ANALYTICAL SOLUTION OF THE BLOCH NMR FLOW EQUATIONS FOR THE ANALYSIS OF BLOCKAGE IN A RADIALLY SYMMETRIC CYLINDRICAL PIPE

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Abstract

The use of various types of wave energy as a probe is an increasingly promising non destructive means of detecting objects and of diagnosing the properties of quite compl

icated materials. An analysis of this technique requires a detailed understanding of how signals evolve in the medium of interest in the absence of inhomogeneities (obstructions or perturbation) and the nature of the emerging signal when the original signal is perturbed by obstructions that might exist in the medium. Properties of the signal are then used to estimate the level of inhomogeneity in the medium. In this study, Magnetic Resonance Imaging (MRI) is used to detect partial blockage of fluid in a cylindrical pipe. The Bloch NMR flow equation is solved analytically in cylindrical coordinates for flow of fluid in a radially symmetric cylindrical pipe. Based on the appropriate boundary conditions, the radial axis was varied to depict free flow and partial blockage in the pipe. The Nuclear Magnetic Resonance (NMR) signals obtained were then analyzed and used to provide information on the type of blockage in the cylindrical pipe.

Keywords: Bloch NMR diffusion equation, Cylindrical pipe, Plaque.

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1.0 Introduction

Magnetic resonance imaging is a recent approach adopted in the diagnosis of ailments and diseases in humans without surgical invasion. It can also be used to determine problems associated with blockage in cylindrical pipes. It provides accurate assessment of the individual component or multi-component systems in a matter of minutes whereas traditional radioactive tracer techniques may take weeks for each component [1]. This is possible because fluids exhibit random molecular motion of spins. Through magnetic resonance coupled with the fact that the molecules of fluids carry magnetic moments with them, the rate of their signal loss or signal attenuation could be easily detected. This goes a long way to signify whether or not a problem exists at any point in any cylindrical pipe.

Many works have been done in the area of fluid flow and blockage in fluid pipeline. Yuan used time splitting algorithms and Godunov mixed format to simulate the pulse propagation in the blocked pipelines. He stated that by using pulse launched at the pipeline inlet, the characteristic data of pipeline blockages could be identified through its propagation characteristics in multiple non-completely blocked natural gas pipeline segments [2]. Another technique used by Sattar is by the system frequency response. This is a technique whereby the frequency response is used in the detection of partial blockages in a pipeline. By this method, a partial blockage increases the amplitude of the pressure oscillations at even harmonics. Such an increase in amplitude has an oscillatory pattern, the frequency and amplitude was then used to predict the location and size of a partial blockage [3]. Similar to this is the method adopted by Mohapatra for the detection of partial blockages in single pipelines by the frequency response method [4]. Wang also investigated analytically the effects of a partial blockage on pipeline transients. A partial blockage is simulated using an orifice equation, and the influence of the blockage on the unsteady pipe flow is considered in the equation using a Dirac delta function [5].

Diffusion Magnetic Resonance Imaging (DMRI), being a viable alternative, is one of the most rapidly evolving techniques in the MRI field. Diffusion and flow can be measured very delicately and accurately using Magnetic Resonance Imaging [6]. Coefficient of diffusion of a substance defined as the amount of material that diffuses in a certain time plays a vital role in the detection of blockage in a pipe using MRI. Random diffusion motion of water molecules has intriguing properties depending on the physiological and anatomical environment of the organisms being studied. This is the principle being exploited by the method of DMRI. Though not widely known, it has been noted for long that nuclear magnetic resonance is capable of quantifying diffusion movement of molecules.

The same principle has been applied to a cylindrical water pipe under the influence of radiofrequency field as a probe to perturb the water molecules. This causes the nuclei to absorb energy from the applied electromagnetic (EM) pulse(s) and radiate this energy at a specific resonance frequency which depends on the strength of the magnetic field and other factors. This allows the observation of specific magnetic properties of an atomic nucleus. A Radio Frequency (RF) transmitter is needed to transmit energy into the fluid under consideration in the cylinder in order to “activate” the nuclei so that they emit a signal [7].

The process undergoes the following four stages: (1) a magnetic field, B_o is applied, (2) the sample responds to B_o (3) a radio frequency pulse or a train of radio frequencies pulses is applied during a limited time and (4) the system relaxes. The Free Induction Decay (FID) is the name given to the time-domain signal obtained during the relaxation process. The relaxation process itself is referred to as the free induction decay. It is the observable NMR signal generated by non-equilibrium nuclear spin magnetization precessing about the magnetic field conventionally along z direction [8]. This time-domain signal is typically digitized and then Fourier transformed in order to obtain a frequency spectrum of the NMR signal i.e. the NMR spectrum [9].

2.0 The Bloch NMR Equations

The x, y, z components of magnetization of fluid flow are given by the Bloch equations which are fundamental to understanding Magnetic Resonance Images:

$$\frac{dM_x}{dt} = -\frac{M_x}{T_2} \quad (1)$$

$$\frac{dM_y}{dt} = \gamma M_z B_1(x) - \frac{M_y}{T_2} \quad (2)$$

$$\frac{dM_z}{dt} = -\gamma M_y B_1(x) - \frac{M_o - M_z}{T_1} \quad (3)$$

where

M_o = equilibrium magnetization

M_x = component of transverse magnetization along the x-axis

M_y = component of transverse magnetization along y-axis

M_z = component of magnetization along the field (z -axis)

γ = gyro-magnetic ratio of fluid spins

B_0 = static magnetic field

$B_1(x,t)$ = radio-frequency (RF) magnetic field

T_1 = Longitudinal or spin lattice relaxation time

T_2 = Transverse or spin-spin relaxation time

V = the flow velocity

From the fundamental Bloch equations stated in equations (1) – (3), the diffusion equation was evolved with the diffusion coefficient D evolving intrinsically without any additional term as done by Torrey [10]. The NMR diffusion equation as derived by Awojoyogbe [11] is given as:

$$\frac{\partial M_y}{\partial t} = D\frac{\partial^2 M_y}{\partial r^2} + \frac{F_o}{T_o}\gamma B_1(r,t) \quad (4)$$

where the diffusion coefficient $D = -\frac{V^2}{T_o}$ was accurately defined in terms of MRI flow parameters fluid velocity, V , T_1 and T_2 relaxation rates (as $T_0 = \frac{1}{T_1} + \frac{1}{T_2}$) and $F_o = \frac{M_o}{T_1}$.

The above diffusion equation with $D = -\frac{V^2}{T_o}$, called coefficient of diffusion was evolved as an intrinsic part of the Bloch Nuclear Magnetic Resonance (NMR) equations.

3.0 Solution of the Diffusion Equation in Radially Symmetric Cylinder

Since the cylinder under consideration is radially symmetric, then it is independent of θ. Therefore M_y can be expressed as

$$M_y = M_y(r, z, t) \tag{5}$$

M_y is the transverse magnetization.

In cylindrical coordinates, equation (4) transforms to

$$\frac{\partial M_y}{\partial t} = D\left(\frac{\partial^2 M_y}{\partial r^2} + \frac{1}{r}\frac{\partial M_y}{\partial r} + \frac{\partial^2 M_y}{\partial z^2}\right) + \frac{F_o}{T_o}\gamma B_1(t) \tag{6}$$

Equation (6) can be expressed in the form:

$$M_y = F(r, z)U(t) + w_c(t) \tag{7}$$

with

$$w_c(\dot{t}) = \frac{F_o}{T_o}\gamma B_1(t) \tag{8}$$

$\Rightarrow$

$$w_c(t) = \int_0^{t_0} \frac{F_o}{T_o}\gamma B_1(t)\, dt \tag{9}$$

Using the method of separation of variables (MSV),

$$M_y = F(r, z)U(t) \tag{10}$$

In order to obtain solution that will not be identically zero, the two expressions on the right hand side of (10) must be equal to a constant, say $-\lambda^2$. Hence, the following two differential equations evolve:

$$\frac{dU(t)}{dt} + \lambda^2 D U(t) = 0 \tag{11}$$

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial z^2} + \lambda^2 F = 0 \tag{12}$$

By integrating equation (11), the general solution below is obtained:

$$U(t) = C_1 e^{-\lambda^2 Dt} \quad \lambda = 1, 2, \dots, \dots, \tag{13}$$

where C_1 is the arbitrary constant of integration

In order to solve (12), the same method of separation of variables is followed:

$$F = Q(r)Z(z) \tag{14}$$

Again the two expressions on the right hand side of (14) must be equal to a constant, say$-\mu^2$, in order to obtain solutions that will not be identically zero. The following two differential equations evolve;

$$\frac{\partial^2 Q}{\partial r^2} + \frac{1}{r}\frac{\partial Q}{\partial r} + \mu^2 Q = 0 \tag{15}$$

and
$$\frac{\partial^2 Z}{\partial z^2} - \beta^2 Z = 0 \tag{16}$$

where we have

$$\beta^2 = \mu^2 - \lambda^2 \tag{17}$$

It could be noted that from equations (15), a Bessel differential equation evolves and its solution is given as

$$F(r) = C_2 J_0(\mu r) + C_3 Y_m(\mu r) \tag{18}$$

where $J_0(\mu r)$ is the Bessel function of the first kind, of order zero and $Y_m(\mu r)$ is the Bessel function of the second kind, of order m. $C_2 and C_3$ are constants.

Also from (16),
$$Z(z) = C_4 e^{\beta z} + C_5 e^{-\beta z} \tag{19}$$

Consequently, the solutions to the equations are:

$$U(t) = C_1 e^{-\lambda^2 Dt} \qquad \lambda = 1, 2, \dots, \dots, \tag{20}$$

$$F(r) = C_2 J_0(\mu r) + C_3 Y_m(\mu r) \tag{21}$$

$$Z(z) = C_4 e^{\beta z} + C_5 e^{-\beta z} \tag{22}$$

Combining the solution to the diffusion equation (6), this gives the product of the quantities in (20), (21) and (22) plus $\int_0^{t_0} w_c(\dot{t})\, dt$ i.e.

$$M_y = M_y\,(r, z, t) = F(r)Z(z)\,U(t) + \int_0^{t_0} w_c(\dot{t})\, dt \tag{23}$$

$$M_y\,(r,z,t) = \{C_2 J_0(\mu r) + C_3 Y_m(\mu r)\}\{C_4 e^{\beta z} + C_5 e^{-\beta z}\}\{C_1 e^{-\lambda^2 D t}\} + \int_0^{t_0} w_c(\dot{t})\,dt \qquad (24)$$

The last function on the right hand side $\int_0^{t_0} \frac{F_o}{T_o} \gamma B_1(t)\,dt = \int_0^{t_0} w_c(\dot{t})\,dt$ is the radio-frequency field applied to perturb the molecules of the fluid. Therefore for the solution of

$$w_c(t) = \int_0^{t_0} \frac{F_o}{T_o} \gamma B_1(t)\,dt \qquad (25)$$

The radio frequency field (rf) field is defined as

$$B_1(t) = bB_1(t)coswt \qquad (26)$$

which implies

$$w_c(t) = \int_0^{t_0} \frac{F_o}{T_o} b\gamma B_1(t)coswt\,dt \qquad (27)$$

$$\int_0^{t_0} \frac{bF_o}{T_o} \cos\,(wt)\,dt = \frac{bF_o}{wT_o} \gamma \sin\,(wt) \qquad (28)$$

Consequently,

$$M_y\,(r,z,t) = \{C_2 J_0(\mu r) + C_3 Y_m(\mu r)\}\{C_4 e^{\beta z} + C_5 e^{-\beta z}\}\{C_1 e^{-\lambda^2 D t}\} + \frac{bF_o}{wT_o} \gamma \sin\,(wt) \qquad (29)$$

3.1 Solution using the Initial and Boundary Conditions

We shall examine the behaviour of diffusion or flow of fluid at the point of free flow and partial blockage as shown in Figures 1a and 1b:

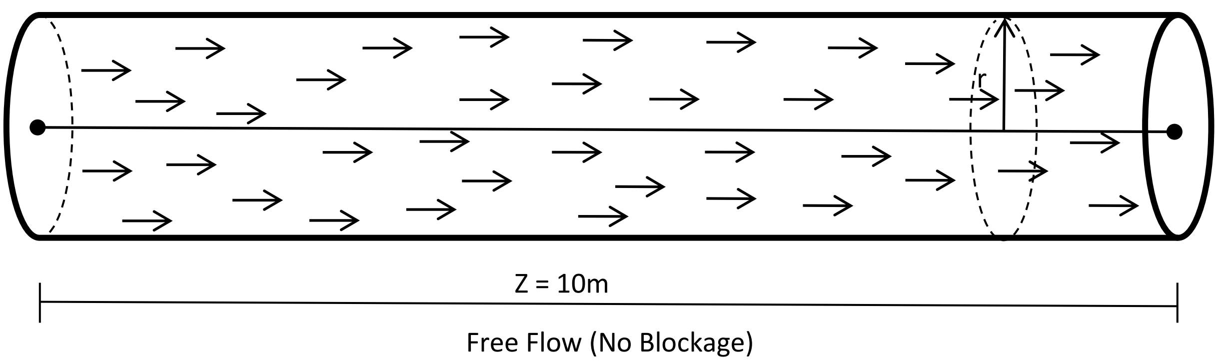

Figure 1a - Free flow in a cylinder

(b)

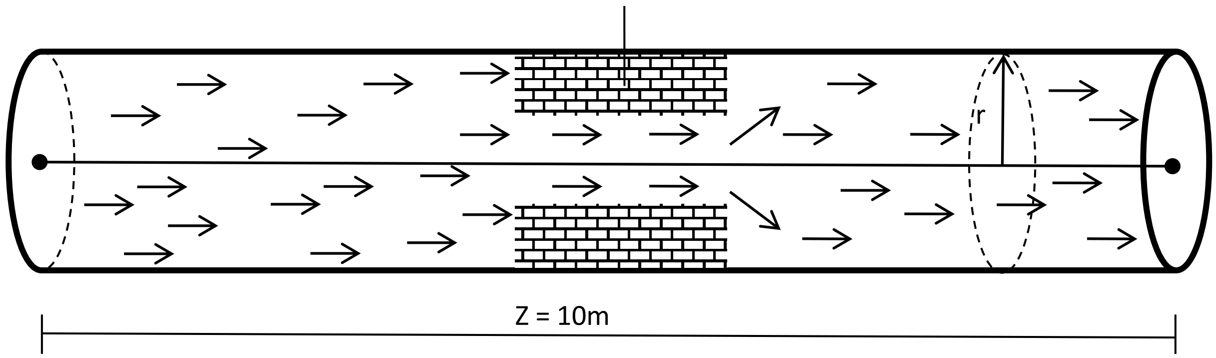

Figure 1b - Partial flow in a cylinder

The following conditions shall be imposed:

$$\left.\begin{array}{ll} i) & M_y\,(r,z,0) = M_i(r,z); \\ ii) & M_y\,(r,0,t) = 0; \\ iii) & M_y\,(r,L,t) = 0; \\ iv) & M_y\,(a,z,t) = 0; \\ v) & \left|M_y(r,z,t)\right| = M, \end{array}\right\} \qquad (30)$$

where r *is the space depicting the blockage* and z is the direction of flow and both are defined as below -

$$0 \le r < a; \quad 0 < z < L; \quad t > 0$$

Firstly, from the boundedness condition, $r = 0, Y_m(\mu r) \to -\infty$; to keep the solution finite, C_3 must be zero. Thus the solution becomes

$$M_y\,(r,z,t) = \left\{e^{-\lambda^2 Dt}\right\}\{C_2 J_0(\mu r)\}\{C_4 e^{\beta z} + C_5 e^{-\beta z}\} \qquad (31)$$

From the second boundary condition, we see that

$$M_y\,(r,0,t) = \left\{e^{-\lambda^2 Dt}\right\}\{J_0(\mu r)\}\{C_4 + C_5\} = 0 \qquad (32)$$

So that we must have $C_4 + C_5 = 0 \; or \; C_5 = -C_4$

then (31) becomes

$$M_y\,(r,z,t) = \left\{e^{-\lambda^2 Dt}\right\}\{J_0(\mu r)\}\{e^{\beta z} - e^{-\beta z}\} = 0 \qquad (33)$$

From the third condition we have

$$M_y\,(r, L, t) = \{e^{-\lambda^2 Dt}\}\{J_0(\mu r)\}\{e^{\beta L} - e^{-\beta L}\} = 0 \tag{34}$$

which can be satisfied with

$$e^{\beta L} - e^{-\beta L} = 0\,, \qquad \Rightarrow e^{\beta L}.\,e^{\beta L} = e^{-\beta L}.\,e^{\beta L} = 1 = e^{2k\pi i} \tag{35}$$

It follows that we must have

$$2\beta L = 2k\pi i \text{ or } \beta = \frac{k\pi i}{L} \quad k = 0,1,2, \dots \dots \tag{36}$$

Using this in equation (34), it becomes

$$M_y\,(r, L, t) = \{Ce^{-\lambda^2 Dt}\}\{J_0(\mu r)\} sin\frac{k\pi z}{L} = 0 \tag{37}$$

where C is a new constant.

From the fourth condition, we obtain

$$M_y\,(a, z, t) = \{Ce^{-\lambda^2 Dt}\}\{J_0(\mu a)\} sin\frac{k\pi z}{L} = 0 \tag{38}$$

which can be satisfied only if $\{J_0(\mu a)\} = 0$ (39)

$$\mu a = s_1, s_2, \dots. \tag{40}$$

$$\mu = \frac{s_1}{a}, \frac{s_2}{a}, \dots \dots. \tag{41}$$

where $\frac{s_m}{a}\,(m = 1,2, \dots.)$ is the positive root of the Bessel function $\{J_0(x)\} = 0$. Now from (17), (36) and (41), it follows that:

$$\lambda^2 = (\frac{s_m}{a})^2 - (\frac{k\pi i}{L})^2 = (\frac{s_m}{a})^2 + (\frac{k\pi}{L})^2 \tag{42}$$

so that a solution satisfying all the boundary conditions except the first is given by

$$M_y\,(r, z, t) = \left\{Ce^{-Dt(\frac{s_m}{a})^2 + (\frac{k\pi}{L})^2}\right\}\left\{J_0\left(\frac{s_m}{a}r\right)\right\} sin\frac{k\pi z}{L} \tag{43}$$

where $k = 1,2,3, \dots..; m = 1,2,3, \dots.$

Replacing $C\ by\ C_{km}$ and summing over $k\ and\ m$ we obtain by the superposition principle the solution

$$M_y\,(r, z, t) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left\{C_{km}e^{-Dt(\frac{s_m}{a})^2 + (\frac{k\pi}{L})^2}\right\}\left\{J_0\left(\frac{s_m}{a}r\right)\right\} sin\frac{k\pi z}{L} \tag{44}$$

The first condition in (30) now leads to

$$M_i\,(r,z) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\{C_{km}\}\left\{J_0\left(\frac{s_m}{a}r\right)\right\} sin\frac{k\pi z}{L} \tag{45}$$

This can be written as

$$M_i\,(r,z) = \sum_{k=1}^{\infty}[\sum_{m=1}^{\infty}\{C_{km}\}\left\{J_0\left(\frac{s_m}{a}r\right)\right\}] sin\frac{k\pi z}{L} = \sum_{k=1}^{\infty} b_k sin\frac{k\pi z}{L} \tag{46}$$

where $$b_k = \sum_{m=1}^{\infty}\{C_{km}\}\left\{J_0\left(\frac{s_m}{a}r\right)\right\} \tag{47}$$

It follows that b_k are the Fourier coefficients obtained when $M_i(r,z)$ is expanded into a Fourier sine series in z (r being kept constant).

Thus $$b_k = \frac{2}{1}\int_0^1 M_i(r,z) sin\frac{k\pi z}{L} dz \tag{48}$$

We now find C_{km} from the expansion in equation (46). Since b_k is a function of r this is simply the expansion of b_k into a Bessel series.

Consequently,

$$C_{km} = \frac{2}{J_1^2(\frac{s_m}{a})}\int_0^1 r b_k J_0\left(\frac{s_m}{a}r\right)\, dr \tag{49}$$

Using (47),

$$C_{km} = \frac{4}{J_1^2(\frac{s_m}{a})}\int_0^1\int_0^1 r M_i(r,z) J_0\left(\frac{s_m}{a}r\right) sin\frac{k\pi z}{L}\, drdz \tag{50}$$

The required solution is

$$M_y\,(\mathrm{r},z,t) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left\{C_{km}e^{-Dt(\frac{s_m}{a})^2+(\frac{k\pi}{L})^2}\right\}\left\{J_0\left(\frac{s_m}{a}r\right)\right\} sin\frac{k\pi z}{L}$$

with C_{km} in (49) as coefficient (51)

With the radio frequency (rf) field, the solution is

$$M_y\,(r,z,t) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left\{C_{km}e^{-Dt(\frac{s_m}{a})^2+(\frac{k\pi}{L})^2}\right\}\left\{J_0\left(\frac{s_m}{a}r\right)\right\} sin\frac{k\pi z}{L} + \frac{aF_o}{wT_o}\gamma\sin\,(wt) \tag{52}$$

Assume $M_i(r,z) = \sigma_0$, a constant.

$$C_{km} = \frac{4\sigma_0}{J_1^2(\frac{s_m}{a})}\int_0^1\int_0^1 r J_0\left(\frac{s_m}{a}r\right) sin\frac{k\pi z}{L} drdz \tag{53}$$

$$C_{km} = \frac{4\sigma_0}{J_1^2(\frac{s_m}{a})}\{\int_0^1 rJ_0(\frac{s_m}{a}r)dr \int_0^1 sin\frac{k\pi z}{L}dz \tag{54}$$

$$= \frac{4\sigma_0}{J_1^2(\frac{s_m}{a})}\left\{\frac{J_1\left(\frac{s_m}{a}\right)}{\frac{s_m}{a}}\right\}\{\frac{1-cosk\pi}{k\pi}\} \tag{55}$$

$$= \frac{4\sigma_0(1-cosk\pi)}{k\pi\frac{s_m}{a}J_1(\frac{s_m}{a})} \tag{56}$$

Substituting for C_{km} in equation (51)

$$M_y\,(r,z,t) = \frac{4\sigma_0}{\pi}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{(1-cosk\pi)}{k\pi\frac{s_m}{a}J_1(\frac{s_m}{a})}e^{-Dt(\frac{s_m}{a})^2+(\frac{k\pi}{L})^2)}\right\}\left\{J_0\left(\frac{s_m}{a}r\right)\right\}sin\frac{k\pi z}{L} \tag{57}$$

Finally, the solution for the magnetization of any molecule of the fluid at any point r and time t is given as:

$$M_y\,(r,z,t) = \frac{4\sigma_0}{\pi}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{(1-cosk\pi)}{k\frac{s_m}{a}J_1(\frac{s_m}{a})}e^{-Dt(\frac{s_m}{a})^2+(\frac{k\pi}{L})^2}\right\}\left\{J_0\left(\frac{s_m}{a}r\right)\right\}sin\frac{k\pi z}{L} + \frac{bF_o}{wT_o}\gamma\sin\,(wt) \tag{58}$$

4.0 Results and Discussion

The fluid under consideration is water and its diffusion coefficient which is almost the same as cerebrospinal fluid is $2.33x10^{-9}$. T_1 and T_2 values of water were used and the following substitution made for both free flow and partial blockage of the pipe:

$$s_m = \gamma G\delta\ and\ F_o = \frac{M_o}{T_1}$$

Based on our computational algorithm, Figures 2a and 2b were obtained for free flow and partial blockage respectively.

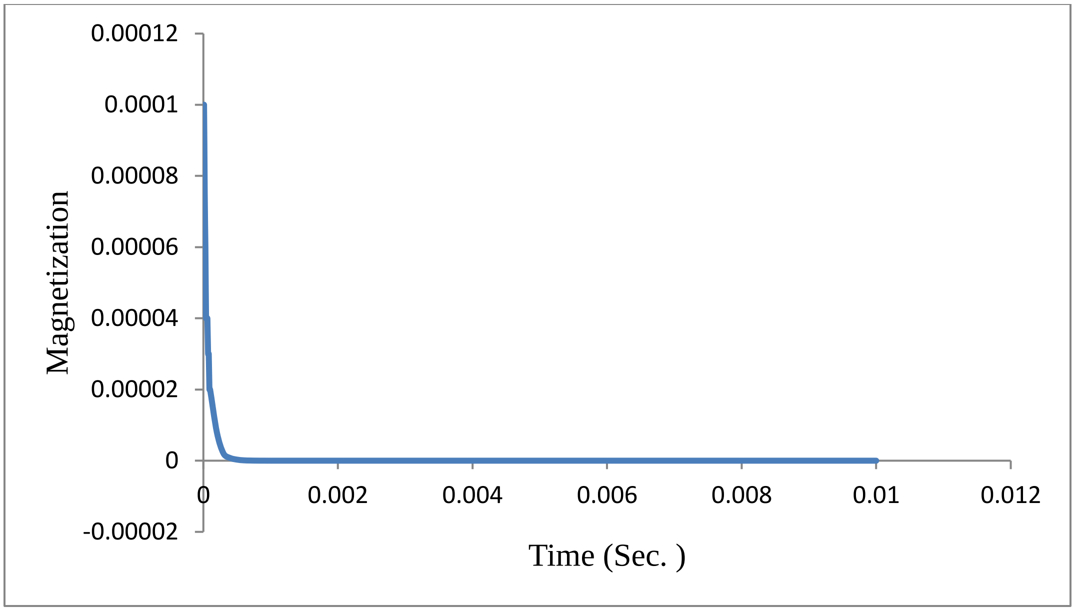

Figure 2a: Free flow of fluid (no blockage) inside the cylinder

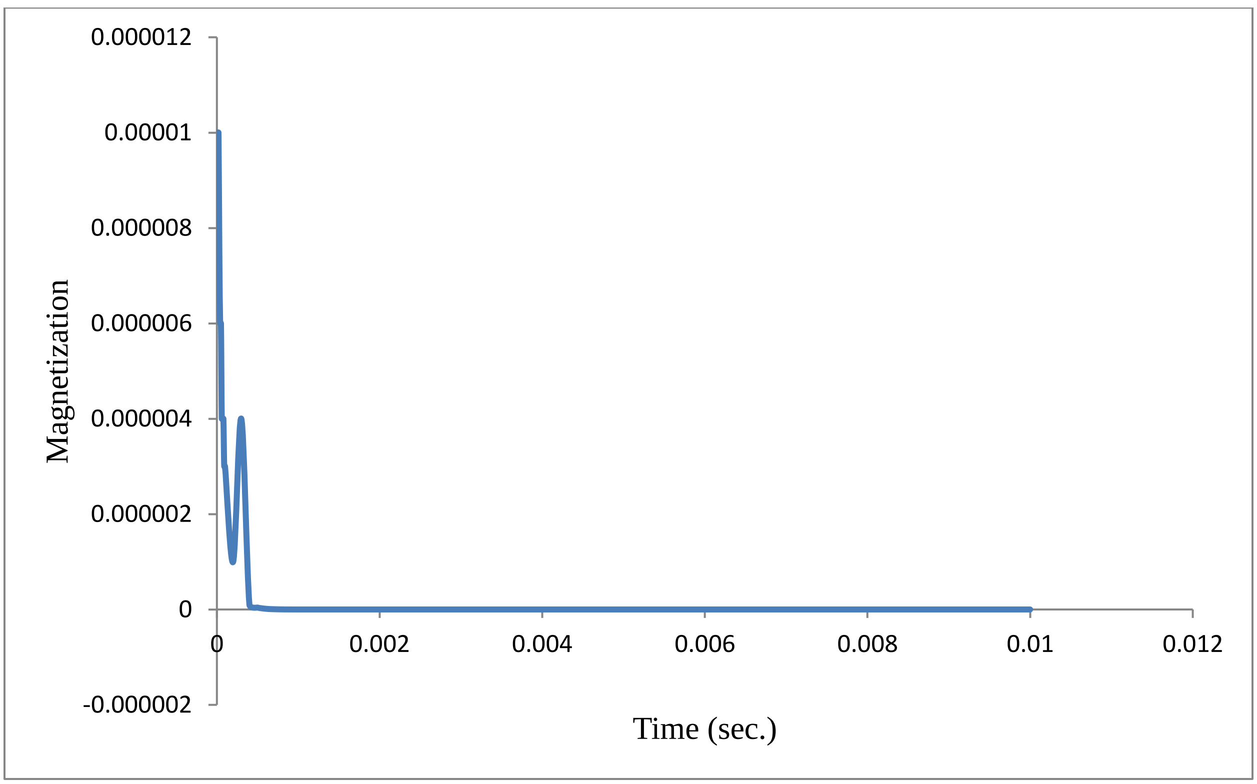

Figure 2b: Partial blockage of fluid inside the cylinder

5.0 Conclusion

The simple analytical expression obtained contains very important magnetic resonance flow parameters which can be useful for the non invasive analysis of fluid flow or blockage. The parameters are T_1, T_2 relaxation rates which are unique to each fluid. Also, the introduction of radio frequency (rf) pulse into the diffusion equation to make the fluid particles precess thereby giving off signal in a matter of microseconds is a great advantage of this study. For example, for free flow condition, the free induction decay (FID) curve is demonstrated signifying no blockage and the magnetization is between 0.0 – 0.006. However, during partial blockage, the magnetization reduces (signal loss) in value (0.0001 – 0.00004). Interestingly, both readings were registered between 0.0 and 0.0005 seconds! Unlike other methods that are slow and may take longer time, this quick identification of problems whenever it arises allows for immediate control and elicitation of appropriate solution before much damage is done. The analysis can also be useful in process industries where different network of pipes are used.

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