& \frac{560139275388255342853316777197477106468416}{283013318506651652028433479272468640351562 5} hf_{n+\frac{45}{116}} + \\
& \frac{560139275388255342853316777197477106468416}{283013318506651652028433479272468640351562 5} hf_{n+\frac{71}{116}} + \\
& \frac{106711123217557710996820477191299233034 7739}{69556520250437710364140437196953109875 00000} hf_{n+\frac{43}{53}} + \\
& \frac{75847977666974768164531250000000}{153238161012921063054386138867139} hf_{n+\frac{19}{20}} + \\
& \frac{181599473993366169835764 7}{183124853801223108489000000} hf_{n+1} + \\
& \frac{4715072213856359}{377753989493034 0000} h^2 g_n + \frac{20180339729328125 00000}{34125944559796068500675 7} h^2 g_{n+\frac{1}{20}} \\
& + \frac{48912498051482023594168827652789}{898921230261501358772448137550 0000} h^2 g_{n+\frac{10}{53}} + \\
& \frac{26930925525708524125106903843 2}{110777486434323239384478171421875} h^2 g_{n+\frac{45}{116}} - \\
& \frac{26930925525708524125106903843 2}{110777486434323239384478171421875} h^2 g_{n+\frac{71}{116}} - \\
& \frac{48912498051482023594168827652789}{898921230261501358772448137550 0000} h^2 g_{n+\frac{43}{53}} - \\
& \frac{20180339729328125 00000}{34125944559796068500675 7} h^2 g_{n+\frac{19}{20}} - \frac{4715072213856359}{377753989493034 0000} h^2 g_{n+1}
\end{aligned}
\right\}
$$

(3.57)

3.3. Analysis of Order of Accuracy and Error Constant

Extending the definition of Fatunla (1991) and Lambert (1973), we define the local truncation error associated with the conventional form of (3.1) to be the linear difference operator

$$
L\left[y(x);h\right] = \sum_{j=0}^{k}\left(\alpha_j y\left(x+jh\right)\right) + h\left(\sum_{j=0}^{k}\beta_j f_{n+j}\right) + h^2\left(\sum_{j=0}^{k}\gamma_j g_{n+j}\right)
$$

(3.58)

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (3.58) as a Taylor series about the point x to obtain the expression

$$L\left[y(x);h\right]=C_0y(x)+C_1hy'(x)+...+C_qh^qy^{(q)}(x)+...$$

(3.59)

Where the constant $C_q, q=0,1,...$ are given as follows

$$\left.\begin{array}{l} C_0=\sum_{j=0}^{k}\alpha_j \\ C_1=\sum_{j=1}^{k} j\alpha_j-\sum_{j=0}^{k}\beta_j \\ C_2=\frac{1}{2!}\sum_{j=1}^{k}(j)^2\alpha_j-\sum_{j=1}^{k} j\beta_j-\sum_{j=0}^{k}\gamma_j \\ \vdots \\ C_q=\frac{1}{\text{q}!}\sum_{j=1}^{k}(j)^q\alpha_j-\frac{1}{(q-1)!}\sum_{j=1}^{k} j^{q-1}\beta_j-\frac{1}{(q-2)!}\sum_{j=1}^{k} j^{q-2}\gamma_j \end{array}\right\}$$

(3.60)

Definition 3.2.1: (Lambert, 1973): A linear multistep method is said to be of order of accuracy p if $C_0=C_1=...C_p=0, C_{p+1}\neq 0.$ C_{p+1} is called the error constant.

3.3.1. Order of accuracy and error constants of (OSDBM2)

For the method in (3.17),

$$\left.\begin{array}{l} \alpha_0=-1,\alpha_{\frac{1}{4}}=1 \\ \beta_0=\frac{20135}{193536},\beta_{\frac{1}{4}}=\frac{3413}{24192},\beta_{\frac{3}{4}}=\frac{11}{24192},\beta_1=\frac{857}{193536} \\ \gamma_0=\frac{233}{71680},\gamma_{\frac{1}{4}}=-\frac{1601}{161280},\gamma_{\frac{3}{4}}=-\frac{289}{161280},\gamma_1=-\frac{23}{71680} \end{array}\right\}$$

(3.61)

$$C_0=\alpha_0+\alpha_{\frac{1}{4}}=0$$

(3.62)

$$C_1 = \frac{1}{4}\alpha_{\frac{1}{4}} - \left(\beta_0 + \beta_{\frac{1}{4}} + \beta_{\frac{3}{4}} + \beta_1\right) = 0$$

(3.63)

$$C_3 = \frac{1}{3!}\left(\left(\frac{1}{4}\right)^3 \alpha_{\frac{1}{4}}\right) - \frac{1}{2!}\left(\left(\frac{1}{4}\right)^2 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^2 \beta_{\frac{3}{4}} + \beta_1\right) - \left(\frac{1}{4}\gamma_{\frac{1}{4}} + \frac{3}{4}\gamma_{\frac{3}{4}} + \gamma_1\right) = 0$$

(3.64)

$$C_4 = \frac{1}{4!}\left(\left(\frac{1}{4}\right)^4 \alpha_{\frac{1}{4}}\right) - \frac{1}{3!}\left(\left(\frac{1}{4}\right)^3 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^3 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{2!}\left(\left(\frac{1}{4}\right)^2 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^2 \gamma_{\frac{3}{4}} + \gamma_1\right) = 0$$

(3.65)

$$C_5 = \frac{1}{5!}\left(\left(\frac{1}{4}\right)^5 \alpha_{\frac{1}{4}}\right) - \frac{1}{4!}\left(\left(\frac{1}{4}\right)^4 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^4 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{3!}\left(\left(\frac{1}{4}\right)^3 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^3 \gamma_{\frac{3}{4}} + \gamma_1\right) = 0$$

(3.66)

$$C_6 = \frac{1}{6!}\left(\left(\frac{1}{4}\right)^6 \alpha_{\frac{1}{4}}\right) - \frac{1}{4!}\left(\left(\frac{1}{4}\right)^5 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^5 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{3!}\left(\left(\frac{1}{4}\right)^4 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^4 \gamma_{\frac{3}{4}} + \gamma_1\right) = 0$$

(3.67)

$$C_7 = \frac{1}{7!}\left(\left(\frac{1}{4}\right)^7 \alpha_{\frac{1}{4}}\right) - \frac{1}{6!}\left(\left(\frac{1}{4}\right)^6 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^6 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{5!}\left(\left(\frac{1}{4}\right)^5 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^5 \gamma_{\frac{3}{4}} + \gamma_1\right) = 0$$

(3.68)

$$C_8 = \frac{1}{8!}\left(\left(\frac{1}{4}\right)^8 \alpha_{\frac{1}{4}}\right) - \frac{1}{7!}\left(\left(\frac{1}{4}\right)^7 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^7 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{6!}\left(\left(\frac{1}{4}\right)^6 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^6 \gamma_{\frac{3}{4}} + \gamma_1\right) = 0$$

(3.69)

$$\left.\begin{aligned} C_9 &= \frac{1}{9!}\left(\left(\frac{1}{4}\right)^9 \alpha_{\frac{1}{4}}\right) - \frac{1}{8!}\left(\left(\frac{1}{4}\right)^8 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^8 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{7!}\left(\left(\frac{1}{4}\right)^7 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^7 \gamma_{\frac{3}{4}} + \gamma_1\right) \\ &= \frac{103}{416179814400} \end{aligned}\right\}$$

(3.70)

Therefore, the method is of order 8 with error constant, $C_9 = \frac{103}{416179814400}$

For the method in (3.18),

$$\left.\begin{aligned}
&\alpha_0 = -1, \alpha_{\frac{3}{4}} = 1 \\
&\beta_0 = \frac{1125}{7168}, \beta_{\frac{1}{4}} = \frac{303}{896}, \beta_{\frac{3}{4}} = \frac{177}{896}, \beta_1 = \frac{411}{7168} \\
&\gamma_0 = \frac{489}{71680}, \gamma_{\frac{1}{4}} = \frac{423}{17920}, \gamma_{\frac{3}{4}} = -\frac{633}{17920}, \gamma_1 = -\frac{279}{71680}
\end{aligned}\right\}$$

(3.71)

$$C_0 = \alpha_0 + \alpha_{\frac{1}{4}} = 0$$

(3.72)

$$C_1 = \frac{1}{4}\alpha_{\frac{1}{4}} - \left(\beta_0 + \beta_{\frac{1}{4}} + \beta_{\frac{3}{4}} + \beta_1\right) = 0$$

(3.73)

$$C_3 = \frac{1}{3!}\left(\left(\frac{1}{4}\right)^3 \alpha_{\frac{1}{4}}\right) - \frac{1}{2!}\left(\left(\frac{1}{4}\right)^2 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^2 \beta_{\frac{3}{4}} + \beta_1\right) - \left(\frac{1}{4}\gamma_{\frac{1}{4}} + \frac{3}{4}\gamma_{\frac{3}{4}} + \gamma_1\right) = 0$$

(3.74)

$$C_4 = \frac{1}{4!}\left(\left(\frac{1}{4}\right)^4 \alpha_{\frac{1}{4}}\right) - \frac{1}{3!}\left(\left(\frac{1}{4}\right)^3 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^3 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{2!}\left(\left(\frac{1}{4}\right)^2 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^2 \gamma_{\frac{3}{4}} + \gamma_1\right) = 0$$

(3.75)

$$C_5 = \frac{1}{5!}\left(\left(\frac{1}{4}\right)^5 \alpha_{\frac{1}{4}}\right) - \frac{1}{4!}\left(\left(\frac{1}{4}\right)^4 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^4 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{3!}\left(\left(\frac{1}{4}\right)^3 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^3 \gamma_{\frac{3}{4}} + \gamma_1\right) = 0$$

(3.76)

$$C_6 = \frac{1}{6!}\left(\left(\frac{1}{4}\right)^6 \alpha_{\frac{1}{4}}\right) - \frac{1}{4!}\left(\left(\frac{1}{4}\right)^5 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^5 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{3!}\left(\left(\frac{1}{4}\right)^4 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^4 \gamma_{\frac{3}{4}} + \gamma_1\right) = 0$$

(3.77)

$$C_7 = \frac{1}{7!}\left(\left(\frac{1}{4}\right)^7 \alpha_{\frac{1}{4}}\right) - \frac{1}{6!}\left(\left(\frac{1}{4}\right)^6 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^6 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{5!}\left(\left(\frac{1}{4}\right)^5 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^5 \gamma_{\frac{3}{4}} + \gamma_1\right) = 0$$

(3.78)

$$C_8 = \frac{1}{8!}\left(\left(\frac{1}{4}\right)^8 \alpha_{\frac{1}{4}}\right) - \frac{1}{7!}\left(\left(\frac{1}{4}\right)^7 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^7 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{6!}\left(\left(\frac{1}{4}\right)^6 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^6 \gamma_{\frac{3}{4}} + \gamma_1\right) = 0$$

(3.79)

$$\left.\begin{aligned}C_9 &= \frac{1}{9!}\left(\left(\frac{1}{4}\right)^9 \alpha_{\frac{1}{4}}\right) - \frac{1}{8!}\left(\left(\frac{1}{4}\right)^8 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^8 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{7!}\left(\left(\frac{1}{4}\right)^7 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^7 \gamma_{\frac{3}{4}} + \gamma_1\right) \\ &= \frac{9}{5138022400}\end{aligned}\right\} \quad (3.80)$$

Hence, the method is of order 8

For the method in (3.19),

$$\left.\begin{aligned}&\alpha_0 = -1, \alpha_1 = 1 \\ &\beta_0 = \frac{61}{378}, \beta_{\frac{1}{4}} = \frac{64}{189}, \beta_{\frac{3}{4}} = \frac{64}{189}, \beta_1 = \frac{61}{378} \\ &\gamma_0 = \frac{1}{40}, \gamma_{\frac{1}{4}} = \frac{8}{315}, \gamma_{\frac{3}{4}} = -\frac{8}{315}, \gamma_1 = -\frac{1}{140}\end{aligned}\right\} \quad (3.81)$$

$$C_0 = \alpha_0 + \alpha_{\frac{1}{4}} = 0 \quad (3.82)$$

$$C_1 = \frac{1}{4}\alpha_{\frac{1}{4}} - \left(\beta_0 + \beta_{\frac{1}{4}} + \beta_{\frac{3}{4}} + \beta_1\right) = 0 \quad (3.83)$$

$$C_3 = \frac{1}{3!}\left(\left(\frac{1}{4}\right)^3 \alpha_{\frac{1}{4}}\right) - \frac{1}{2!}\left(\left(\frac{1}{4}\right)^2 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^2 \beta_{\frac{3}{4}} + \beta_1\right) - \left(\frac{1}{4}\gamma_{\frac{1}{4}} + \frac{3}{4}\gamma_{\frac{3}{4}} + \gamma_1\right) = 0 \quad (3.84)$$

$$C_4 = \frac{1}{4!}\left(\left(\frac{1}{4}\right)^4 \alpha_{\frac{1}{4}}\right) - \frac{1}{3!}\left(\left(\frac{1}{4}\right)^3 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^3 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{2!}\left(\left(\frac{1}{4}\right)^2 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^2 \gamma_{\frac{3}{4}} + \gamma_1\right) = 0 \quad (3.85)$$

$$C_5 = \frac{1}{5!}\left(\left(\frac{1}{4}\right)^5 \alpha_{\frac{1}{4}}\right) - \frac{1}{4!}\left(\left(\frac{1}{4}\right)^4 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^4 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{3!}\left(\left(\frac{1}{4}\right)^3 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^3 \gamma_{\frac{3}{4}} + \gamma_1\right) = 0 \quad (3.86)$$

$$C_6 = \frac{1}{6!}\left(\left(\frac{1}{4}\right)^6 \alpha_{\frac{1}{4}}\right) - \frac{1}{4!}\left(\left(\frac{1}{4}\right)^5 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^5 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{3!}\left(\left(\frac{1}{4}\right)^4 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^4 \gamma_{\frac{3}{4}} + \gamma_1\right) = 0$$

(3.87)

$$C_7 = \frac{1}{7!}\left(\left(\frac{1}{4}\right)^7 \alpha_{\frac{1}{4}}\right) - \frac{1}{6!}\left(\left(\frac{1}{4}\right)^6 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^6 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{5!}\left(\left(\frac{1}{4}\right)^5 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^5 \gamma_{\frac{3}{4}} + \gamma_1\right) = 0$$

(3.88)

$$C_8 = \frac{1}{8!}\left(\left(\frac{1}{4}\right)^8 \alpha_{\frac{1}{4}}\right) - \frac{1}{7!}\left(\left(\frac{1}{4}\right)^7 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^7 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{6!}\left(\left(\frac{1}{4}\right)^6 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^6 \gamma_{\frac{3}{4}} + \gamma_1\right) = 0$$

(3.89)

$$\left.\begin{aligned} C_9 &= \frac{1}{9!}\left(\left(\frac{1}{4}\right)^9 \alpha_{\frac{1}{4}}\right) - \frac{1}{8!}\left(\left(\frac{1}{4}\right)^8 \beta_{\frac{1}{4}} + \left(\frac{3}{4}\right)^8 \beta_{\frac{3}{4}} + \beta_1\right) - \frac{1}{7!}\left(\left(\frac{1}{4}\right)^7 \gamma_{\frac{1}{4}} + \left(\frac{3}{4}\right)^7 \gamma_{\frac{3}{4}} + \gamma_1\right) \\ &= \frac{13}{6502809600} \end{aligned}\right\}$$

(3.90)

The method methods (3.16), (3.17) and (3.18) are of order 8. Therefore, the one-step second derivative block method with two intra-step points is of a uniform order 8 and the error constants given as $\left(\frac{103}{416179814400}, \frac{9}{5138022400}, \frac{13}{6502809600}\right)^T$

Similar procedure is applied to the cases of OSDBM3, OSDBM3, OSDBM4, OSDBM5 and OSDBM6 and the summary of the order and error constants obtained is presented in Tables 3.2,3.3,3.4, 3.5, 3.6 respectively.

Table 3.2 Order and Error Constants for the Proposed (OSDBM2)

Equation	Order p	Error constants, C_{p+1}
3.17	8	$\frac{103}{416179814400}$
3.18	8	$\frac{9}{5138022400}$

3.19	8	$\frac{13}{6502809600}$

Table 3.3 Order and Error Constants for the Proposed (OSDBM3)

Equation	Order p	Error constants, C_{p+1}
3.39	10	$\frac{54355055225}{88253992062514248568 0128}$
3.40	10	$\frac{186341}{268844974497792000}$
3.41	10	$\frac{146131455427861}{110317490078142810710016000}$
3.42	10	$\frac{186341}{134422487248896000}$

Table 3.4 Order and Error Constants for the Proposed (OSDBM4)

Equation	Order p	Error constants, C_{p+1}
3.39	12	$\frac{39331308300539375731 25}{3101837300463558404959051145119468 09344}$
3.40	12	$\frac{1085620410122054 9375}{622583425411184020191466912967 39328}$
3.41	12	$\frac{1860296420252348580289}{383128261791497858579364254133780 4800}$
3.42	12	$\frac{5019144627163010954107523}{77545932511588960123976278627986 70233600}$
3.43	12	$\frac{2265698353}{343325104164130577448 9600}$

Table 3.5 Order and Error Constants for the Proposed (OSDBM5)

Equation	Order p	Error constants, C_{p+1}
3.44	14	$\frac{213743559100493024975}{98073727507265805514550207132752882434048}$
3.45	14	$\frac{4754047589}{1411151632466261606934773760 00}$
3.46	14	$\frac{222373}{4409848851457067521671168 0000}$
3.47	14	$\frac{12402993}{645245373784298860052480 00}$
3.48	14	$\frac{250824629782342469253}{11210988512490375573222474523634302 97600}$
3.49	14	$\frac{222373}{9843412614859525718016 00}$

Table 3.6 Order and Error Constants for the Proposed (OSDBM6)

Equation	Order, p	Error constants, C_{p+1}
3.51	16	$\dfrac{287274354210728059365516603 1}{91561001222167410502075919892480000000000000000000}$
3.52	16	$\dfrac{39806544218069223800229061043241 1}{76286346497654031883858780099732915625575515979807457280}$
3.53	16	$\dfrac{289647468279310318545692852289 9}{15014621244595116588704167260624601303079222773809152 0}$
3.54	16	$\dfrac{609250024915265883890645700510873851893}{153923328102794624903887564682746889296098094717127884800000 00}$
3.55	16	$\dfrac{2046548331663368127001493773853407891049}{38143173248827015941929390049866457812787757989903728640000000}$
3.56	16	$\dfrac{536169061329179050656344833969}{91561001222167410502075919892480000000000000000000}$
3.57	16	$\dfrac{86246688779405813}{14649760195546785680332147182796800000000}$

3.4 Consistency

Definition: A linear multistep method is said to be consistent if the following conditions are satisfied.

(i) the order of accuracy $p > 1$

(ii) $\sum_{j=0}^{k} \alpha_j = 0$

(iii) $\rho'(1) = \sigma(1)$, where $\rho(r)$ *and* $\sigma(r)$, are respectively first and second characteristic polynomials of the methods.

From section 3.3.1, conditions (i) and (ii) are satisfied for all the proposed methods, since in each case the order $p > 1$ and $C_0 = \sum_{j=0}^{k} \alpha_j = 0$.

For the third condition, the first and second characteristic polynomials are obtained and evaluated in what follows. For all the methods, conditions for consistency are satisfied. Hence they are consistent with uniform order of accuracy $p > 1$.

The summary of parameters for measuring consistency is presented in Tables 3.7, 3.8,3.9,3.10.

Table 3.7 Parameters for Determining Consistency of (OSDBM2)

Equation	Order p	$\sum \alpha_j$	$\rho'(1)$	$\sigma(1)$
3.17	8	0	$\frac{1}{4}$	$\frac{1}{4}$
3.18	8	0	$\frac{3}{4}$	$\frac{3}{4}$
3.19	8	0	1	1

Table 3.8 Parameters for determining consistency of (OSDBM3)

Equation	Order, p	$\sum \alpha_j$	$\rho'(1)$	$\sigma(1)$
3.39	4	0	$\frac{5}{34}$	$\frac{5}{34}$
3.40	4	0	$\frac{1}{2}$	$\frac{1}{2}$
3.41	4	0	$\frac{29}{34}$	$\frac{29}{34}$
3.42	4	0	1	1

Table 3.9 Parameters for Determining Consistency of (OSDBM4)

Equation	Order, p	$\sum \alpha_j$	$\rho'(1)$	$\sigma(1)$
3.39	3	0	$\frac{5}{52}$	$\frac{5}{52}$
3.40	3	0	$\frac{10}{29}$	$\frac{10}{29}$
3.41	3	0	$\frac{19}{29}$	$\frac{19}{29}$
3.42	3	0	$\frac{47}{52}$	$\frac{47}{52}$
3.43	3	0	1	1

Table 3.10 Parameters for Determining Consistency of (OSDBM5)

Equation	Order, p	$\sum \alpha_j$	$\rho'(1)$	$\sigma(1)$
3.44	14	0	$\frac{5}{74}$	$\frac{5}{74}$
3.45	14	0	$\frac{1}{4}$	$\frac{1}{4}$
3.46	14	0	$\frac{1}{2}$	$\frac{1}{2}$
3.47	14	0	$\frac{3}{4}$	$\frac{3}{4}$
3.48	14	0	$\frac{69}{74}$	$\frac{69}{74}$
3.49	14	0	1	1

Table 3.11 Parameters for determining consistency of (OSDBM6)

Equation	Order, p	$\sum \alpha_j$	$\rho'(1)$	$\sigma(1)$
3.51	16	0	$\frac{45}{116}$	$\frac{45}{116}$
3.52	16	0	$\frac{1}{20}$	$\frac{1}{20}$
3.53	16	0	$\frac{10}{53}$	$\frac{10}{53}$
3.54	16	0	$\frac{45}{116}$	$\frac{45}{116}$
3.55	16	0	$\frac{71}{116}$	$\frac{71}{116}$
3.56	16	0	$\frac{43}{53}$	$\frac{43}{53}$
3.57	16	0	$\frac{19}{20}$	$\frac{19}{20}$
3.51	16	0	1	1

3.5 Zero Stability

In what follows, the one-step second derivative block methods can generally be written as a matrix difference equation as follows

$$A^{(1)}Y_w = A^{(0)}Y_{w-1} + h\left[B^{(0)}F_{w-1} + B^{(1)}F_w\right] + h^2\left[C^{(0)}G_{w-1} + C^{(1)}G_w\right] \tag{3.91}$$

And the matrices $A^{(1)}, A^{(0)}, \mathrm{B}^{(1)}, \mathrm{B}^{(0)}, \mathrm{C}^{(1)}$ and $C^{(0)}$ are matrices whose entries are given by the coefficients of the method, whose first characteristic polynomial is given as

$$\rho(\lambda) = \left|\lambda A^{(1)} - A^{0}\right|$$

(3.92)

Definition (Zero-stability): The block method (3.1) is said to be zero stable the roots of the first characteristic polynomial $\rho(\lambda)$ satisfies $\left|\lambda_j\right| \le 1,\ j = 1,2,3,...$ and for those roots with $\left|\lambda_j\right| = 1$, the multiplicity must not exceed 1.

3.5.1 Zero stability of (OSDBM2)

Equations (3.17) to (3.19) are written in the form (3.91) and

$$A^{(0)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad A^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B^{(0)} = \begin{pmatrix} 0 & 0 & \frac{20135}{193536} \\ 0 & 0 & \frac{1125}{7168} \\ 0 & 0 & \frac{61}{378} \end{pmatrix} \quad B^{(1)} = \begin{pmatrix} \frac{3413}{24192} & \frac{11}{24192} & \frac{857}{193536} \\ \frac{303}{896} & \frac{177}{896} & \frac{411}{7168} \\ \frac{64}{189} & \frac{64}{189} & \frac{61}{378} \end{pmatrix}$$

$$C^{(0)} = \begin{pmatrix} 0 & 0 & \frac{233}{71680} \\ 0 & 0 & \frac{489}{71680} \\ 0 & 0 & \frac{1}{140} \end{pmatrix} \quad C^{(1)} = \begin{pmatrix} -\frac{1601}{161280} & -\frac{289}{161280} & -\frac{23}{71680} \\ -\frac{423}{17920} & -\frac{633}{17920} & -\frac{279}{71680} \\ -\frac{8}{315} & -\frac{8}{315} & -\frac{1}{140} \end{pmatrix}$$

Using (3.92), we have

$$\rho(\lambda) = \lambda^2(\lambda - 1) = 0$$
$$\lambda = \{0, 0, 1\}$$

(3.93)

Therefore, the method is zero stable since is satisfies $\left|\lambda_j\right| \leq 1$.

3.5.2 Zero stability of (OSDBM3)

$$A^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B^{(0)} = \begin{pmatrix} 0 & 0 & 0 & \frac{48236526812245}{791200932938304} \\ 0 & 0 & 0 & \frac{2229002917}{20486760000} \\ 0 & 0 & 0 & \frac{2618145660737}{20275557960000} \\ 0 & 0 & 0 & \frac{83566921}{640211250} \end{pmatrix}$$

$$B^{(1)} = \begin{pmatrix} \frac{5815453137955}{69329555914752} & \frac{575288125}{463713937344} & -\frac{29906050595}{69329555914752} & \frac{1109311642315}{791200932938304} \\ \frac{531827571502403}{2548880732160000} & \frac{269281}{1632960} & -\frac{2082565315751}{509776146432000} & \frac{89027711}{4097352000} \\ \frac{364210994519}{1776660480000} & \frac{761804585009}{2318569686720} & \frac{214416025769}{1776660480000} & \frac{1410449179487}{20275557960000} \\ \frac{1018388173679}{4978282680000} & \frac{269281}{816480} & \frac{1018388173679}{4978282680000} & \frac{83566921}{640211250} \end{pmatrix}$$

$$C^{(0)} = \begin{pmatrix} 0 & 0 & 0 & \frac{181522121275}{163696744745856} \\ 0 & 0 & 0 & \frac{2540591}{847728000} \\ 0 & 0 & 0 & \frac{94010208599}{24330669552000} \\ 0 & 0 & 0 & \frac{94010208599}{24330669552000} \end{pmatrix}$$

The first characteristic polynomial is given as

$$\rho(\lambda)=\lambda^3(\lambda-1)=0 \\ \lambda=\{0,0,0,1\} \tag{3.94}$$

Therefore, the method is zero stable since is satisfies $|\lambda_j| \leq 1$.

3.5.3 Zero stability of (OSDBM4)

Following similar approach in the above 2 cases, the first characteristic polynomial for the method is given in (3.95).

$$\rho(\lambda)=\lambda^4(\lambda-1)=0 \\ \lambda=\{0,0,0,0,1\} \tag{3.95}$$

Therefore, the method is zero stable since is satisfies $|\lambda_j| \leq 1$.

3.5.4 Zero stability of (OSDBM5)

The first characteristic polynomial is given as

$$\rho(\lambda)=\lambda^5(\lambda-1)=0 \\ \lambda=\{0,0,0,0,0,1\} \tag{3.96}$$

Therefore, the method is zero stable since is satisfies $|\lambda_j| \leq 1$.

3.5.5 Zero stability of (OSDBM6)

The first characteristic polynomial is given as

$$\rho(\lambda)=\lambda^6(\lambda-1)=0 \\ \lambda=\{0,0,0,0,0,0,1\} \tag{3.97}$$

Therefore, the method is zero stable since is satisfies $|\lambda_j| \leq 1$.

3.6 Convergence

The necessary and sufficient condition for linear multistep method to be convergent is for it to be consistent and zero stable (Lambert, 1973). Following this theorem, each of the block methods developed are convergent.

3.7 Region of Absolute Stability

Following Akinfenwa *et al*, (2014) the region of absolute stability is determined by obtaining the stability polynomial of the form:

$$\sigma(z) = \left(A^{(1)} - zB^{(1)} - z^2C^{(1)}\right)^{-1}\left(A^{(0)} + zB^{(0)} + z^2C^{(0)}\right) \quad (3.98)$$

where $z = \lambda h$

The matrix $\sigma(z)$ has eigenvalues $\{0,0,0,...,\lambda_k\}$, and the dominant eigenvalue $\lambda_k : \mathbb{C} \to \mathbb{C}$ is a rational function (called the stability function) with real coefficients given by

$$\lambda_k = \frac{P(z)}{P(-z)} \quad (3.99)$$

For the proposed methods, the stability functions $\lambda_k = \dfrac{P(z)}{P(-z)}$ are given in the Appendix A. It is clear from the stability functions that for $\mathrm{Re}(z) < 0, \quad |\lambda_k| \le 1$. Below are the region of absolute stability regions of the proposed methods.

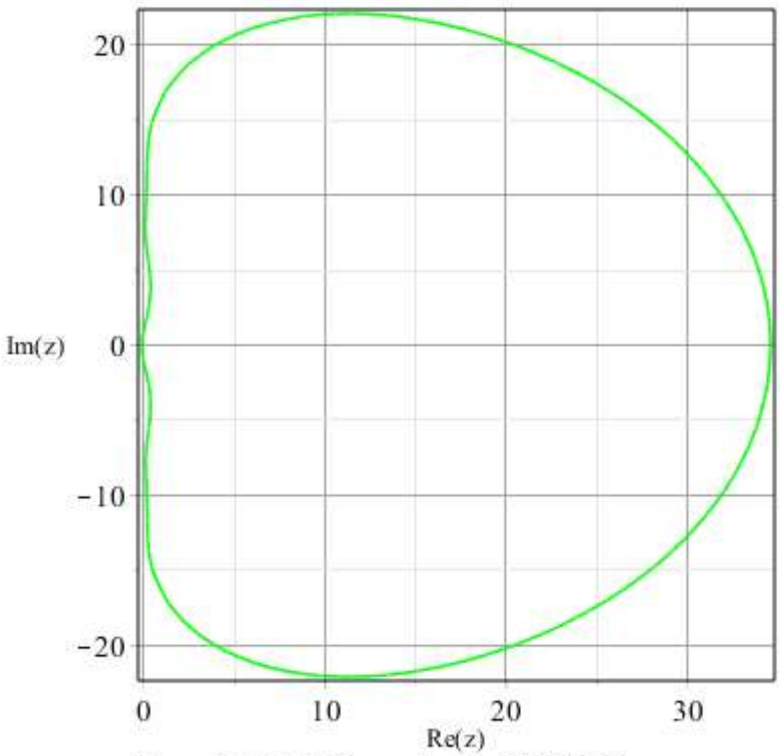

Figure 3.1: Stability region for OSDBM2

Figure 3.1 shows the stability region of OSDBM2 and found to be an A-stable method since its region of absolute stability contains the left half-plane $\mathbb{C}^{-}$.

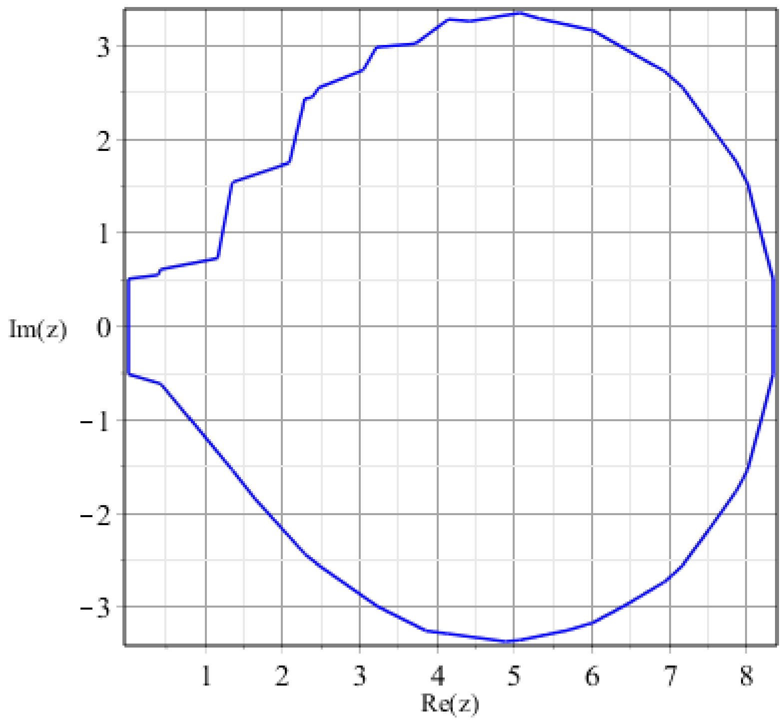

Figure 3.2: Stability Region for OSBDM3

Figure 3.2 shows the stability region of OSDBM3 and found to be an A-stable method since its region of absolute stability contains the left half-plane $\mathbb{C}^{-}$.

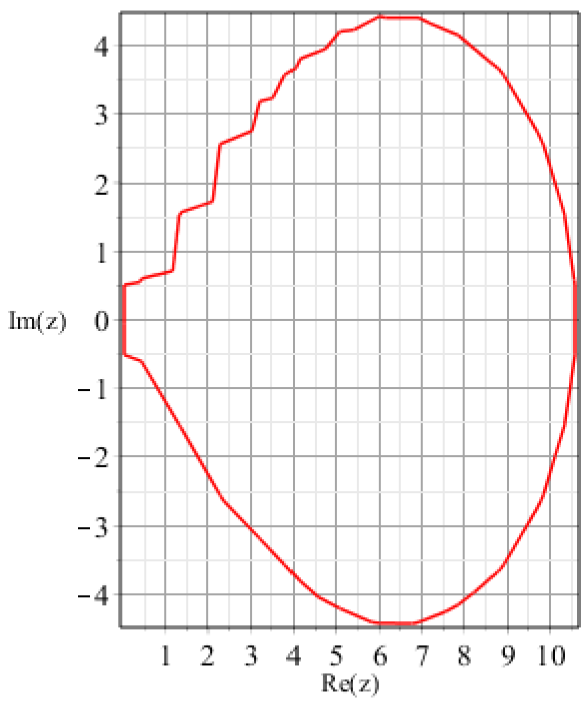

Figure 3.3: Stability Region for OSBDM4

Figure 3.3 shows the stability region of OSDBM4 and found to be an A-stable method since its region of absolute stability contains the left half-plane $\mathbb{C}^-$.

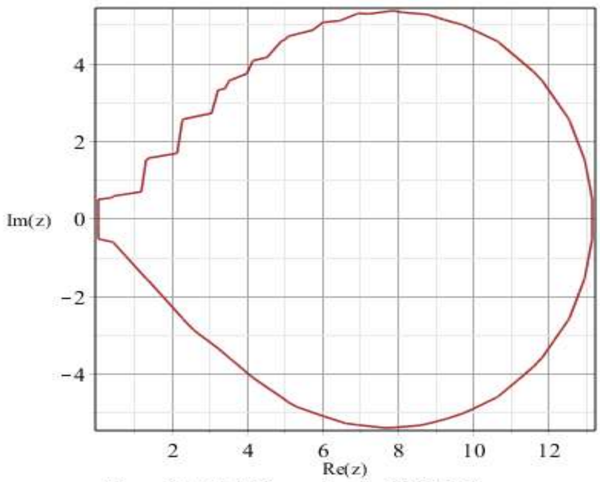

Figure 3.4: Stability region for OSDBM5

Figure 3.4 shows the stability region of OSDBM5 and found to be an A-stable method since its region of absolute stability contains the left half-plane $\mathbb{C}^-$.

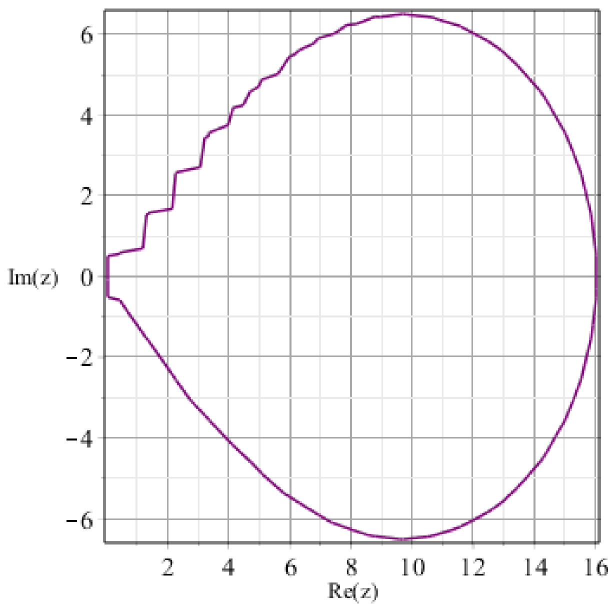

Figure 3.5: Stability region of OSDBM6

Figure 3.5 shows the stability region of OSDBM6 and found to be an A-stable method since its region of absolute stability contains the left half-plane $\mathbb{C}^-$.

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

4.1 Implementation of the Proposed Methods

This section discusses the implementation of the proposed one-step second derivative block intra-step methods derived in chapter three on some standard stiff systems of ordinary differential equations. The continuous representation generates a main discrete one-step second derivative block intra-step method (OSDBM) and additional methods which are combined and used as a block method to simultaneously produce approximations $\{y_{n+\eta}, y_{n+1}\}$ at a block points $\{x_{n+\eta}, x_{n+1}\}$, $h = x_{n+1} - x_n$, $n = 0, ..., \mathrm{N}-1$, on a partition $[a,b]$, where $a,b \in \mathbb{R}$ is the interval of integration, $\eta \in (0,1)$ are the intra-step points, h is the constant step-size, n is a grid index and $N > 0$ is the number of steps. We obtain initial conditions at $x_{n+1}, n = 0,1,..., N-1$, using the computed values y_{n+1} over sub-intervals $[x_0, x_1],...,[x_{N-1}, x_N]$. For instance when $n = 0, [y_\eta, y_1]$ are obtained simultaneously over the sub-interval $[x_0, x_1]$, as y_0 is known from the IVP, for $n = 1, [y_{\eta+1}, y_2]$ are also obtained simultaneously over the sub-interval $[x_1, x_2]$, as y_1 is now known from the previous block, and so on. Therefore, the sub-interval $[x_n, x_{n+1}]$ do not over-lap and the solutions obtained in this manner are more accurate than those obtained in the conventional way of predictor-corrector methods. The implementation code for problem 1 using OSDBM2 block is included for illustration in the appendix B.

4.1.2 Numerical Experiments

In this section, some numerical examples are given to illustrate the effectiveness of the proposed methods. We find the absolute errors of the approximate solution on the partition π_N as $|y(x)-y(x_n)|$ and also make comparisons with some existing methods in the literature. For the purpose of comparative analysis of performance of the new methods on various numerical examples, the following notations are used:

- CEA: The method of Abhulimen (2014), method of order p=5
- EOS: The method of Ehieie *et al.* (2013), (a 2-step method of order p=3).
- A&U: The method of Abhulimen and Ukpebor (2019), (a 4 step method of order p= 6)
- MOS: The method of Mohammed *et.al.* (2019), (a 2-step method of order p= 4)
- ABG: The method of Akintububo (2019), (a 4-step method of order p= 8).
- AAK: The method of Akinfenwa (2014), (a 2-step method of order p=8).
- NJ: The method of Ngwane and Jator (2012), (a one-step method of order p= 6).
- AJY: The method of Akinfenwa *et al.* (2013), (a 6-step method of order p=6).
- N.O.: The method of Nwachukwu and Okor (2018), (a 4-step method of order p=5).
- AAO: The method of Akinfenwa *et al.* (2017), (a 3-step method of order p= 8).
- HBSDBDF: The method Akinfenwa *et al.* (2020), (a 3-step method of order p=7)

Problem 1. We consider the following linear stiff system of IVP on the range $0 \le x \le 1$. Source: Ngwane and Jator (2012).

$$y' = -y + 95z, \qquad y(0) = 1$$
$$z' = -y - 97z, \qquad z(0) = 1$$

Exact solution:

$$y(x)=\frac{95}{47}e^{-2x}-\frac{48}{47}e^{-96x}$$
$$z(x)=\frac{48}{47}e^{-96x}-\frac{1}{47}e^{-2x}$$

Problem 2. We consider the following nonlinear IVP over the range $0\le x\le 1$. Source: Akinfenwa *et al.* (2013).

$$y'=-1002y+1000z^2, \qquad y(0)=1$$
$$z'=y-z(1+z), \qquad z(0)=1$$

Exact solution: $y(x)=e^{-2x}$, $z(x)=e^{-x}$

Problem 3. We consider the following system of three linear equations. Source: Akinfenwa *et al.* (2011).

$$y'=-21y+19z-20w, \qquad y(0)=1,$$
$$z'=19y-20z+20w, \qquad z(0)=0,$$
$$w'=40y-40z-20w, \qquad w(0)=-1$$
$$x\in[0,1]$$

Exact solution:
$$y(x)=\frac{1}{2}e^{-2x}+\frac{1}{2}e^{-40x}\sin(40x)+\frac{1}{2}e^{-40x}\cos(40x)$$
$$z(x)=\frac{1}{2}e^{-2x}-\frac{1}{2}e^{-40x}\sin(40x)-\frac{1}{2}e^{-40x}\cos(40x)$$
$$w(x)=e^{-40x}\sin(40x)-e^{-40x}\cos(40x)$$

Problem 4. We also consider the Van Der Pol Oscillatory Problem. Source: Mohammed *et al.* (2019).

$$y''-2\xi\left(1-y^2\right)y'+y=0,\ y(0)=0, y'(0)=0.5,\ x\in[0,10],\ \xi=0.025$$

The above second order differential equation is reduced to its corresponding first order system of equations:

$$y' = z, \qquad y(0) = 0$$
$$z' = -y + 2\xi\left(1 - y^2\right)z, \qquad z(0) = 0.5$$

The solution to this problem is validated with RK5 method in Maple (2015) and Mohammed *et al.* (2019).

Problem 5. We consider a singularly perturbed problem. Source: Nwanchukwu and Okor (2018).

$$y' = -\left(2 + 10^4\right)y + 10^4 z^2, \qquad y(0) = 1$$
$$z' = y - z - z^2, \qquad z(0) = 1$$

Analytical solution: $y(x) = e^{-2x}, \; z(x) = e^{-x}$

Problem 6. We also consider the nonlinear system of differential equation in the range $0 \le x \le 10$ Source: Akinfewan *et al*. (2017).

$$y' = \mu y + z^2, \qquad y(0) = -\frac{1}{(\mu + 2)}$$
$$z' = -z, \qquad z(0) = 1$$

Where $\mu = 10000$, The exact solution is $y(x) = -\frac{e^{-2x}}{(\mu + 2)}, \; z(x) = e^{-t}$

Problem 7. We also consider the following nearly sinusoidal problem. Source: (Akinfenwa *et. al.*, 2020)

$$y' = -21y + z + 2\sin x, \qquad y(0) = 2$$
$$z' = 998y - 999z + 999\cos x - 999\sin x, \qquad z(0) = 3$$

Exact solution: $y(x) = 2e^{-x} + \sin x, \; z(x) = 2e^{-x} + \cos x$

4.2 Results

Table 4.1 Comparing the Exact Solution with the Proposed Methods for Problem 1 at the Point $x = 1$ for h = 0.03125

	y(1) z(1)
Exact	0.27355004058464267510489259622097911
	-0.002879474111417291316893606276010306
OSDBM2	0.27355004058464267535957869372847955
	-0.002879474111417291319574512565562942
OSDBM3	0.27355004058464267510489328607856833
	-0.002879474111417291316893613537669140
OSDBM4	0.27355004058464267510489259622226162
	-0.002879474111417291316893606276023806
OSDBM5	0.27355004058464267510489259622097866
	-0.002879474111417291316893606276010302
OSDBM6	0.27355004058464267510489259622097910
	-0.002879474111417291316893606276010307

Problem 1 is solved on the interval $0 \le x \le 1$ for h=0.03125 using the proposed methods and the solutions at point $x = 1$ is compared with the exact solution as shown in table 4.1 which shows that the numerical methods agree with the exact solution.

Table 4.2a Comparison of Absolute Error of the Methods for Problem 1 h=0.03125

x	OSDBM2 y(*x*)	OSDBM3 y(*x*)	OSDBM4 y(*x*)	OSDBM5 y(*x*)	OSDBM6 y(*x*)
0.0625	2.68*10^(-07)	1.66*10^(-09)	7.01*10^(-12)	2.14*10^(-14)	4.98*10^(-17)

0.1250	1.33*10^(-09)	8.23*10^(-12)	3.74*10^(-14)	1.06*10^(-16)	2.47*10^(-19)
0.1875	4.94*10^(-12)	3.06*10^(-14)	1.29*10^(-16)	3.95*10^(-19)	9.19*10^(-22)
0.2500	1.63*10^(-14)	1.01*10^(-16)	4.27*10^(-19)	1.31*10^(-21)	3.04*10^(-24)
0.3125	5.06*10^(-17)	3.13*10^(-19)	1.32*10^(-21)	4.04*10^(-24)	9.41*10^(-27)
0.3750	1.47*10^(-19)	9.31*10^(-22)	3.93*10^(-24)	1.20*10^(-26)	2.80*10^(-29)
0.4375	3.18*10^(-21)	1.77*10^(-24)	1.14*10^(-26)	3.48*10^(-29)	8.09*10^(-32)
0.5000	3.64*10^(-21)	9.30*10^(-25)	3.05*10^(-29)	9.85*10^(-32)	2.30*10^(-34)
0.5625	3.62*10^(-21)	9.31*10^(-25)	1.64*10^(-30)	2.81*10^(-34)	1.80*10^(-36)
0.6250	3.55*10^(-21)	9.13*10^(-25)	1.70*10^(-30)	7.20*10^(-36)	8.00*10^(-37)
0.6875	3.44*10^(-21)	8.86*10^(-25)	1.65*10^(-30)	5.90*10^(-36)	7.00*10^(-37)
0.7500	3.12*10^(-21)	8.53*10^(-25)	1.59*10^(-30)	5.80*10^(-36)	5.00*10^(-37)
0.8125	3.17*10^(-21)	8.16*10^(-25)	1.52*10^(-30)	5.30*10^(-36)	3.00*10^(-37)
0.8750	3.01*10^(-21)	7.75*10^(-25)	1.44*10^(-30)	5.10*10^(-36)	4.00*10^(-37)
0.9375	2.85*10^(-21)	7.32*10^(-25)	1.36*10^(-30)	4.70*10^(-36)	1.00*10^(-37)
1.0000	2.68*10^(-21)	6.90*10^(-25)	1.28*10^(-30)	4.40*10^(-36)	3.00*10^(-37)

Table 4.2b Comparison of Absolute Errors of the Methods for Problem 1 h=0.03125

x	OSDBM2 z(*x*)	OSDBM3 z(*x*)	OSDBM4 z(*x*)	OSDBM5 z(*x*)	OSDBM6 z(*x*)
0.0625	1.64*10^(-4)	3.76*10^(-06)	7.01*10^(-12)	7.30*10^(-10)	6.65*10^(-12)
0.1250	8.42*10^(-7)	1.87*10^(-08)	3.47*10^(-14)	3.62*10^(-12)	3.30*10^(-14)
0.1875	3.23*10^(-9)	6.94*10^(-11)	1.29*10^(-16)	1.35*10^(-14)	1.23*10^(-16)
0.2500	1.10*10^(-11)	2.30*10^(-13)	4.27*10^(-19)	4.45*10^(-17)	4.06*10^(-19)
0.3125	3.53*10^(-14)	7.12*10^(-16)	1.32*10^(-21)	1.38*10^(-19)	1.26*10^(-21)
0.3750	1.07*10^(-16)	2.12*10^(-18)	3.93*10^(-24)	4.10*10^(-22)	3.74*10^(-24)
0.4375	6.00*10^(-19)	6.13*10^(-21)	1.14*10^(-26)	1.19*10^(-24)	1.08*10^(-26)
0.5000	9.26*10^(-19)	7.28*10^(-24)	3.22*10^(-29)	3.36*10^(-27)	3.06*10^(-29)
0.5625	9.08*10^(-19)	9.99*10^(-24)	7.17*10^(-32)	9.33*10^(-30)	8.54*10^(-32)
0.6250	8.82*10^(-19)	9.84*10^(-24)	1.76*10^(-32)	2.54*10^(-32)	2.32*10^(-34)
0.6875	8.49*10^(-19)	9.55*10^(-24)	1.73*10^(-32)	3.15*10^(-34)	3.10*10^(-36)
0.7500	8.12*10^(-19)	9.20*10^(-24)	1.67*10^(-32)	3.71*10^(-34)	4.32*10^(-35)
0.8125	7.71*10^(-19)	8.79*10^(-24)	1.60*10^(-32)	3.55*10^(-34)	3.10*10^(-36)
0.8750	7.33*10^(-19)	8.36*10^(-24)	1.52*10^(-32)	3.38*10^(-34)	2.20*10^(-36)
0.9375	7.29*10^(-19)	7.90*10^(-24)	1.43*10^(-32)	3.19*10^(-34)	2.80*10^(-36)
1.0000	6.87*10^(-19)	7.44*10^(-24)	1.35*10^(-32)	3.01*10^(-34)	2.80*10^(-36)

The absolute errors $\left(\left|y(x_n)-y_n\right|\right)$ and $\left(\left|z(x_n)-z_n\right|\right)$ of the numerical methods on problem 1 for h=0.03125 are presented in tables 4.2a and 4.2b respectively, which shows that the methods have relatively small errors. It is evident from the tables that OSDBM6 of the highest order of accuracy 16, has the least error which shows the effect of the intra-step points on the performance of the proposed methods. It is observed that as the number of selected intra-step points increases, the performance of the methods gets better. Each of the methods approaches the exact solution faster as the number of iteration increases within the interval of integration.

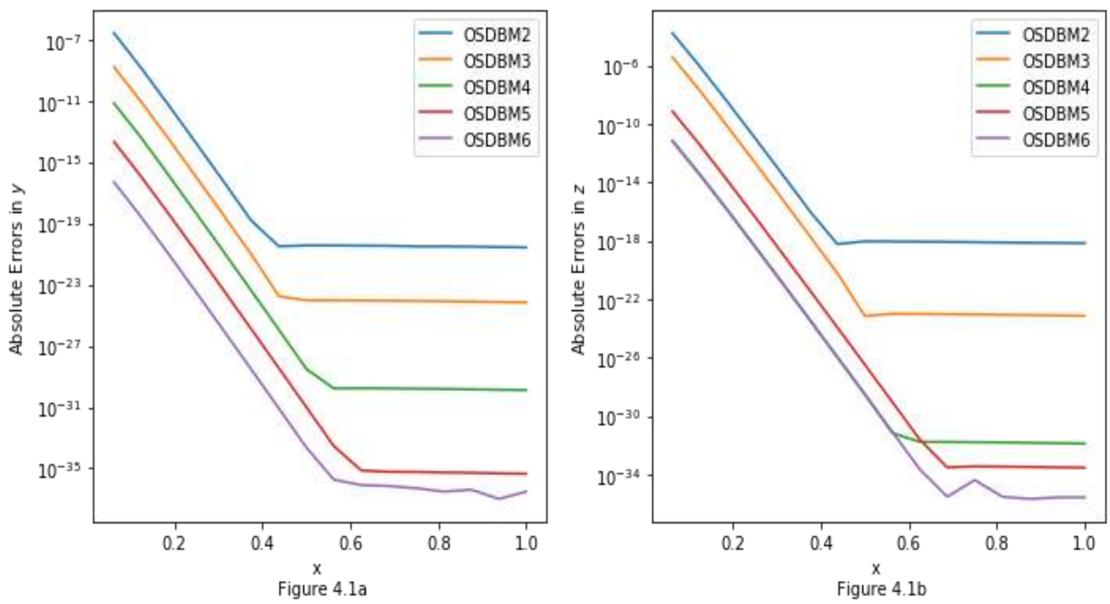

Figure 4.1. Absolute Errors of the Proposed Methods for Problem 1 (h=0.03125)

Figures 4.1a and 4.1b show the absolute errors $\left(\left|y\left(x_n\right)-y_n\right|\right)$ and $\left(\left|z\left(x_n\right)-z_n\right|\right)$ of the proposed methods in problem 1 with h=0.03125. The graphical representation of tabulated results in tables 4.2a and 4.2b further elaborates the effectiveness of the methods. The method OSDBM6 of order 16 has the least error followed by the OSDBM5 of order 14, OSDBM4 of order 12, OSDBM3 of order 10 and OSDBM2 of order 8 with relatively small errors.

Table 4.3 Comparing the Exact Solution of with the Proposed Methods for Problem 1 at the Point $x=1$ for h = 0.0625

	y(1) z(1)
Exact	0.2735500405846426751048925962209
	-0.00287947411141729131689360627601030 6
OSDBM2	0.2735500405846427403297629990886
	-0.002879474111417292003471189464090788
OSDBM3	0.2735500405846426751055992836930
	-0.002879474111417291316901045091505350
OSDBM4	0.2735500405846426751048926014774
	-0.002879474111417291316893606331341707
OSDBM5	0.2735500405846426751048925962210
	-0.002879474111417291316893606276010607
OSDBM6	0.2735500405846426751048925962209
	-0.002879474111417291316893606276010309

Problem 1 is also solved on the interval $0 \le x \le 1$ for h=0.0625 using the proposed methods and the solutions at point $x=1$ are compared with the exact solution as shown in table 4.3 which shows that the numerical methods agree with the exact solution.

Table 4.4a Comparison of Absolute Errors for y(x) of the Methods for Problem 1

h=0.0625

x	OSDBM2 y(*x*)	OSDBM3 y(*x*)	OSDBM4 y(*x*)	OSDBM5 y(*x*)	OSDBM6 y(*x*)
0.0625	1.64*10^(-4)	3.76*10^(-6)	6.12*10^(-08)	7.30*10^(-10)	6.65*10^(-12)
0.1250	8.42*10^(-7)	1.86*10^(-8)	3.03*10^(-10)	3.62*10^(-12)	3.30*10^(-14)
0.1875	3.23*10^(-9)	6.94*10^(-11)	1.13*10^(-12)	1.35*10^(-14)	1.23*10^(-16)
0.2500	1.10*10^(-11)	2.30*10^(-13)	3.73*10^(-15)	4.45*10^(-17)	4.06*10^(-19)
0.3125	3.53*10^(-14)	7.12*10^(-16)	1.16*10^(-17)	1.38*10^(-19)	1.26*10^(-21)
0.3750	2.34*10^(-17)	2.12*10^(-18)	3.44*10^(-20)	4.10*10^(-22)	3.74*10^(-24)
0.4375	8.76*10^(-17)	5.18*10^(-21)	9.94*10^(-23)	1.19*10^(-24)	1.08*10^(-26)
0.5000	8.86*10^(-17)	9.43*10^(-22)	2.74*10^(-25)	3.36*10^(-27)	3.06*10^(-29)
0.5625	8.80*10^(-17)	9.54*10^(-22)	6.31*10^(-27)	9.33*10^(-30)	8.51*10^(-32)
0.6250	8.63*10^(-17)	9.35*10^(-22)	6.95*10^(-27)	1.20*10^(-32)	7.00*10^(-35)
0.6875	8.38*10^(-17)	9.08*10^(-22)	6.75*10^(-27)	3.66*10^(-32)	2.90*10^(-34)
0.7500	8.07*10^(-17)	8.74*10^(-22)	6.50*10^(-27)	3.53*10^(-32)	2.70*10^(-34)
0.8125	7.71*10^(-17)	8.35*10^(-22)	6.21*10^(-27)	3.38*10^(-32)	2.30*10^(-34)
0.8750	7.33*10^(-17)	7.94*10^(-22)	5.91*10^(-27)	3.21*10^(-32)	2.30*10^(-34)
0.9375	6.93*10^(-17)	7.51*10^(-22)	5.58*10^(-27)	3.04*10^(-32)	2.60*10^(-34)
1.0000	6.52*10^(-17)	7.07*10^(-22)	5.26*10^(-27)	2.86*10^(-32)	2.5*10^(-34)

Table 4.4b Comparison of Absolute Errors for z(x) of the Methods for Problem 1 h=0.0625

x	OSDBM2 z(*x*)	OSDBM3 z(*x*)	OSDBM4 z(*x*)	OSDBM5 z(*x*)	OSDBM6 z(*x*)
0.0625	1.64*10^(-4)	3.76*10^(-06)	6.12*10^(-08)	7.30*10^(-10)	6.65*10^(-12)
0.1250	8.42*10^(-7)	1.87*10^(-08)	3.03*10^(-10)	3.62*10^(-12)	3.30*10^(-14)
0.1875	3.23*10^(-9)	6.94*10^(-11)	1.13*10^(-12)	1.35*10^(-14)	1.23*10^(-16)
0.2500	1.10*10^(-11)	2.30*10^(-13)	3.73*10^(-15)	4.45*10^(-17)	4.06*10^(-19)
0.3125	3.53*10^(-14)	7.12*10^(-16)	1.16*10^(-17)	1.38*10^(-19)	1.26*10^(-21)
0.3750	1.07*10^(-16)	2.12*10^(-18)	3.44*10^(-20)	4.10*10^(-22)	3.74*10^(-24)
0.4375	6.00*10^(-19)	6.13*10^(-21)	9.94*10^(-23)	1.19*10^(-24)	1.08*10^(-26)
0.5000	9.26*10^(-19)	7.28*10^(-24)	2.81*10^(-25)	3.36*10^(-27)	3.06*10^(-29)
0.5625	9.08*10^(-19)	9.99*10^(-24)	7.10*10^(-28)	9.33*10^(-30)	8.54*10^(-32)
0.6250	8.82*10^(-19)	9.84*10^(-24)	7.10*10^(-29)	2.54*10^(-32)	2.32*10^(-34)
0.6875	8.49*10^(-19)	9.55*10^(-24)	7.11*10^(-29)	3.15*10^(-34)	3.10*10^(-36)
0.7500	8.12*10^(-19)	9.20*10^(-24)	6.84*10^(-29)	3.71*10^(-34)	4.32*10^(-35)
0.8125	7.71*10^(-19)	8.79*10^(-24)	6.54*10^(-29)	3.55*10^(-34)	3.10*10^(-36)
0.8750	7.33*10^(-19)	8.36*10^(-24)	6.22*10^(-29)	3.38*10^(-34)	2.20*10^(-36)
0.9375	7.29*10^(-19)	7.90*10^(-24)	5.88*10^(-29)	3.19*10^(-34)	2.80*10^(-36)
1.0000	6.87*10^(-19)	7.44*10^(-24)	5.53*10^(-29)	3.01*10^(-34)	2.80*10^(-36)

The absolute errors $\left(\left|y\left(x_n\right)-y_n\right|\right)$ and $\left(\left|z\left(x_n\right)-z_n\right|\right)$ of the numerical methods on problem 1 for h=0.0625 are presented in tables 4.4a and 4.4b respectively, which shows that the methods have relatively small errors. It is evident from the tables that OSDBM6 of the highest order of accuracy 16, has the least error which shows the effect of the intra-step points on the performance of the proposed methods. It is observed that as the number of selected intra-step points increases, the performance of the methods gets better. Each of the methods approaches the exact solution faster as the number of iteration increases within the interval of integration.

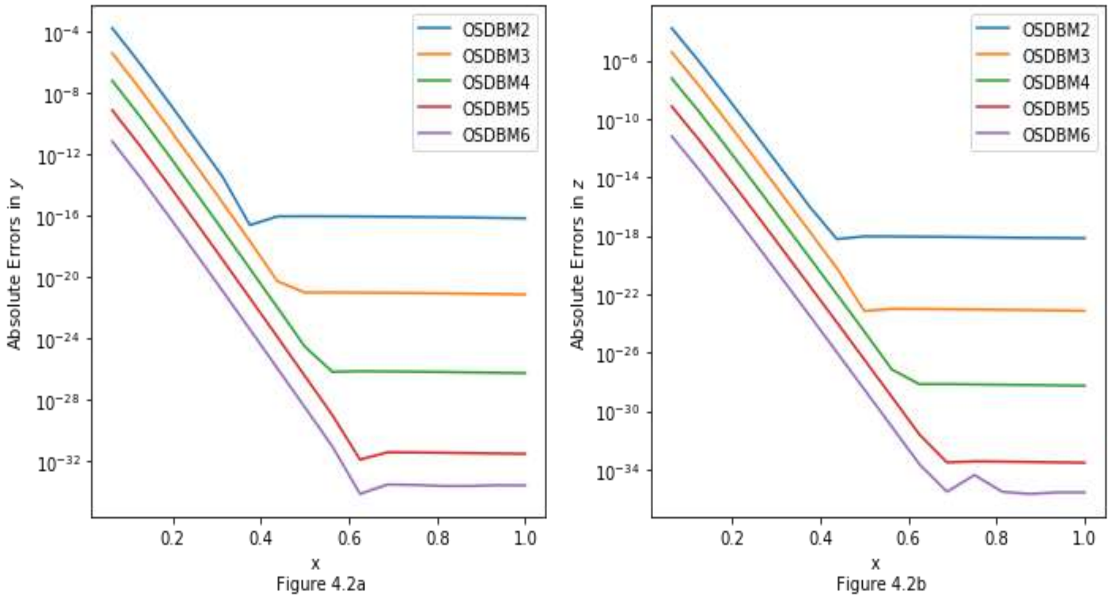

Figure 4.2 Absolute Error of the Methods for Problem 1 (h=0.0625)

Figures 4.2a and 4.2b show the absolute errors $\left(\left|y(x_n)-y_n\right|\right)$ and $\left(\left|z(x_n)-z_n\right|\right)$ of the proposed methods in problem 1 with h=0.03125. The graphical representation of tabulated results in tables 4.4a and 4.4b further elaborates the effectiveness of the methods. The method OSDBM6 of order 16 has the least error followed by the OSDBM5 of order 14, OSDBM4 of order 12, OSDBM3 of order 10 and OSDBM2 of order 8 with relatively small errors.

Table 4.5 Comparative Analysis of Error in Problem 1 at $x=1$.

h	Methods	$y(1)\|error\|$	$z(1)\|error\|$
0.03125	CEA	1.91×10^{-9}	2.01×10^{-9}
	EOS	3.40×10^{-9}	3.60×10^{-9}
	A&U	3.0×10^{-8}	1.4×10^{-9}
	ABG	7.8×10^{-13}	1.1×10^{-16}
	NJ	5.0×10^{-14}	5.0×10^{-14}
	AAK	5.0×10^{-19}	5.0×10^{-20}
	OSDBM2	**6.87*10^(-19)**	**2.68*10^(-21)**
0.0625	CEA	1.91×10^{-9}	2.01×10^{-9}
	EOS	3.40×10^{-9}	3.40×10^{-9}
	A&U	3.0×10^{-8}	1.4×10^{-9}
	ABG	9.25×10^{-11}	9.56×10^{-11}
	NJ	3.0×10^{-12}	3.0×10^{-12}
	AAK	1.0×10^{-16}	1.0×10^{-17}
	OSDBM2	**6.52*10^(-17)**	**6.87*10^(-19)**

Table 4.5 above presents the comparative analysis of absolute errors in problem 1 at different step sizes *"h"* at point $x=1$. Some methods from available literatures are compared with OSDBM2 which is the least effective of our proposed methods. However, the OSDBM2 outperform all the existing methods considered in the literature. This further proves the efficiency and effectiveness of the proposed methods.

Table 4.6 Comparing the Exact Solution of with the Proposed Methods for Problem 2 at the Point $x = 1$ for h = 0.02

	y(1) z(1)
Exact	0.135335283236612691893999494972484400 0.367879441171442321595523770161460867
OSDBM2	0.135335283236612691894021487301455710 0.367879441171442321595548649284463680
OSDBM3	0.135335283236612691893999494983981640 0.367879441171442321595523770173418140
OSDBM4	0.135335283236612691893999494972484409 0.367879441171442321595523770161460870
OSDBM5	0.135335283236612691893999494972484410 0.367879441171442321595523770161460860
OSDBM6	0.135335283236612691893999494972484380 0.367879441171442321595523770161460850

Problem 2 is solved on the interval $0 \le x \le 1$ for h=0.02 using the proposed methods and the solutions at point $x = 1$ are compared with the exact solution as shown in table 4.6 which shows that the numerical methods agree with the exact solution.

Table 4.7a Comparison of Absolute Errors for y(x) of the Methods for Problem 2 h=0.02

x	OSDBM2 y(*x*)	OSDBM3 y(*x*)	OSDBM4 y(*x*)	OSDBM5 y(*x*)	OSDBM6 y(*x*)
0.1	3.47*10^(-23)	2.31*10^(-29)	6.58*10^(-35)	6.63*10^(-42)	2.60*10^(-48)
0.2	3.83*10^(-23)	2.42*10^(-29)	1.38*10^(-35)	6.88*10^(-42)	2.75*10^(-48)
0.3	3.92*10^(-23)	2.38*10^(-29)	1.35*10^(-35)	6.71*10^(-42)	2.67*10^(-48)
0.4	3.83*10^(-23)	2.26*10^(-29)	1.27*10^(-35)	6.30*10^(-42)	2.50*10^(-48)
0.5	3.64*10^(-23)	2.09*10^(-29)	1.16*10^(-35)	5.75*10^(-42)	2.30*10^(-48)
0.6	3.38*10^(-23)	1.90*10^(-29)	1.05*10^(-35)	5.14*10^(-42)	2.08*10^(-48)
0.7	3.08*10^(-23)	1.70*10^(-29)	9.27*10^(-36)	4.54*10^(-42)	1.83*10^(-48)
0.8	2.78*10^(-23)	1.51*10^(-29)	8.12*10^(-36)	3.96*10^(-42)	1.59*10^(-48)
0.9	2.48*10^(-23)	1.32*10^(-29)	7.05*10^(-36)	3.43*10^(-42)	1.38*10^(-48)
1.0	2.20*10^(-23)	1.15*10^(-29)	6.08*10^(-36)	2.94*10^(-42)	1.17*10^(-48)

Table 4.7b Comparison of Absolute Errors for z(x) of the Methods for Problem 2 h=0.02

x	OSDBM2 z(*x*)	OSDBM3 z(*x*)	OSDBM4 z(*x*)	OSDBM5 z(*x*)	OSDBM6 z(*x*)
0.1	6.85*10^(-24)	3.76*10^(-30)	2.14*10^(-36)	1.07*10^(-42)	4.30*10^(-49)
0.2	1.22*10^(-23)	6.61*10^(-30)	3.71*10^(-36)	1.85*10^(-42)	7.60*10^(-49)
0.3	1.64*10^(-23)	8.71*10^(-30)	4.83*10^(-36)	2.40*10^(-42)	9.70*10^(-49)
0.4	1.95*10^(-23)	1.02*10^(-29)	5.60*10^(-36)	2.77*10^(-42)	1.11*10^(-48)
0.5	2.17*10^(-23)	1.12*10^(-29)	6.08*10^(-36)	2.99*10^(-42)	1.22*10^(-48)
0.6	2.33*10^(-23)	1.18*10^(-29)	6.35*10^(-36)	3.11*10^(-42)	1.28*10^(-48)
0.7	2.43*10^(-23)	1.22*10^(-29)	6.46*10^(-36)	3.14*10^(-42)	1.27*10^(-48)
0.8	2.48*10^(-23)	1.23*10^(-29)	6.43*10^(-36)	3.12*10^(-42)	1.27*10^(-48)
0.9	2.50*10^(-23)	1.22*10^(-29)	6.32*10^(-36)	3.05*10^(-42)	1.23*10^(-48)
1.0	2.49*10^(-23)	1.20*10^(-29)	6.13*10^(-36)	2.95*10^(-42)	1.19*10^(-48)

The absolute errors $\left(\left|y(x_n)-y_n\right|\right)$ and $\left(\left|z(x_n)-z_n\right|\right)$ of the numerical methods on problem 2 for h=0.02 are presented in tables 4.7a and 4.7b respectively, which shows that the methods have relatively small errors. It is evident from the tables that OSDBM6 of the highest order of accuracy 16, has the least error which shows the effect of the intra-step points on the performance of the proposed methods. It is observed that as the number of selected intra-step points increases, the performance of the methods gets better. Each of the methods approaches the exact solution faster as the number of iteration increases within the interval of integration.

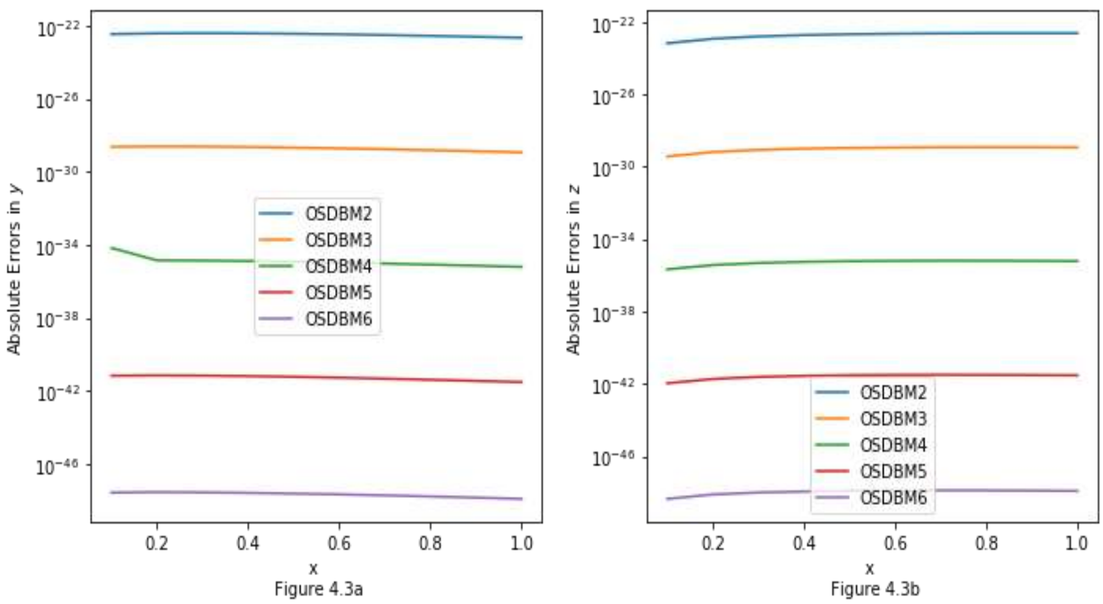

Figure 4.3 Absolute Error of the Proposed Methods for Problem 2

Figures 4.3a and 4.3b show the absolute errors $\left(\left|y\left(x_n\right)-y_n\right|\right)$ and $\left(\left|z\left(x_n\right)-z_n\right|\right)$ respectively, of the proposed methods in problem 2 with h=0.02. The graphical representation of the tabulated results in tables 4.7a and 4.7b further elaborates the effectiveness of the methods. The method OSDBM6 of order 16 has the least error followed by the OSDBM5 of order 14, OSDBM4 of order 12, OSDBM3 of order 10 and OSDBM2 of order 8 with relatively small errors.

Table 4.8 Comparative Analysis of Result of Problem 2 at $x=1 and\ x=10.$

x	h	N	Methods	$y\|error\|$	$z\|error\|$
1	0.02	50	AYJ k=6	9.11×10^{-13}	1.25×10^{-12}
	0.1	10	NJ	5.68×10^{-13}	6.57×10^{-13}
	0.008	125	N.O.	1.80×10^{-15}	6.11×10^{-16}
	0.1	**10**	**OSDBM2**	**$9.52*10^{-18}$**	**$9.72*10^{-18}$**
10	0.02	500	AJY k=6	2.20×10^{-20}	1.25×10^{-12}
	0.01	1000	NJ	7.10×10^{-22}	7.82×10^{-18}
	0.02	500	ABG	2.12×10^{-21}	7.98×10^{-17}
	0.2	**50**	**OSDBM2**	**$4.52*10^{-22}$**	**$2.44*10^{-18}$**

From the table 4.8, the numerical results reveal that the method OSDBM2 of order 8 with higher step size h =0.1 and 0.2 is superior in terms of accuracy when compared with other existing methods with lower step sizes 0.008, 0.02, and 0.01.

Table 4.9 Comparing the Exact Solution of with the Proposed Methods for Problem 3 at the Point $x=3$ for h = 0.02

	y(3)
Exact	0.00123937608833317921152258371540833394575323980
OSDBM2	0.00123937608833317921162001525122699933747822035
OSDBM3	0.00123937608833317921152258382350535077827583291
OSDBM4	0.00123937608833317921152258371540841628226798188
OSDBM5	0.00123937608833317921152258371540833394579833579
OSDBM6	0.00123937608833317921152258371540833394575323892

Problem 3 is solved on the interval $0 \le x \le 3$ for h=0.02 using the proposed methods and the solutions at point $x = 3$ are compared with the exact solution as shown in table 4.9 which shows that the numerical methods agree with the exact solution.

Table 4.10a Comparison of Absolute Errors for y(x) of the Methods for Problem 3 h=0.02

x	OSDBM2	OSDBM3	OSDBM4	OSDBM5	OSDBM6
	y(*x*)	y(*x*)	y(*x*)	y(*x*)	y(*x*)
0.2	1.43*10^(-11)	1.32*10^(-15)	7.75*10^(-18)	3.65*10^(-22)	1.13*10^(-24)
0.4	2.39*10^(-15)	8.31*10^(-18)	1.30*10^(-21)	2.22*10^(-24)	1.92*10^(-28)
0.6	4.48*10^(-18)	1.66*10^(-21)	2.42*10^(-24)	4.48*10^(-28)	3.54*10^(-31)
0.8	3.24*10^(-21)	1.65*10^(-24)	6.09*10^(-28)	4.41*10^(-31)	8.93*10^(-35)
1.0	1.77*10^(-21)	2.48*10^(-27)	3.79*10^(-31)	1.38*10^(-34)	5.55*10^(-38)
1.2	1.43*10^(-21)	1.58*10^(-27)	1.35*10^(-33)	5.76*10^(-38)	2.16*10^(-41)
1.4	1.12*10^(-21)	1.24*10^(-27)	9.43*10^(-34)	5.45*10^(-40)	6.07*10^(-45)
1.6	8.55*10^(-22)	9.48*10^(-28)	7.22*10^(-34)	3.96*10^(-40)	1.69*10^(-46)
1.8	6.44*10^(-22)	7.15*10^(-28)	5.45*10^(-34)	2.98*10^(-40)	1.24*10^(-46)
2.0	4.80*10^(-22)	5.32*10^(-28)	4.06*10^(-34)	2.22*10^(-40)	9.26*10^(-47)
2.2	3.54*10^(-22)	3.93*10^(-28)	2.99*10^(-34)	1.64*10^(-40)	6.83*10^(-47)
2.4	2.59*10^(-22)	2.87*10^(-28)	2.19*10^(-34)	1.20*10^(-40)	4.99*10^(-47)
2.6	1.88*10^(-22)	2.08*10^(-28)	1.59*10^(-34)	8.70*10^(-41)	3.63*10^(-47)
2.8	1.36*10^(-22)	1.51*10^(-28)	1.15*10^(-34)	6.28*10^(-41)	2.62*10^(-47)
3.0	9.74*10^(-23)	1.08*10^(-28)	8.23*10^(-35)	4.51*10^(-41)	1.88*10^(-47)

Table 4.10b Comparison of Absolute Errors for z(x) of the Methods for Problem 3 h=0.02

x	OSDBM2 z(*x*)	OSDBM3 z(*x*)	OSDBM4 z(*x*)	OSDBM5 z(*x*)	OSDBM6 z(*x*)
0.2	3.32*10^(-04)	3.32*10^(-04)	3.32*10^(-04)	3.32*10^(-04)	3.32*10^(-04)
0.4	3.24*10^(-08)	3.24*10^(-08)	3.24*10^(-08)	3.24*10^(-08)	3.24*10^(-08)
0.6	3.42*10^(-11)	3.42*10^(-11)	3.42*10^(-11)	3.42*10^(-11)	3.42*10^(-11)
0.8	6.98*10^(-15)	6.98*10^(-15)	6.98*10^(-15)	6.98*10^(-15)	6.98*10^(-15)
1.0	3.16*10^(-18)	3.17*10^(-18)	3.17*10^(-18)	3.17*10^(-18)	3.17*10^(-18)
1.2	2.52*10^(-21)	1.09*10^(-21)	1.09*10^(-21)	1.09*10^(-21)	1.09*10^(-21)
1.4	1.12*10^(-21)	2.51*10^(-25)	2.49*10^(-25)	2.49*10^(-25)	2.49*10^(-25)
1.6	8.54*10^(-22)	8.01*10^(-28)	1.47*10^(-28)	1.47*10^(-28)	1.47*10^(-28)
1.8	6.44*10^(-22)	7.15*10^(-28)	1.31*10^(-32)	1.31*10^(-32)	1.31*10^(-32)
2.0	4.80*10^(-22)	5.32*10^(-28)	4.24*10^(-34)	1.79*10^(-35)	1.79*10^(-35)
2.2	3.54*10^(-22)	3.93*10^(-28)	2.99*10^(-34)	5.05*10^(-41)	2.14*10^(-40)
2.4	2.59*10^(-22)	2.87*10^(-28)	2.19*10^(-34)	1.18*10^(-40)	2.00*10^(-42)
2.6	1.88*10^(-22)	2.08*10^(-28)	1.59*10^(-34)	8.70*10^(-41)	2.55*10^(-46)
2.8	1.36*10^(-22)	1.51*10^(-28)	1.15*10^(-34)	6.28*10^(-41)	2.64*10^(-46)
3.0	9.74*10^(-22)	1.08*10^(-28)	8.23*10^(-35)	4.51*10^(-41)	1.88*10^(-47)

Table 4.10c Comparison of Absolute Errors for w(x) of the Methods for Problem 3 h=0.02

x	OSDBM2 w(*x*)	OSDBM3 w(*x*)	OSDBM4 w(*x*)	OSDBM5 w(*x*)	OSDBM6 w(*x*)
0.2	2.93*10^(-12)	2.54*10^(-14)	1.65*10^(-18)	6.79*10^(-21)	2.46*10^(-25)
0.4	1.87*10^(-14)	4.24*10^(-18)	1.01*10^(-20)	1.15*10^(-24)	1.48*10^(-27)
0.6	3.75*10^(-18)	7.97*10^(-21)	2.04*10^(-24)	2.12*10^(-27)	3.00*10^(-31)
0.8	3.73*10^(-21)	1.99*10^(-24)	2.01*10^(-27)	5.35*10^(-31)	2.94*10^(-34)
1.0	1.16*10^(-24)	1.25*10^(-27)	6.28*10^(-31)	3.33*10^(-34)	9.21*10^(-38)
1.2	4.93*10^(-28)	4.83*10^(-31)	2.66*10^(-34)	1.29*10^(-37)	3.88*10^(-41)
1.4	2.39*10^(-31)	1.42*10^(-34)	1.29*10^(-37)	3.78*10^(-41)	1.89*10^(-44)
1.6	4.74*10^(-35)	8.83*10^(-38)	2.55*10^(-41)	2.36*10^(-44)	3.71*10^(-48)
1.8	3.97*10^(-38)	1.09*10^(-41)	2.15*10^(-44)	2.87*10^(-48)	3.15*10^(-51)
2.0	2.36*10^(-42)	1.36*10^(-44)	1.26*10^(-48)	3.63*10^(-51)	1.81*10^(-55)
2.2	5.76*10^(-45)	2.28*10^(-47)	3.09*10^(-51)	4.69*10^(-57)	4.52*10^(-58)
2.4	2.08*10^(-47)	6.81*10^(-48)	1.60*10^(-55)	4.91*10^(-58)	2.37*10^(-62)
2.6	1.90*10^(-47)	3.99*10^(-48)	3.94*10^(-58)	5.13*10^(-62)	5.76*10^(-65)
2.8	1.67*10^(-47)	6.31*10^(-48)	6.24*10^(-62)	5.91*10^(-65)	9.17*10^(-69)
3.0	1.30*10^(-47)	1.40*10^(-47)	4.46*10^(-65)	1.28*10^(-68)	6.52*10^(-72)

The absolute errors $\left(\left|y(x_n)-y_n\right|\right)$, $\left(\left|z(x_n)-z_n\right|\right)$ and $\left(\left|w(x_n)-w_n\right|\right)$ of the numerical methods on problem 3 for h=0.02 are presented in tables 4.10a, 4.10b and 4.10c respectively, which shows that the methods have relatively small errors. It is evident from the tables that OSDBM6 of the highest order of accuracy 16, has the least error which shows the effect of the intra-step points on the performance of the proposed methods. It is observed that as the number of selected intra-step points increases, the performance of the methods gets better. Each of the methods approaches the exact solution faster as the number of iteration increases within the interval of integration.

Table 4.11 Comparative Analysis of Results of Problem 3.

x	*h*	N	Error in NJ	**Error in OSDBM2**
3	0.02	150	9.34×10^{-7}	9.74×10^{-23}
	0.01	300	1.40×10^{-8}	3.81×10^{-25}
	0.005	600	2.31×10^{-10}	1.49×10^{-27}
	0.0025	1200	3.60×10^{-12}	5.81×10^{-30}

From the table 4.11, the numerical results reveal that our method OSDBM2 of order 8 is superior in terms of accuracy when compared with the Hybrid-Second Derivative Method of order 10 in Ngwane and Jator (2012) with different step sizes.

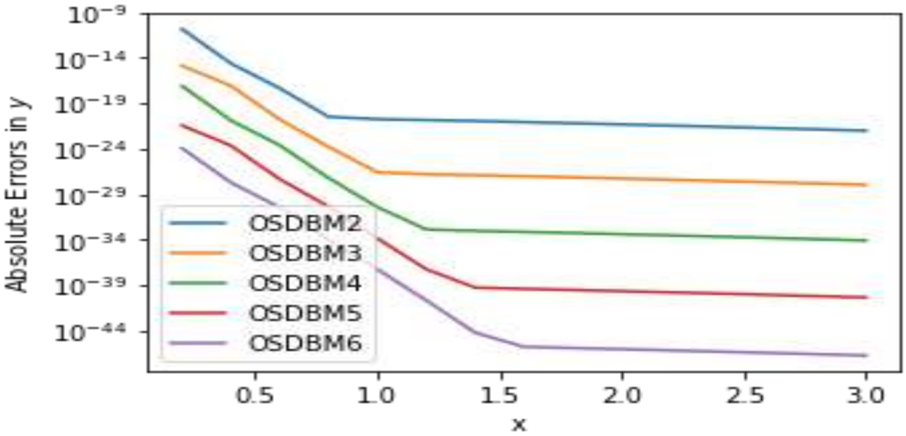

Figure 4.4a

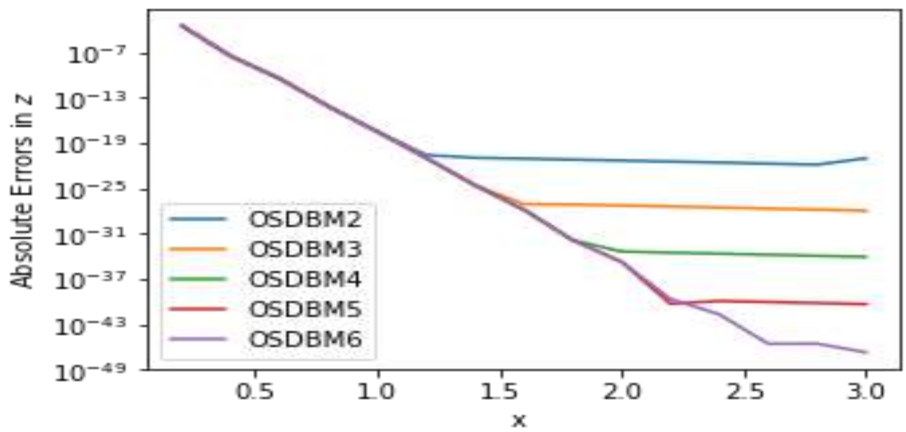

Figure 4.4b

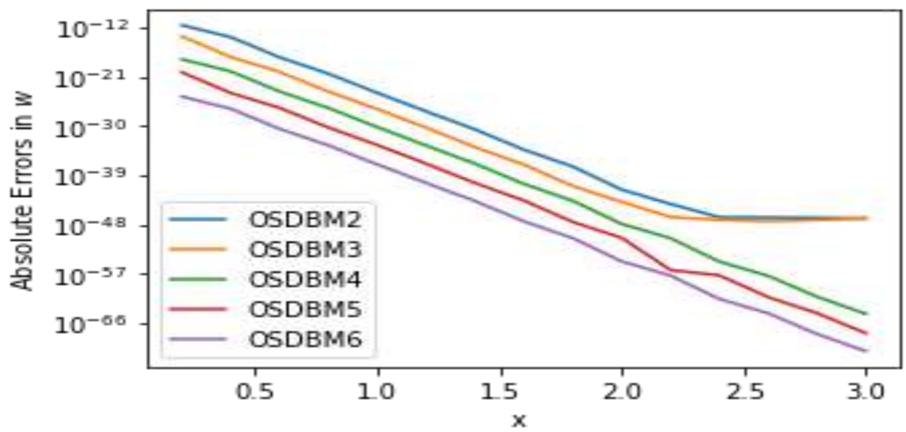

Figure 4.4c

Figure 4.4 Absolute Error of the Proposed Methods in Problem 3

Figures 4.4a, 4.4b and 4.4c show the absolute errors $\left(\left|y\left(x_n\right)-y_n\right|\right)$, $\left(\left|z\left(x_n\right)-z_n\right|\right)$ and $\left(\left|w\left(x_n\right)-w_n\right|\right)$ respectively, of the proposed methods in problem 3 with h=0.02. The graphical representation of tabulated results in tables 4.10a and 4.10b further elaborates the effectiveness of the methods. The method OSDBM6 of order 16 has the least error followed by the OSDBM5 of order 14, OSDBM4 of order 12, OSDBM3 of order 10 and OSDBM2 of order 8 with relatively small errors.

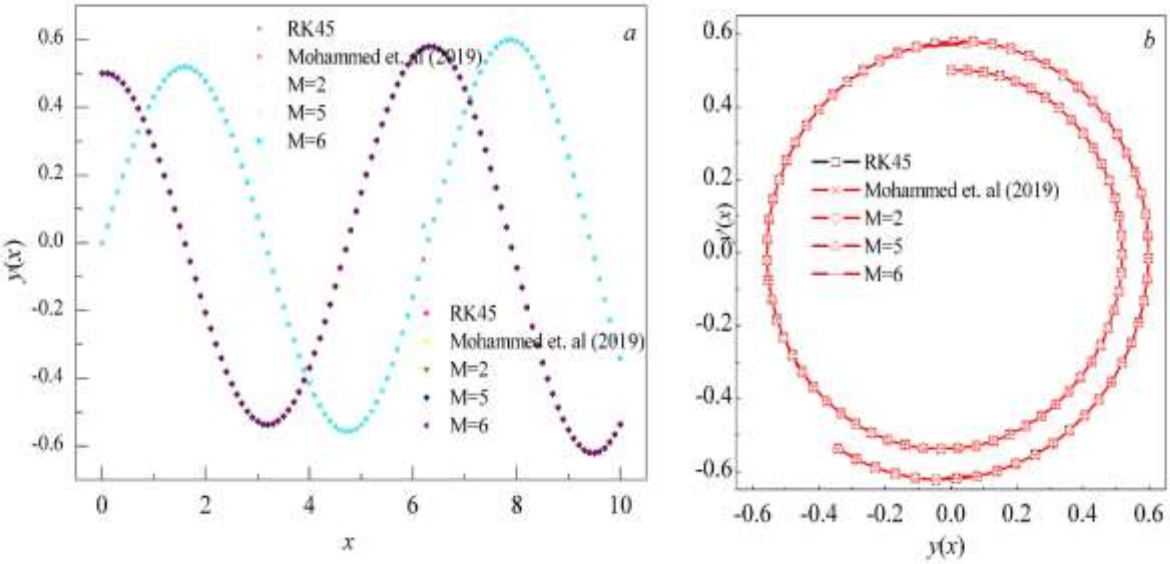

Figure 4.5 Comparison of Numerical Solutions of Problem 4

Figure 4.5 shows the graphs of the numerical solution of problem 4. We subject the methods OSDBM2, OSDBM5 and OSDBM6 to the van der pol oscillation problem which has no known analytical solution. However, we validate the efficacy of the methods with the numerical solution from the RK45 method in Maple package and also with the numerical method of Mohammed *et al.* (2019). We see that the solutions from our methods are in agreement with the known existing numerical methods.

Table 4.12 Comparing the Exact Solution with The Proposed Methods for Problem 5 at the Point $x = 10$ for h =0.1

	y(1) z(1)
Exact	2.061153622438557827965940380*10^(-9) 4.53999297624848515355915*10^(-5)
OSDBM2	2.061153622438957813262146230*10^(-9)

	4.5399929762484848606608926*10^(-5)
OSDBM3	2.0611536224385578280848431986*10^(-9)
	4.5399929762484848515356558*10^(-5)
OSDBM4	2.0611536224385578279659404276*10^(-9)
	4.5399929762484848515355915*10^(-5)
OSDBM5	2.0611536224385578279659403801*10^(-9)
	4.5399929762484848515355915*10^(-5)
OSDBM6	2.0611536224385578279659403801*10^(-9)
	4.5399929762484848515355915*10^(-5)

Problem 5 is solved on the interval $0 \le x \le 10$ for h=0.1 using the proposed methods and the solutions at point $x = 10$ is compared with the exact solution as shown in table 4.12 which shows that the numerical methods agree with the exact solution.

Table 4.13a Comparison of Absolute Errors for y(x) of the Methods for Problem 5, h=0.1

x	OSDBM2 y(*x*)	OSDBM3 y(*x*)	OSDBM4 y(*x*)	OSDBM5 y(*x*)	OSDBM6 y(*x*)
1	5.85*10^(-18)	5.02*10^(-23)	4.06*10^(-28)	3.44*10^(-33)	2.79*10^(-38)
2	1.66*10^(-18)	1.42*10^(-23)	1.01*10^(-28)	6.96*10^(-34)	4.91*10^(-39)
3	3.82*10^(-19)	3.19*10^(-24)	1.96*10^(-29)	1.13*10^(-34)	7.32*10^(-40)
4	9.22*10^(-20)	6.76*10^(-25)	3.41*10^(-30)	1.73*10^(-35)	1.05*10^(-40)
5	2.70*10^(-20)	1.45*10^(-25)	5.62*10^(-31)	2.56*10^(-36)	1.48*10^(-41)
6	9.82*10^(-21)	3.21*10^(-26)	8.94*10^(-32)	3.56*10^(-37)	2.08*10^(-42)
7	4.13*10^(-21)	7.49*10^(-27)	1.39*10^(-32)	5.47*10^(-38)	2.91*10^(-43)
8	1.86*10^(-21)	1.82*10^(-27)	2.12*10^(-33)	7.91*10^(-39)	4.07*10^(-44)
9	8.59*10^(-22)	4.60*10^(-28)	3.19*10^(-34)	1.14*10^(-39)	5.69*10^(-45)
10	4.00*10^(-22)	1.19*10^(-28)	4.74*10^(-30)	1.64*10^(-40)	7.95*10^(-46)

Table 4.13b Comparison of Absolute Errors for z(x) of the Methods for Problem 5, h=0.1

x	OSDBM2 z(*x*)	OSDBM3 z(*x*)	OSDBM4 z(*x*)	OSDBM5 z(*x*)	OSDBM6 z(*x*)
1	7.59*10^(-18)	5.76*10^(-23)	3.68*10^(-28)	2.55*10^(-33)	2.01*10^(-38)
2	5.53*10^(-18)	4.09*10^(-23)	2.42*10^(-28)	1.48*10^(-33)	1.06*10^(-38)
3	3.048*10^(-18)	2.21*10^(-23)	1.24*10^(-28)	6.88*10^(-34)	4.53*10^(-39)
4	1.480*10^(-18)	1.07*10^(-23)	5.81*10^(-29)	2.99*10^(-34)	1.82*10^(-39)
5	6.80*10^(-19)	4.86*10^(-24)	2.59*10^(-29)	1.26*10^(-34)	7.15*10^(-40)
6	3.00*10^(-19)	2.13*10^(-24)	1.12*10^(-29)	5.19*10^(-35)	2.78*10^(-40)
7	1.28*10^(-19)	9.11*10^(-25)	4.71*10^(-30)	2.12*10^(-35)	1.08*10^(-40)
8	5.40*10^(-20)	3.82*10^(-25)	1.95*10^(-30)	8.55*10^(-36)	4.17*10^(-41)
9	2.23*10^(-20)	1.58*10^(-25)	8.00*10^(-30)	3.42*10^(-36)	1.61*10^(-41)
10	9.13*10^(-21)	6.42*10^(-26)	3.24*10^(-30)	1.36*10^(-36)	6.18*10^(-42)

The absolute errors $\left(\left|y\left(x_n\right)-y_n\right|\right)$ and $\left(\left|z\left(x_n\right)-z_n\right|\right)$ of the numerical methods on problem 5 for h=0.1 are presented in tables 4.13a and 4.13b respectively, which shows that the methods have relatively small errors. It is evident from the tables that OSDBM6 of the highest order of accuracy 16, has the least error which shows the effect of the intra-step points on the performance of the proposed methods. It is observed that as the number of selected intra-step points increases, the performance of the methods gets better. Each of the methods approaches the exact solution faster as the number of iteration increases within the interval of integration.

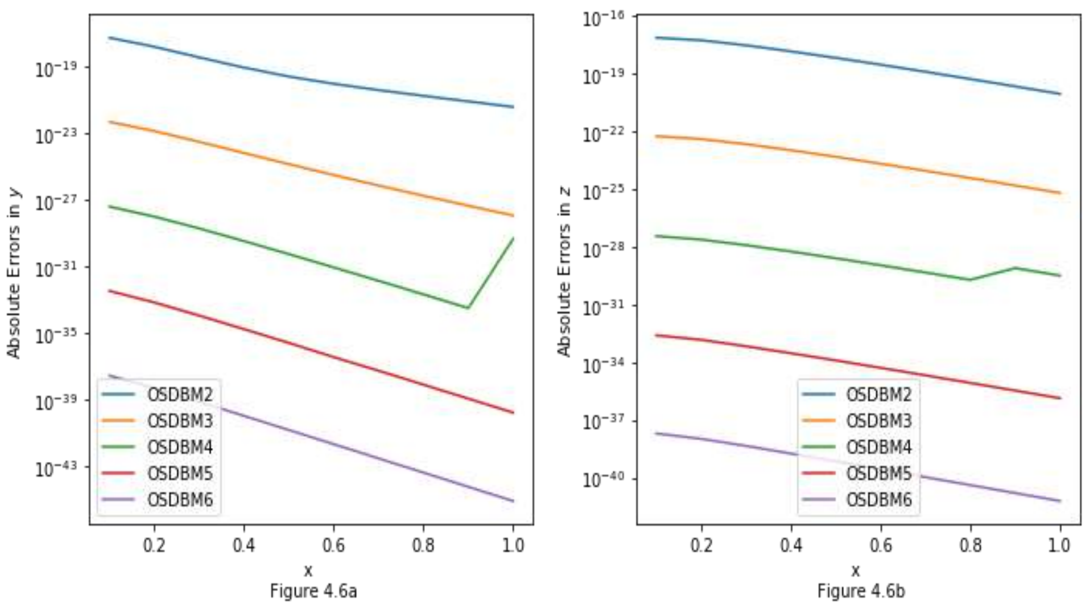

Figure 4.6. Absolute Error of the Proposed Methods in Problem 5

Figures 4.6a and 4.6b show the absolute errors $\left(\left|y\left(x_n\right)-y_n\right|\right)$ and $\left(\left|z\left(x_n\right)-z_n\right|\right)$ respectively, of the proposed methods in problem 5 with h=0.1. The graphical representation of the tabulated results in tables 4.13a and 4.13b further elaborates the effectiveness of the methods. The method OSDBM6 of order 16 has the least error followed by the OSDBM5 of order 14, OSDBM4 of order 12, OSDBM3 of order 10 and OSDBM2 of order 8 with relatively small errors.

Table 4.14 Comparative Analysis of Result of Problem 5.

x	Error in N.O. (order5) h=0.01	**OSDBM2 (order 8) h=0.1**	Error in N.O. (order5) h=0.01	**OSDBM2 (order 8) h=0.1**
		y(x)	z(x)	
1	3.43×10^{-11}	5.85*10^(-18)	4.96×10^{-11}	7.59*10^(-18)
2	5.16×10^{-12}	1.66*10^(-18)	1.91×10^{-11}	5.53*10^(-18)
3	6.49×10^{-13}	3.82*10^(-19)	6.79×10^{-12}	3.048*10^(-18)
4	9.57×10^{-14}	9.22*10^(-20)	2.61×10^{-12}	1.480*10^(-18)
5	1.20×10^{-14}	2.70*10^(-20)	9.29×10^{-13}	6.80*10^(-19)
6	1.77×10^{-15}	9.82*10^(-21)	3.57×10^{-13}	3.00*10^(-19)
7	2.22×10^{-16}	4.13*10^(-21)	1.27×10^{-13}	1.28*10^(-19)
8	3.28×10^{-17}	1.86*10^(-21)	4.89×10^{-14}	5.40*10^(-20)
9	4.11×10^{-18}	8.59*10^(-22)	1.74×10^{-14}	2.23*10^(-20)
10	6.07×10^{-19}	4.00*10^(-22)	6.68×10^{-15}	9.12*10^(-21)

From table 4.14, the numerical results reveal that our method OSDBM2 of order 8 with higher step size h =0.1 is superior in terms of accuracy when compared with the method Second Derivative Generalized Backward Differentiation Formula with order of accuracy 5 with lower step size of h=0.01 in Nwachukwu and Okor (2018).

Table 4.15 Comparing the Exact Solution with the Proposed Methods for Problem 6 at the Point $x = 10$ for h = 0.1

	y(10) z(10)
Exact	- 2.060741474143729082149510478 *10^(-13)
	0.0000453999297624848515355915155605506102379180
OSDBM2	-2.060742187381118884560167662*10^(-13)
	0.000045399929762484860614651231076720193
OSDBM3	- 2.06074148046744979651634499*10^(-13)
	0.00004539992976248485153565447125673563206855184
OSDBM4	-2.060741474310177447492212073*10^(-13)
	0.00004539992976248485153559151586025098134437
OSDBM5	-2.060741474158181748795032032*10^(-13)
	0.00004539992976248485153559151556055163618133
OSDBM6	2.060741474148171047317907051 *10^(-13)
	0.00004539992976248485153559151556055061024059167

Problem 6 is solved on the interval $0 \le x \le 10$ for h=0.1 using the proposed methods and the solutions at point $x = 10$ is compared with the exact solution as shown in table 4.15 which shows that the numerical methods agree with the exact solution.

Table 4.16a Comparison of Absolute Errors for y(x) of the Methods for Problem 6, h=0.1

x	OSDBM2 y(x)	OSDBM3 y(x)	OSDBM4 y(x)	OSDBM5 y(x)	OSDBM6 y(x)
1	6.13*10^(-22)	8.11*10^(-27)	2.05*10^(-31)	6.46*10^(-36)	1.86*10^(-40)
2	3.09*10^(-22)	1.78*10^(-26)	1.46*10^(-30)	1.18*10^(-34)	9.54*10^(-39)
3	3.79*10^(-22)	6.31*10^(-26)	1.11*10^(-29)	2.15*10^(-33)	4.88*10^(-37)
4	7.52*10^(-22)	2.35*10^(-25)	8.46*10^(-29)	3.92*10^(-32)	2.49*10^(-35)
5	1.60*10^(-21)	8.75*10^(-24)	6.45*10^(-28)	7.15*10^(-31)	1.27*10^(-33)
6	3.41*10^(-21)	3.26*10^(-24)	4.92*10^(-27)	1.30*10^(-29)	1.27*10^(-32)
7	7.30*10^(-21)	1.22*10^(-23)	3.75*10^(-26)	2.38*10^(-28)	3.33*10^(-30)
8	1.56*10^(-20)	4.54*10^(-23)	2.86*10^(-25)	4.34*10^(-27)	1.70*10^(-28)
9	3.34*10^(-20)	1.70*10^(-22)	2.18*10^(-24)	9.72*10^(-26)	8.69*10^(-27)
10	7.12*10^(-20)	6.32*10^(-22)	1.66*10^(-23)	1.45*10^(-24)	4.44*10^(-25)

Table 4.16b Comparison of Absolute Errors for z(x) of the Methods for Problem 6, h=0.1

x	OSDBM2 z(x)	OSDBM3 z(x)	OSDBM4 z(x)	OSDBM5 z(x)	OSDBM6 z(x)
1	7.36*10^(-18)	5.10*10^(-23)	2.43*10^(-28)	8.31*10^(-34)	2.17*10^(-39)
2	5.41*10^(-18)	3.75*10^(-23)	1.79*10^(-28)	6.12*10^(-34)	1.59*10^(-39)
3	2.998*10^(-18)	20.7*10^(-23)	9.86*10^(-29)	3.38*10^(-34)	8.80*10^(-40)
4	1.47*10^(-18)	1.02*10^(-23)	4.84*10^(-29)	1.66*10^(-34)	4.31*10^(-40)
5	6.74*10^(-19)	4.67*10^(-24)	2.22*10^(-29)	7.61*10^(-35)	1.98*10^(-40)
6	2.97*10^(-19)	2.06*10^(-24)	9.82*10^(-30)	3.36*10^(-35)	8.76*10^(-41)
7	1.28*10^(-19)	8.85*10^(-25)	42.1*10^(-30)	1.44*10^(-35)	3.76*10^(-41)
8	5.37*10^(-20)	3.72*10^(-25)	1.77*10^(-30)	6.06*10^(-36)	1.58*10^(-41)
9	2.22*10^(-20)	1.54*10^(-25)	7.33*10^(-31)	2.51*10^(-36)	6.54*10^(-42)
10	9.08*10^(-21)	6.30*10^(-26)	3.00*10^(-31)	1.03*10^(-36)	2.67*10^(-42)

The absolute errors $\left(\left|y\left(x_n\right)-y_n\right|\right)$ and $\left(\left|z\left(x_n\right)-z_n\right|\right)$ of the numerical methods on problem 6 for h=0.1 are presented in tables 4.15a and 4.15b respectively, which shows that the methods have relatively small errors. It is evident from the tables that OSDBM6 of the highest order of accuracy 16, has the least error which shows the effect of the intra-step points on the performance of the proposed methods. It is observed that as the number of selected intra-step points increases, the performance of the methods gets better. Each of the methods approaches the exact solution faster as the number of iteration increases within the interval of integration.

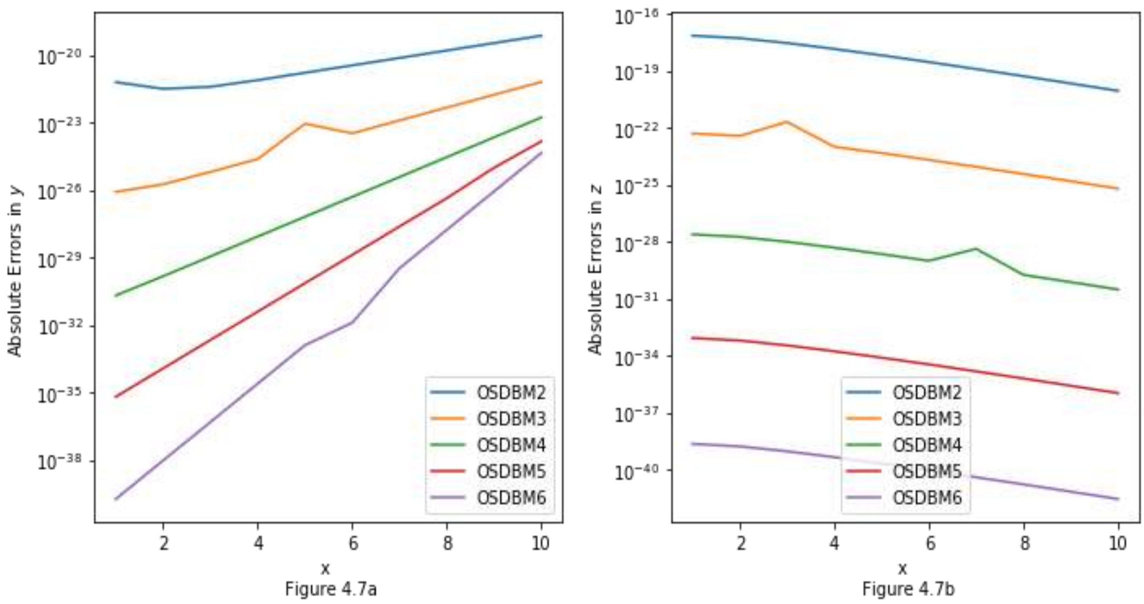

Figure 4.7 Absolute Error of the Proposed Methods in Problem 6

Figures 4.7a and 4.7b show the absolute errors $\left(\left|y\left(x_n\right)-y_n\right|\right)$ and $\left(\left|z\left(x_n\right)-z_n\right|\right)$ respectively, of the proposed methods in problem 6 with h=0.1. The graphical representation of the tabulated results in tables 4.15a and 4.15b further elaborates the effectiveness of the methods. The method OSDBM6 of order 16 has the least error followed by the OSDBM5 of order 14, OSDBM4 of order 12, OSDBM3 of order 10 and OSDBM2 of order 8 with relatively small errors.

Table 4.17 Comparative Analysis of Result of Problem 6.

x	Error in AAO (order8) h=0.01	**M=2 (order 8)** h=0.1	Error in AAO (order8) h=0.01	**M=2 (order 8)** h=0.1
	y(x)		z(x)	
3	2.03×10^{-19}	**3.79*10^(-22)**	1.44×10^{-14}	**2.998*10^(-18)**
5	1.20×10^{-20}	**1.60*10^(-21)**	3.21×10^{-15}	**6.74*10^(-19)**
10	1.11×10^{-20}	**7.12*10^(-20)**	4.38×10^{-17}	**9.08*10^(-21)**

From table 4.17, the numerical results revealed that our method OSDBM2 of order 8 is superior in terms of accuracy even with higher step size h=0.1 when compared with the method $\frac{3}{8}$ - type block method for stiff systems in Akinfenwa *et al.* (2017) with a lower step size h=0.01.

Table 4.18 Comparative Analysis of Absolute Error for Problem 7

	HBSDBDF		OSDBM2	
h	Maximum error	Relative error	Maximum error	Relative error
0.4	8.9924×10^{-7}	3.6279×10^{-7}	1.1528×10^{-12}	4.0743×10^{-11}
0.2	5.9042×10^{-9}	2.6294×10^{-9}	4.5164×10^{-15}	1.3583×10^{-13}
0.1	4.5695×10^{-11}	1.8848×10^{-11}	1.777×10^{-17}	6.4596×10^{-16}

For this problem, the accuracy of OSDBM2 is demonstrated in table 4.18. For different step sizes h, the maximum error and relative error are calculated with the formula $\max_n \frac{|y_n - y(x_n)|}{|y_n + y(x_n)|}$. The proposed method OSDBM2 is superior in terms of accuracy compared with HBSDBDF.

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

In this project, a modified single-step numerical scheme is proposed to improve the order of accuracy by imposing varieties of countable intra-step points for one-step methods from the Bhaskara cosine approximation formula, and incorporating higher derivatives in the derivation process of our algorithms for the solution of systems of first order initial value problems of ordinary differential equations. Analysis of basic properties of numerical methods was carried out and findings show that the methods are of higher order and convergent. The stability analysis of the methods were carried out in the spirit of Akinfenwa *et al*, (2014) and are all found to be A-stable which make them good instruments for solving stiff systems of ordinary differential equations.

The derived methods are self-starting and were implemented as block methods to simultaneously produce approximations $\left\{y_{n+\eta}, y_{n+1}\right\}$ at a block points $\left\{x_{n+\eta}, x_{n+1}\right\}$, on non-overlapping interval.

The effectiveness of the derived methods is further demonstrated by considering four test problems for stiff systems, the singularly perturbed problem and the Van Der Pol Oscillatory problem. The desirable property of a numerical solution is to behave like that of the exact solution of the problem which can be seen in the tables and figures presented. From the results obtained, it is observed that **(OSDBM2), (OSDBM3), (OSDBM4), (OSDBM5)** and **(OSDBM6)** effectively handled all the problems considered in this thesis with **(OSDBM6)** having the overall least error. Also for the purpose of comparison, **(OSDBM2)** which is of order 8 (the least accurate of the methods derived) was compared with some existing methods in the literature and found that **(OSDBM2)** shows superiority over the existing methods (even with higher step sizes) in the literature.

5.2 Recommendation

The main idea in this research work is the introduction of intra-step points and multi-derivatives (in this case second derivative) to modify single-step method for solving stiff system of first order initial value problems. This is achieved by interpolating the power series function at the origin and collocating its first and second derivatives at some carefully selected intra-step points to generate the continuous form of the method, in which the block method is obtained. Therefore, the followings are recommended for future research purpose.

1. That the methods be employed in solving related real life problems especially those with no exact solutions.
2. That the methods be extended to higher step number to handle other classes of IVPs in ordinary differential equations such as singular initial value problems, oscillatory problems and fuzzy differential equation.

5.3 Contributions to Knowledge

The modification of a class of one-step numerical method through the use of some carefully selected number of off-step points $(m = 2, 3, 4, 5, 6)$ and incorporating second derivatives in the derivation process of the new one-step methods, have yielded a higher order of accuracy $(8, 10, 12, 14, 16)$ respectively. The methods derived successfully solved some stiff systems of ordinary differential equations and the numerical results yielded, have minimal errors compared with the existing methods considered in the literature.

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APPENDICES

Stability function of OSDBM2

$$\lambda(z) - \frac{3z^6 + 114z^5 + 2212z^4 + 26240z^3 + 195840z^2 + 860160z + 1720320}{3z^6 - 114z^5 + 2212z^4 - 26240z^3 + 195840z^2 - 860160z + 1720320}$$

Appendix B

Stability function of OSDBM2

$$\lambda(z) - \frac{3z^6 + 114z^5 + 2212z^4 + 26240z^3 + 195840z^2 + 860160z + 1720320}{3z^6 - 114z^5 + 2212z^4 - 26240z^3 + 195840z^2 - 860160z + 1720320}$$

Stability function of OSDBM3

$$\begin{aligned}&\frac{1}{1663499961502784160}\big(33627190836857898777375z^8 + 234230372103851017163840 0z^7\\
&+ 781044824408749858395241 20z^6 + 17374381910780109534216606 72z^5\\
&+ 25791922532600978895713002752z^4 + 269334086656443554609452154880z^3\\
&+ 1881426567705763005194032281600z^2 + 8066803837072402259365183488000z\\
&+ 16133607674144804518730366976000\big) \big/ \big(21025z^8 - 1384170z^7 + 47476092z^6\\
&- 1038752880z^5 + 15555540960z^4 - 161689812480z^3 + 1131595637760z^2\\
&- 4849296076800z + 9698592153600\big)\end{aligned}$$

Stability function of OSDBM4

$$\begin{aligned}&-\big(99681125z^{10} + 10127222775z^9 + 541039862535z^8 + 18788175851880z^7\\
&+ 462013430575428z^6 + 8343718070571576z^5 + 111519896632971120z^4\\
&+ 1085055826208739840z^3 + 7330722284995200000z^2 + 30963663795466321920z\\
&+ 61927327590932643840\big) \big/ \big(99681125z^{10} - 10127222775z^9 + 541039862535z^8\\
&- 18788175851880z^7 + 462013430575428z^6 - 8343718070571576z^5\\
&+ 111519896632971120z^4 - 1085055826208739840z^3 + 7330722284995200000z^2\\
&- 30963663795466321920z + 61927327590932643840\big)\end{aligned}$$

Stability function of OSDBM5

$$\begin{aligned}&-\big(1950105600z^{12} + 279322951680z^{11} + 20897182896768z^{10} + 1015352172233984z^9\\
&+ 35101727342868783z^8 + 903878100820094306z^7 + 17724686412082932100z^6\\
&+ 266407560800727978240z^5 + 3047378336600274270720z^4\\
&+ 25923149620627061698560z^3 + 156382175616219435294720z^2\\
&+ 609147730495666755993600z + 1218295460991333511987200\big) \big/ \big(1950105600z^{12}\\
&- 279322951680z^{11} + 20897182896768z^{10} - 1015352172233984z^9\\
&+ 35101727342868783z^8 - 903878100820094306z^7 + 17724686412082932100z^6\\
&- 266407560800727978240z^5 + 3047378336600274270720z^4\\
&- 25923149620627061698560z^3 + 156382175616219435294720z^2\\
&- 609147730495666755993600z + 1218295460991333511987200\big)\end{aligned}$$

Stability function of OSDBM6

$$\begin{aligned}&-\big(11356240665375z^{14} + 2234688100339725z^{13} + 233060693487675605z^{12}\\
&+ 16051110304222521080z^{11} + 804715337510257806156z^{10}\\
&+ 30909829439677166074152z^9 + 935272197554552490818640z^8\\
&+ 22595458445517006370298880z^7 + 437133164547728581867468800z^6\\
&+ 6727170489686829437444259840z^5 + 80933323572727023984099655680z^4\\
&+ 737007271342534619008598016000z^3 + 4795348300509213521550508032000z^2\\
&+ 19928002546944865701199872000000z + 39856005093889731402399744000000\big) \big/\\
&\big(11356240665375z^{14} - 2234688100339725z^{13} + 233060693487675605z^{12}\\
&- 16051110304222521080z^{11} + 804715337510257806156z^{10}\\
&- 30909829439677166074152z^9 + 935272197554552490818640z^8\\
&- 22595458445517006370298880z^7 + 437133164547728581867468800z^6\\
&- 6727170489686829437444259840z^5 + 80933323572727023984099655680z^4\\
&- 737007271342534619008598016000z^3 + 4795348300509213521550508032000z^2\\
&- 19928002546944865701199872000000z + 39856005093889731402399744000000\big)\end{aligned}$$

for n **from** 0 **to** N **do**

$h := 0.0625;$

$y[0] := 1; z[0] := 1;$

$f[n+0] := -y[n+0] + 95\cdot z[n+0]; f\left[n+\frac{3}{4}\right] := -y\left[n+\frac{3}{4}\right] + 95\cdot z\left[n+\frac{3}{4}\right]; f\left[n+\frac{1}{4}\right]$

\> $N := 15;$

$:= -y\left[n+\frac{1}{4}\right] + 95\cdot z\left[n+\frac{1}{4}\right]; f[n+1] := -y[n+1] + 95\cdot z[n+1];$

$N := 15$

\> $g[n+0] := -94\cdot y[n+0] - 9310\cdot z[n+0]; g\left[n+\frac{3}{4}\right] := -94\cdot y\left[n+\frac{3}{4}\right] - 9310\cdot z\left[n+\frac{3}{4}\right]; g\left[n+\frac{1}{4}\right] := -94\cdot y\left[n+\frac{1}{4}\right] - 9310\cdot z\left[n+\frac{1}{4}\right]; g[n+1] := -94\cdot y[n+1] - 9310\cdot z[n+1]; p[n+0] := -y[n+0] - 97\cdot z[n+0]; p\left[n+\frac{3}{4}\right] := -y\left[n+\frac{3}{4}\right] - 97\cdot z\left[n+\frac{3}{4}\right]; p\left[n+\frac{1}{4}\right] := -y\left[n+\frac{1}{4}\right] - 97\cdot z\left[n+\frac{1}{4}\right]; p[n+1] := -y[n+1] - 97\cdot z[n+1]; q[n+0] := 98\cdot y[n+0] + 9314\cdot z[n+0]; q\left[n+\frac{3}{4}\right] := 98\cdot y\left[n+\frac{3}{4}\right] + 9314\cdot z\left[n+\frac{3}{4}\right]; q\left[n+\frac{1}{4}\right] := 98\cdot y\left[n+\frac{1}{4}\right] + 9314\cdot z\left[n+\frac{1}{4}\right]; q[n+1] := 98\cdot y[n+1] + 9314\cdot z[n+1];$

$$A := \Big(y_{n+\frac{3}{4}} = y_n + \frac{1125}{7168}hf_n + \frac{303}{896}hf_{n+\frac{1}{4}} + \frac{177}{896}hf_{n+\frac{3}{4}} + \frac{411}{7168}hf_{n+1} + \frac{489}{71680}h^2 g_n + \frac{423}{17920}h^2 g_{n+\frac{1}{4}} - \frac{633}{17920}h^2 g_{n+\frac{3}{4}} - \frac{279}{71680}h^2 g_{n+1}, y_{n+\frac{1}{4}} = y_n + \frac{20135}{193536}hf_n + \frac{3413}{24192}hf_{n+\frac{1}{4}} + \frac{11}{24192}hf_{n+\frac{3}{4}} + \frac{857}{193536}hf_{n+1} + \frac{233}{71680}h^2 g_n - \frac{1601}{161280}h^2 g_{n+\frac{1}{4}} - \frac{289}{161280}h^2 g_{n+\frac{3}{4}} - \frac{23}{71680}h^2 g_{n+1}, y_{n+1} = y_n + \frac{61}{378}hf_n + \frac{64}{189}hf_{n+\frac{1}{4}} + \frac{64}{189}hf_{n+\frac{3}{4}} + \frac{61}{378}hf_{n+1} + \frac{1}{140}h^2 g_n + \frac{8}{315}h^2 g_{n+\frac{1}{4}} - \frac{8}{315}h^2 g_{n+\frac{3}{4}} - \frac{1}{140}h^2 g_{n+1}, z\left[n+\frac{3}{4}\right] = z[n] + \frac{1125}{7168}\cdot h\cdot p[n] + \frac{303}{896}\cdot h\cdot p\left[n+\frac{1}{4}\right] + \frac{177}{896}\cdot h\cdot p\left[n+\frac{3}{4}\right] + \frac{411}{7168}\cdot h\cdot p[n+1] + \frac{489}{71680}h^2\cdot q[n] + \frac{423}{17920}h^2\cdot q\left[n+\frac{1}{4}\right] - \frac{633}{17920}h^2\cdot q\left[n+\frac{3}{4}\right] - \frac{279}{71680}h^2\cdot q[n+1], z\left[n+\frac{1}{4}\right] = z[n] + \frac{20135}{193536}\cdot h\cdot p[n] + \frac{3413}{24192}\cdot h\cdot p\left[n+\frac{1}{4}\right] + \frac{11}{24192}\cdot h\cdot p\left[n+\frac{3}{4}\right] + \frac{857}{193536}\cdot h\cdot p[n+1] + \frac{233}{71680}h^2\cdot q[n] - \frac{1601}{161280}h^2\cdot q\left[n+\frac{1}{4}\right] - \frac{289}{161280}h^2\cdot q\left[n+\frac{3}{4}\right] - \frac{23}{71680}h^2\cdot q[n+1], z[n+1] = z[n] + \frac{61}{378}\cdot h\cdot p[n] + \frac{64}{189}\cdot h\cdot p\left[n+\frac{1}{4}\right] + \frac{64}{189}\cdot h\cdot p\left[n+\frac{3}{4}\right] + \frac{61}{378}\cdot h\cdot p[n+1] + \frac{1}{140}h^2\cdot q[n] + \frac{8}{315}h^2\cdot q\left[n+\frac{1}{4}\right] - \frac{8}{315}h^2\cdot q\left[n+\frac{3}{4}\right] - \frac{1}{140}h^2\cdot q[n+1]\Big); P := fsolve(\{A\});$$

end do:

$$y_0 := 1$$
$$y_1 := 1.7810743524323604550204 76$$
$$y_2 := 1.5741646786711707653188 81$$
$$y_3 := 1.3892017149391282371852 87$$
$$y_4 := 1.2259662270291321724076 88$$
$$y_5 := 1.0819113980701685840703 20$$
$$y_6 := 0.9547834576680081903601794$$
$$y_7 := 0.8425934440310277092312317$$
$$y_8 := 0.7435861044954686110213115$$
$$y_9 := 0.6562124340221963503680792$$
$$y_{10} := 0.5791054404620864592977648$$
$$y_{11} := 0.5110587574776791346706169$$
$$y_{12} := 0.4510077705127837774332570$$
$$y_{13} := 0.3980129605191157102101759$$
$$y_{14} := 0.3512452048466444782745303$$
$$y_{15} := 0.3099728053248554738237061$$
$$y_{16} := 0.2735500405846427403297618$$
$$z_0 := 1$$
$$z_1 := -0.01608054726316962298803285$$
$$z_2 := -0.01656311252836098240451160$$
$$z_3 := -0.01462315735718377864128950$$
$$z_4 := -0.01290490760386525288995013$$
$$z_5 := -0.01138854103218802039023278$$
$$z_6 := -0.01005035218597869161129619$$
$$z_7 := -0.008869404674010817089165924$$
$$z_8 := -0.007827222152583880113630733$$
$$z_9 := -0.006907499305496803688078755$$
$$z_{10} := -0.006095846741706173255765930$$
$$z_{11} := -0.005379565868186096154427547$$
$$z_{12} := -0.004747450215924039762455337$$
$$z_{13} := -0.004189610110727533791686063$$
$$z_{14} := -0.003697317945754152402889793$$
$$z_{15} := -0.003262871634998478671828485$$
$$z_{16} := -0.002879474111417292003471176$$