18 A MODIFIED SOR ITERATIVE SCHEME FOR SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS WITH M −MATRICES

[1]**Abdulrahman Ndanusa** and [2]**Kuluwa Adamu Al-Mustapha**
[1]*Department of Mathematics, Federal University of Technology, Minna, Nigeria*
[2]*Department of Mathematical Sciences, Baze University, Abuja, Nigeria*
Email: as.ndanusa@futminna.edu.ng

Abstract
This present work concerns the theoretical and numerical investigation of a new combination preconditioner applied to the successive overrelaxation (SOR) method, in order to speed up its convergence for solving large sparse linear algebraic systems whose coefficient matrix is an L −matrix. The structures and properties of some lower triangular, upper triangular and combination preconditioners are studied, and a new combination preconditioner is proposed. The results of comparison theorems and numerical experiments revealed that the proposed preconditioned SOR scheme is the best one among all corresponding results, as it maintains faster convergence rate.

Keywords: *Successive overrelaxation (SOR), Linear system, Preconditioner, L −matrix, Convergence.*

Introduction

In the mathematical modelling of natural and social sciences such as economic modelling, numerical weather forecasting, optimization, simulated nuclear explosion, electrical networks, oil and gas resource development, it is not uncommon to resort to partial differential equations. The approximation of such partial differential equations by finite element or finite differences more often than not leads to an associated large sparse linear system of equations. However, the direct solution methods such as Gaussian elimination is not always applicable because of enormous storage space requirement; hence the recourse to iterative solution methods such as Jacobi, Gauss-Seidel and Successive Overrelaxation (SOR) methods. Typically, the large sparse linear system of equations is expressed in matrix form as

$$Ax = b \quad (1)$$

where $A = (a_{i,j}) \in \mathbb{R}^{n\times n}$ is a square matrix while $b = (b_j), x = (x_j) \in \mathbb{R}^{n\times n}$ are known and unknown vectors respectively. It is assumed, for simplicity, without loss of generality, that the diagonal entries of A are unit elements and A has the usual splitting $A = I - E - F$, where $-E$ and $-F$ are strictly lower and strictly upper triangular matrices respectively. For any regular splitting

$$A = M - N \quad (2)$$

with $\det(M) \neq 0$ and $Q \neq 0$, then a linear stationary iterative method for the solution of (1) takes the form

$$x^{(n+1)} = Tx^{(n)} + k, n = 0,1,\cdots \quad (3)$$

where $T = M^{-1}N$ is the iteration matrix and $k = M^{-1}b$ is the iteration vector.

For $\omega \in \mathbb{R}\backslash\{0\}$, and based on the splitting

$$A = M_\omega - N_\omega = \frac{1}{\omega}(I - \omega E) - \frac{1}{\omega}[(1-\omega)I + \omega F] \quad (4)$$

the SOR method introduced by Young (1950) for solving (1) is defined by

$$x^{(n+1)} = \mathcal{L}_\omega x^{(n)} + (I - \omega E)^{-1}\omega b, n = 0,1,\cdots \quad (5)$$

where

$$\mathcal{L}_\omega = (I - \omega E)^{-1}\{(1-\omega)I + \omega F\}$$

is the SOR iteration matrix. The splitting (4) is also called the SOR splitting of A.

Convergence is a basic criterion required of any iterative method before it can be used to solve a linear system of equations. If A is nonsingular, then the stationary linear iteration (3) is guaranteed to converge if and only if the spectral radius of T, $\rho(T)$, is less than 1 (Ames, 1977). The convergence speed of the iterative method is determined by the magnitude of $\rho(T)$, and the smaller it is, the faster the method converges (Song, 2020). Therefore, in order to improve the convergence of an iterative method, there is the need to decrease the spectral radius of its iteration matrix. Preconditioning is one technique for achieving just that. It involves transformation of original linear system (1) into an equivalent preconditioned linear system with more favourable properties for iterative methods, by application of a transformation matrix P thus:

$$PAx = Pb \quad (6)$$

Here P, which is nonsingular, is called the preconditioner. The preconditioned linear system (6) has the same solution as the original system (1). The preconditioned linear system (6) with different forms of preconditioner P have been investigated have applied for Jacobi, Gauss-Seidel, SOR and AOR methods in Milaszewicz (1987), Gunawardena *et al.* (1991), Li and Evans (1994), Kohno *et al.* (1997), Li and Sun (2000), Kotakemori *et al.* (2002), Morimoto *et al.* (2004), Kotakemori *et al.* (1996), Ndanusa and Adeboye (2012), Bai and Wang (2015), Mayaki and Ndanusa (2019), Faruk and Ndanusa (2019), Abdullahi and Ndanusa (2020), Ndanusa (2020) and Ndanusa *et al.* (2020).

In this paper, an investigation of properties and behaviours of some lower triangular, upper triangular and combination preconditioners is undertaken. We propose a new combination preconditioner for the SOR method with convergence results that surpass corresponding schemes.

Materials and Methods

Preliminaries

A great many researchers have proposed the preconditioner P in (6) by choosing $P = I + Q$, $Q > 0$, where $I \in I^{n \times n}$, the set of $n \times n$ identity matrices, and Q is a sparse matrix whose nonzero entries are the negatives of the corresponding entries of A. Therefore, the corresponding matrix splitting is defined by $PA = \widetilde{M} - \widetilde{N}$. And the corresponding preconditioned iterative method is defined by

$$x^{(n+1)} = \tilde{T}x^{(n)} + \tilde{k}, n = 0,1,\cdots \quad (7)$$

where $\tilde{T} = \widetilde{M}^{-1}\widetilde{N}$ is the iteration matrix and $\tilde{k} = \widetilde{M}^{-1}b$ is the iteration vector.

Milaszewicz (1987) is a lower triangular preconditioner of the form $P = I + Q$, where

$$Q = (q_{ij}) = \begin{cases} -a_{i1}, i = 2, \cdots, n \\ 0, \text{otherwise} \end{cases}$$

The preconditioner of Morimoto *et al.* (2003) is another lower triangular preconditioner that attempts to provide the preconditioned effect on the last row of A. It takes the form $P = I + Q$, where Q is defined as

$$Q = (q_{nj}) = \begin{cases} -a_{nj}, 1 \leq j \leq n-1 \\ 0, \text{otherwise } 00 \end{cases}$$

An example of upper triangular preconditioner can be found in the works of Gunawardena *et al.* (1991) and Dehghan and Hajarian (2009) with corresponding Q matrices defined by

$$Q = (q_{ij}) = \begin{cases} -a_{ii+1}, i = 1,2, \cdots, n-1, j = i+1 \\ 0, \text{otherwise} \end{cases}$$

and

$$Q = (q_{in}) \begin{cases} -(a_{in} + \delta_i), \text{for } i = 1, \cdots, n-1 \\ 0, \text{otherwise} \end{cases}$$

respectively. Evans *et al.* (2001) proposed a lower triangular and an upper triangular preconditioner defined by

$$Q = (q_{ij}) = \begin{cases} -a_{1n} \\ 0, \text{otherwise} \end{cases}$$

and

$$Q = (q_{ij}) = \begin{cases} -a_{n1} \\ 0, \text{otherwise } 00 \end{cases}$$

respectively. The preconditioners of Ndanusa and Adeboye (2012) and Abdullahi and Ndanusa (2020) with the Q matrix defined by

$$Q = (q_{ij}) = \begin{cases} -a_{i1}, i = 2, \cdots, n \\ -a_{i,i+1}, i = 1, \cdots, n-1 \\ 0, \text{otherwise} \end{cases}$$

and

$$Q = (q_{ij}) = \begin{cases} -a_{1n} \\ -a_{i1}, i = 2, \cdots, n \\ -a_{i,i+1}, i = 1, \cdots, n-1 \\ 0, \text{otherwise} \end{cases}$$

respectively, are examples of combination preconditioners.

The Proposed Preconditioned SOR Schemes

Following (6) and a careful analysis of the foregoing lower triangular, upper triangular and combination preconditioners, a new combination preconditioner $P = I + Q$ is proposed, where Q is defined by

$$Q = (q_{ij}) \begin{cases} -a_{ij};\ j = i+1, j = i-1 \\ -a_{i1};\ i = 3(1)n \\ -a_{1n} \\ 0;\ \text{otherwise} \end{cases} \quad (8)$$

The application of the preconditioner P to the linear system (1) results in the corresponding preconditioned system (6), which is more succinctly written as

$$\tilde{A}x = \tilde{b} \quad (9)$$

where,

$$\begin{aligned} \tilde{A} = PA &= (I + Q)(I - E - F) \\ &= I - E - F + Q - QE - QF \\ &= I - E - F - E_Q - F_Q + D_* - E_* - F_* \end{aligned}$$

where $Q = -E_Q - F_Q$ and $-QE - QF = D_* - E_* - F_*$. Therefore,

$$\tilde{A} = (I + D_*) - (E + E_Q + E_*) - (F + F_Q + F_*)$$

That is, $\tilde{A} = \tilde{D} - \tilde{E} - \tilde{F}$, where $\tilde{D} = I + D_*$, $\tilde{E} = E + E_Q + E_*$ and $\tilde{F} = F + F_Q + F_*$. The effect of the preconditioner P on the coefficient matrix A is the reflected on the corresponding preconditioned coefficient matrix $\tilde{A}$. Unlike the behaviour of many preconditioners where some entries of the coefficient matrix are eliminated from the preconditioned coefficient matrix, no entry gets eliminated in $\tilde{A}$; rather, each entry of the new matrix is a scaled down entry of the corresponding original matrix. Now, from (9) results the following

$$\omega\tilde{A}x = \omega\tilde{b} \quad (10)$$

Thus,

$$\begin{aligned}\omega\tilde{A} &= \omega(\tilde{D} - \tilde{E} - \tilde{F}) \\ &= \omega(I + D_* - \tilde{E} - \tilde{F}) = \omega I + \omega D_* - \omega\tilde{E} - \omega\tilde{F} \\ &= I - \omega\tilde{E} + \omega D_* - I + \omega I - \omega\tilde{F} \\ &= [I - \omega(\tilde{E} - D_*)] - [(1-\omega)I + \omega\tilde{F}] \\ &= \tilde{M} - \tilde{N}\end{aligned}$$

And the preconditioned SOR scheme is takes the form

$$x^{(k+1)} = [I - \omega(\tilde{E} - D_*)]^{-1}\{(1-\omega)I + \omega\tilde{F}\}x^{(k)} + [I - \omega(\tilde{E} - D_*)]^{-1}\omega\tilde{b}$$

That is,

$$x^{(k+1)} = \mathcal{L}_{\omega(1)}x^{(k)} + [I - \omega(\tilde{E} - D_*)]^{-1}\omega\tilde{b} \quad (11)$$

where $\mathcal{L}_{\omega(1)} = [I - \omega(\tilde{E} - D_*)]^{-1}\{(1-\omega)I + \omega\tilde{F}\}$ is the preconditioned SOR iteration matrix. Also, from (10)

$$\begin{aligned}\omega\tilde{A} &= \omega(\tilde{D} - \tilde{E} - \tilde{F}) = \omega\tilde{D} - \omega\tilde{E} - \omega\tilde{F} \\ &= (\tilde{D} - \omega\tilde{E}) - [(1-\omega)\tilde{D} + \omega\tilde{F}]\end{aligned}$$

is another splitting of the preconditioned coefficient matrix $\omega\tilde{A} = \tilde{M} - \tilde{N}$, where $\tilde{M} = (\tilde{D} - \omega\tilde{E})$ and $\tilde{N} = [(1-\omega)\tilde{D} + \omega\tilde{F}]$, from whence the second preconditioned SOR iterative scheme is defined as

$$x^{(k+1)} = \mathcal{L}_{\omega(2)}x^{(k)} + (\tilde{D} - \omega\tilde{E})^{-1}\omega\tilde{b} \quad (12)$$

where $\mathcal{L}_{\omega(2)} = (\tilde{D} - \omega\tilde{E})^{-1}[(1-\omega)\tilde{D} + \omega\tilde{F}]$.

Convergence Analysis

Lemma 1 (Varga (1981)) Let $A \geq 0$ be an irreducible $n \times n$ matrix. Then,

i. A has a positive real eigenvalue equal to its spectral radius.
ii. To $\rho(A)$ there corresponds an eigenvector $x > 0$.
iii. $\rho(A)$ increases when any entry of A increases.
iv. $\rho(A)$ is a simple eigenvalue of A.
v.

Lemma 2 (Varga (1981))

i. Let A be a nonnegative matrix. Then
If $\alpha x \leq Ax$ for some nonnegative vector $x, x \neq 0$, then $\alpha \leq \rho(A)$.
ii. If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector x, then $\alpha \leq \rho(A) \leq \beta$ and x is a positive vector.

Lemma 3 (Li and Sun(2000)) Let $A = M - N$ be an M −splitting of A. Then the splitting is convergent, i.e., $\rho(M^{-1}N < 1)$, if and only if A is a nonsingular M −matrix.

Theorem 1 Let $\mathcal{L}_\omega = (I - \omega E)^{-1}\{(1-\omega)I + \omega F\}$, $\mathcal{L}_{\omega(1)} = [I - \omega(\tilde{E} - D_*)]^{-1}\{(1-\omega)I + \omega\tilde{F}\}$ and $\mathcal{L}_{\omega(2)} = (\tilde{D} - \omega\tilde{E})^{-1}[(1-\omega)\tilde{D} + \omega\tilde{F}]$ be the SOR, first preconditioned SOR and second preconditioned SOR iteration matrices respectively. If A is an irreducible M −matrix with $0 \leq a_{1,2}a_{2,1} + a_{1,n}a_{n,1} < 1$, $0 \leq a_{1,n}a_{n,1} + a_{n-1,n}a_{n,n-1} < 1$, $0 \leq a_{1,2}a_{2,1} + a_{2,3}a_{3,2} < 1$, $0 \leq a_{i,1}a_{1,i} + a_{i-1,i}a_{i,i-1} + a_{i,i+1}a_{i+1,i} < 1$, $i = 3,4,\cdots(n-1)$ and $0 < \omega < 1$, then $\mathcal{L}_\omega$, $\mathcal{L}_{\omega(1)}$ and $\mathcal{L}_{\omega(2)}$ are nonnegative and irreducible matrices.

Proof: The matrices $\mathcal{L}_\omega$, $\mathcal{L}_{\omega(1)}$ and $\mathcal{L}_{\omega(2)}$ reduce to I when $\omega = 0$. For $\omega < 0$ and $\omega > 1$, negative entries appear in these matrices. Thus, the range of values of ω that ensures nonnegativity of these matrices is, $0 < \omega < 1$.
Given $0 < \omega < 1$, $(1-\omega)I + \omega F \geq 0$, since F≥ 0. Also, $(I - \omega E)^{-1} = I + \omega E + \omega^2E^2 + \cdots + \omega^{n-1}E^{n-1} \geq 0$, since $E \geq 0$. Hence $\mathcal{L}_\omega = (I - \omega E)^{-1}\{(1-\omega)I + \omega F\} \geq 0$, that is, a nonnegative matrix. For $0 < \omega < 1$,

$$\begin{aligned}\mathcal{L}_\omega &= [I + \omega E + \omega^2E^2 + \cdots + \omega^{n-1}E^{n-1}][(1-\omega)I + \omega F] \\ &= (1-\omega)I + \omega(1-\omega)E + \omega F + \omega^2EF + \omega^2(1-\omega)E^2 + \omega^3E^2F + \cdots \\ &= (1-\omega)I + \omega(1-\omega)E + \omega F + \text{nonnegative terms}\end{aligned}$$

Since $A = I - E - F$ is irreducible, so also is the matrix $(1-\omega)I + \omega(1-\omega)E + \omega F$ because the coefficients of I, E and F are not zero and less than 1 in absolute value. Hence, $\mathcal{L}_\omega$ is an irreducible matrix. The iteration matrix $\mathcal{L}_{\omega(1)}$ is defined by

$$\mathcal{L}_{\omega(1)} = \left[I - \omega\left(\tilde{E} - D_*\right)\right]^{-1}\{(1-\omega)I + \omega\tilde{F}\}$$

Since $\tilde{E} \geq 0$, $F \geq 0$, $-D_* \geq 0$, and for $0 < \omega < 1$, $(1-\omega)I + \omega\tilde{F} \geq 0$ and $[I - \omega(\tilde{E} - D_*]^{-1} = I + \omega(\tilde{E} - D_*) + \omega^2\left(\tilde{E} - D_*\right)^2 + \cdots + \omega^{n-1}\left(\tilde{E} - D_*\right)^{n-1} \geq 0$.Hence, $\mathcal{L}_{\omega(1)} = \left[I - \omega\left(\tilde{E} - D_*\right)\right]^{-1}\{(1-\omega)I + \omega\tilde{F}\} \geq 0$, and therefore $\mathcal{L}_{\omega(1)}$ is a nonnegative matrix.

Let the coefficient matrix $A = I - E - F$ be an irreducible matrix; then the preconditioned matrix $\tilde{A}$ is defined by

$$\begin{aligned}
\tilde{A} = PA = (I+S)A &= \left(I - E_Q - F_Q\right)A \\
&= \left(I - E_Q - F_Q\right)(I - E - F) \\
&= I - E_Q - F_Q - E + E_QE + F_QE - F + E_QF + F_QF \\
&= I - E_Q - F_Q - E + E_QE - (F_QE)_E - (F_QE)_F - F - (E_QF)_E - (E_QF)_F + F_QF \\
&= I - E - E_Q + E_QQ - (E_QF)_E - (F_QE)_E - F - F_Q + F_QF - (Q_QE)_F - (E_QF)_F \\
&= I - \left(E + E_Q - E_{Qs}E + \left(E_QF\right)_E + \left(F_QE\right)_E\right) - \left(F + F_Q - F_QF + \left(F_QE\right)_F + \left(E_QF\right)_F\right) \\
&= I - \tilde{E} - \tilde{F}
\end{aligned}$$

where $\tilde{E} = E + E_Q - E_QE + \left(E_QF\right)_E + \left(F_QE\right)_E$,$\tilde{F} = F + F_Q - F_QF + \left(F_QE\right)_F + \left(E_QF\right)_F$ and $-(X)_E$ and $-(X)_F$ denote the strictly lower and strictly upper parts of the matrix X respectively. Since A is irreducible, it is obvious that $\tilde{A} = I - \tilde{E} - \tilde{F}$ is irreducible, since it inherits the nonzero structure of the irreducible matrix A. Now,

$$\begin{aligned}
\mathcal{L}_{\omega(1)} &= \left[I - \omega\left(\tilde{E} - D_*\right)\right]^{-1}\{(1-\omega)I + \omega\tilde{F}\} \\
&= [I + \omega(\tilde{E} - D_*) + \omega^2\left(\tilde{E} - D_*\right)^2 + \cdots + \omega^{n-1}\left(\tilde{E} - D_*\right)^{n-1}]\{(1-\omega)I + \omega\tilde{F}\} \\
&= (1-\omega)I + \omega\tilde{F} + \omega(1-\omega)(\tilde{E} - D_*) + \omega^2(\tilde{E} - D_*)\tilde{F} + \omega^2(1-\omega)\left(\tilde{E} - D_*\right)^2 + \cdots \\
&= (1-\omega)I + \omega(1-\omega)\tilde{E} + \omega\tilde{F} + \omega(1-\omega)(-D_*) + \omega^2\left(\tilde{E} - D_*\right)\tilde{F} + \omega^2(1-\omega)\left(\tilde{E} - D_*\right)^2 + \cdots \\
&= (1-\omega)I + \omega(1-\omega)\tilde{E} + \omega\tilde{F} + \text{nonnegative terms}
\end{aligned}$$

Since $\tilde{A} = I - \tilde{E} - \tilde{F}$ is irreducible, it implies, for $0 < \omega < 1$, the matrix $(1-\omega)I + \omega(1-\omega)\tilde{E} + \omega\tilde{F}$ is also irreducible, because the coefficients of I, $\tilde{E}$ and $\tilde{F}$ are different from zero and less than one in absolute value. Therefore, the matrix $\mathcal{L}_{\omega(1)} = \left[I - \omega\left(\tilde{E} - D_*\right)\right]^{-1} \times \{(1-\omega)I + \omega\tilde{F}\}$ is irreducible. Hence $\mathcal{L}_{\omega(1)}$ is a nonnegative and irreducible matrix.

Similarly,

$$\begin{aligned}
\mathcal{L}_{\omega(2)} &= \left(\widetilde{D} - \omega\tilde{E}\right)^{-1}\left[(1-\omega)\widetilde{D} + \omega\tilde{F}\right] \\
&= [\widetilde{D}\left(I - \omega\widetilde{D}^{-1}\tilde{E}\right)]^{-1}\left[(1-\omega)\widetilde{D} + \omega\tilde{F}\right] \\
&= [\widetilde{D}\left(I - \omega\widetilde{D}^{-1}\tilde{E}\right)]^{-1}\left[(1-\omega)\widetilde{D} + \omega\tilde{F}\right] \\
&= \left(I - \omega\widetilde{D}^{-1}\tilde{E}\right)^{-1}\widetilde{D}^{-1}\left[(1-\omega)\widetilde{D} + \omega\tilde{F}\right] \\
&= \left(I - \omega\widetilde{D}^{-1}\tilde{E}\right)^{-1}\left[(1-\omega)I + \omega\widetilde{D}^{-1}\tilde{F}\right] \\
&= [I + \omega\widetilde{D}^{-1}\tilde{E} + \omega^2(\widetilde{D}^{-1}\tilde{E})^2 + \cdots + \omega^{n-1}(\widetilde{D}^{-1}\tilde{E})^{n-1}]\left[(1-\omega)I + \omega\widetilde{D}^{-1}\tilde{F}\right] \\
&= (1-\omega)I + \omega(1-\omega)\widetilde{D}^{-1}\tilde{E} + \omega\widetilde{D}^{-1}\tilde{F} + \text{nonnegative terms}
\end{aligned}$$

Similarly, it is conclusive that $\mathcal{L}_{\omega(2)} = \left(\widetilde{D} - \omega\tilde{E}\right)^{-1}\left[(1-\omega)\widetilde{D} + \omega\tilde{F}\right]$ is a nonnegative and irreducible matrix.

Theorem 2: Let $\mathcal{L}_\omega = (I - \omega E)^{-1}\{(1-\omega)I + \omega F\}$ and $\mathcal{L}_{\omega(1)} = \left[I - \omega\left(\tilde{E} - D_*\right)\right]^{-1}\{(1-\omega)I + \omega\tilde{F}\}$ be the SOR and the preconditioned SOR iteration matrices respectively. If $0 < \omega < 1$ and A is an irreducible M −matrix with $0 \leq a_{1,2}a_{2,1} + a_{1,n}a_{n,1} < 1$, $0 \leq a_{1,n}a_{n,1} + a_{n-1,n}a_{n,n-1} < 1$, $0 \leq a_{1,2}a_{2,1} + a_{2,3}a_{3,2} < 1$, $0 \leq a_{i,1}a_{1,i} + a_{i-1,i}a_{i,i-1} + a_{i,i+1}a_{i+1,i} < 1$, $i = 3,4,\cdots(n-1)$, then

a) $\rho\left(\mathcal{L}_{\omega(1)}\right) < \rho(\mathcal{L}_\omega)$, if $\rho(\mathcal{L}_\omega) < 1$
b) $\rho\left(\mathcal{L}_{\omega(1)}\right) = \rho(\mathcal{L}_\omega)$, if $\rho(\mathcal{L}_\omega) = 1$
c) $\rho\left(\mathcal{L}_{\omega(1)}\right) > \rho(\mathcal{L}_\omega)$, if $\rho(\mathcal{L}_\omega) > 1$

Proof: It is established in Theorem 1 that the $\mathcal{L}_\omega$ and $\mathcal{L}_{\omega(1)}$ are nonnegative and irreducible matrices. Now, suppose that $\rho(\mathcal{L}_\omega) = \lambda$, then there exists a positive vector $x = (x_1, x_2, \cdots, x_n)$ such that

$$\mathcal{L}_\omega x = \lambda x$$

which implies

$$(I - \omega E)^{-1}\{(1-\omega)I + \omega F\}x = \lambda x$$

$$(1-\omega)I+\omega F=\lambda(I-\omega E)$$

And for this $x>0$

$$\mathcal{L}_{\omega(1)}x-\lambda x=\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\{(1-\omega)I+\omega\tilde{F}\}x-\lambda x$$
$$=\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\{(1-\omega)I+\omega\tilde{F}\}x-\lambda\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\left[I-\omega\left(\tilde{E}-D_*\right)\right]x$$
$$=\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\{(1-\omega)I+\omega\tilde{F}-\lambda\omega D_*-\lambda(I-\omega\tilde{E})\}x$$
$$=\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\{(1-\omega-\lambda)I+\omega\tilde{F}+\lambda\omega\tilde{E}-\gamma\omega D_*\}x$$
$$=\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\left\{(1-\omega-\lambda)I+\omega(F+F_Q+F_*)+\lambda\omega(E+E_Q+E_*)-\lambda\omega D_*\right\}x$$
$$=\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\left\{(1-\omega-\lambda)I+(\omega F+\lambda\omega E)+\left(\omega F_Q+\omega E_Q\right)+\left(\lambda\omega E_Q-\omega E_Q\right)+(-\omega D_*+\omega E_*+\omega F_*)\right.$$
$$\left.+(\lambda\omega E_*-\omega E_*)+(-\lambda\omega D_*+\omega D_*)\right\}x$$
$$=\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\left\{-\omega Q+\omega QE+\omega QF+(\lambda-1)\omega E_Q+(\lambda-1)\lambda\omega E_*+(\lambda-1)(-D_*)\right\}x$$
$$=\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\left\{\omega(\lambda-1)\left(E_*+E_Q-D_*\right)+Q-\omega Q-Q+\omega QE+\omega QF\right\}x$$
$$=\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\left\{\omega(\lambda-1)\left(E_*+E_Q-D_*\right)+(1-\omega)Q-Q(I-\omega)E+\omega QF\right\}x$$
$$=\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\left\{\omega(\lambda-1)\left(E_*+E_Q-D_*\right)+Q[(1-\omega)I+\omega F-(I-\omega)E]\right\}x$$
$$=\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\left\{\omega(\lambda-1)\left(E_*+E_Q-D_*\right)+Q[\lambda(I-\omega E)-(I-\omega)E]\right\}x$$
$$=\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\left\{\omega(\lambda-1)\left(E_*+E_Q-D_*\right)+\lambda Q(I-\omega E)-Q(I-\omega)E\right\}x$$
$$=(\lambda-1)\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\left\{\omega\left(E_*+E_Q-D_*\right)+Q(I-\omega E)\right\}x$$
$$=[(\lambda-1)/\lambda]\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\left\{\lambda\omega\left(E_*+E_Q-D_*\right)+(1-\omega)Q+\omega QF\right\}x$$

Suppose $T=Vx$, where $V=\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\left\{\lambda\omega\left(E_*+E_Q-D_*\right)+(1-\omega)Q+\omega QF\right\}$. Then $V=\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\left\{\lambda\omega\left(E_*+E_Q-D_*\right)+(1-\omega)Q+\omega QF\right\}\geq 0$, since $\lambda\omega(E_*-D_*)\geq 0$, $\lambda\omega E_Q+(1-\omega)Q\geq 0$ and $\omega QF\geq 0$. Also, $\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}=I+\omega(\tilde{E}-D_*)+\omega^2\left(\tilde{E}-D_*\right)^2+\cdots+\omega^{n-1}\left(\tilde{E}-D_*\right)^{n-1}\geq 0$, since $\tilde{E}\geq 0$ and $-D_*\geq 0$. Therefore, $V=\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\left\{\lambda\omega\left(E_*+E_Q-D_*\right)+(1-\omega)Q+\omega QF\right\}\geq 0$. Consequently, $T=\left[I-\omega\left(\tilde{E}-D_*\right)\right]^{-1}\left\{\lambda\omega\left(E_*+E_Q-D_*\right)+(1-\omega)Q+\omega QF\right\}x\geq 0$, since $x>0$.

a) If $\lambda<1$, then $\mathcal{L}_{\omega(1)}x-\lambda x\leq 0$ but not equal to 0. Therefore, $\mathcal{L}_{\omega(1)}x\leq\lambda x$. Hence,
$$\rho\left(\mathcal{L}_{\omega(1)}\right)<\lambda=\rho(\mathcal{L}_\omega)$$
b) If $\lambda=1$, then $\mathcal{L}_{\omega(1)}x-\lambda x=0$. Therefore, $\mathcal{L}_{\omega(1)}x=\lambda x$. Hence,
$$\rho\left(\mathcal{L}_{\omega(1)}\right)=\lambda=\rho(\mathcal{L}_\omega)$$
c) If $\lambda>1$, then $\mathcal{L}_{\omega(1)}x-\lambda x\geq 0$ but not equal to 0. Therefore, $\mathcal{L}_{\omega(1)}x\geq\lambda x$. Hence,
$$\rho\left(\mathcal{L}_{\omega(1)}\right)>\lambda=\rho(\mathcal{L}_\omega)$$

Theorem 3: Let $\mathcal{L}_\omega=(I-\omega E)^{-1}\{(1-\omega)I+\omega F\}$ and $\mathcal{L}_{\omega(2)}=\left(\tilde{D}-\omega\tilde{E}\right)^{-1}[(1-\omega)\tilde{D}+\omega\tilde{F}]$ be the SOR and preconditioned SOR iteration matrices respectively. If $0<\omega<1$ and $A\in\mathbb{R}^{nxn}$ is an irreducible $M-$matrix with $0\leq a_{1,2}a_{2,1}+a_{1,n}a_{n,1}<1$, $0\leq a_{1,n}a_{n,1}+a_{n-1,n}a_{n,n-1}<1$, $0\leq a_{1,2}a_{2,1}+a_{2,3}a_{3,2}<1$, $0\leq a_{i,1}a_{1,i}+a_{i-1,i}a_{i,i-1}+a_{i,i+1}a_{i+1,i}<1$, $i=3,4,\cdots(n-1)$, then

a) $\rho\left(\mathcal{L}_{\omega(2)}\right)<\rho(\mathcal{L}_\omega)$, if $\rho(\mathcal{L}_\omega)<1$
b) $\rho\left(\mathcal{L}_{\omega(2)}\right)=\rho(\mathcal{L}_\omega)$, if $\rho(\mathcal{L}_\omega)=1$
c) $\rho\left(\mathcal{L}_{\omega(2)}\right)>\rho(\mathcal{L}_\omega)$, if $\rho(\mathcal{L}_\omega)>1$

Proof: Theorem 1 established that $\mathcal{L}_\omega$ and $\mathcal{L}_{\omega(2)}$ are nonnegative and irreducible matrices. Let $\rho(\mathcal{L}_\omega)=\lambda$, then there exists a positive vector $x=(x_1,x_2,\cdots,x_n)^T$, such that
$$\mathcal{L}_\omega x=\lambda x$$
Or,
$$(I-\omega E)^{-1}\{(1-\omega)I+\omega F\}x=\lambda x$$
$$(1-\omega)I+\omega F=\gamma(I-\omega E)$$
Therefore, for this $x>0$,
$$\mathcal{L}_{\omega(2)}x-\lambda x=\left(\tilde{D}-\omega\tilde{E}\right)^{-1}\{(1-\omega)\tilde{D}+\omega\tilde{F}\}x-\lambda x$$
$$=\left(\tilde{D}-\omega\tilde{E}\right)^{-1}\{(1-\omega)\tilde{D}+\omega\tilde{F}\}x-\left(\tilde{D}-\omega\tilde{E}\right)^{-1}\left(\tilde{D}-\omega\tilde{E}\right)\lambda x$$
$$=\left(\tilde{D}-\omega\tilde{E}\right)^{-1}\{(1-\omega)\tilde{D}+\omega\tilde{F}-\lambda(\tilde{D}-\omega\tilde{E})\}x$$
$$=\left(\tilde{D}-\omega\tilde{E}\right)^{-1}\{(1-\omega-\lambda)\tilde{D}+\lambda\omega\tilde{E}+\omega\tilde{F}\}x$$
$$=\left(\tilde{D}-\omega\tilde{E}\right)^{-1}\{(1-\omega-\lambda)(I+D_*)+\lambda\omega(E+E_Q+E_*)+\omega(F+F_Q+F_*)\}x$$

$$= \left(\tilde{D} - \omega\tilde{E}\right)^{-1}\{(1-\omega-\lambda)D_* + \lambda\omega E_* + \lambda\omega E_Q + \lambda\omega E + \omega F_Q + \omega F_* + (1-\omega-\lambda)I + \omega F\}x$$
$$= \left(\tilde{D} - \omega\tilde{E}\right)^{-1}\{(1-\omega-\lambda)D_* + \lambda\omega E_* + \lambda\omega E_Q + \omega F_Q + \omega F_*\}x$$
$$= \left(\tilde{D} - \omega\tilde{E}\right)^{-1}\{(1-\omega-\lambda)D_* + \lambda\omega E_* + \lambda\omega E_Q + \omega F_Q + \omega F_*\}x$$
$$= \left(\tilde{D} - \omega E\right)^{-1}\{(\lambda-1)(-D_*) + (\lambda-1)\omega E_* - \omega(D_* - E_* - F_*) + \lambda\omega E_Q + \omega F_Q\}x$$
$$= \left(\tilde{D} - \omega\tilde{E}\right)^{-1}\{(\lambda-1)(-D_* + \omega E_*) + \omega QE + \omega QF + (\lambda-1)\omega E_Q + \omega(E_Q + F_Q)\}x$$
$$= \left(\tilde{D} - \omega\tilde{E}\right)^{-1}\{(\lambda-1)(-D_* + \omega E_* + \omega E_Q) + (1-\omega)Q + \omega QF - Q(I-\omega E)\}x$$
$$= \left(\tilde{D} - \omega\tilde{E}\right)^{-1}\{(\lambda-1)(-D_* + \omega E_* + \omega E_Q) + Q[(1-\omega)I + \omega F] - Q(I-\omega E)\}x$$
$$= \left(\tilde{D} - \omega\tilde{E}\right)^{-1}\{(\lambda-1)\left(-D_* + \omega E_* + \omega E_Q\right) + \lambda Q(I-\omega E) - Q(I-\omega E)\}x$$
$$= \left(\tilde{D} - \omega\tilde{E}\right)^{-1}\{(\lambda-1)\left(-D_* + \omega E_* + \omega E_Q\right) + (\lambda-1)Q(I-\omega E)\}x$$
$$= (\lambda-1)\left(\tilde{D} - \omega\tilde{E}\right)^{-1}\{\left(-D_* + \omega E_* + \omega E_Q\right) + [(\lambda-1)Q(I-\omega E)]/\lambda\}x$$
$$= [(\lambda-1)/\lambda]\left(\tilde{D} - \omega\tilde{E}\right)^{-1}\{\lambda\left(-D_* + \omega E_* + \omega E_Q\right) + (1-\omega)Q + \omega QF\}x$$

Let $T = Vx$, where $V = \left(\tilde{D} - \omega\tilde{E}\right)^{-1}\{\lambda\left(-D_* + \omega E_* + \omega E_Q\right) + (1-\omega)Q + \omega QF\}$. It is obvious that $\lambda\left(-D_* + \omega E_* + \omega E_Q\right) + (1-\omega)Q + \omega QF \geq 0$, $(1-\omega)Q \geq 0$ and $\lambda\left(-D_* + \omega E_* + \omega E_Q\right) \geq 0$. Since $\tilde{D}$ is a nonsingular matrix, we let $\tilde{D} - \omega\tilde{E}$ be a splitting of some matrix J, i.e., $J = \tilde{D} - \omega\tilde{E}$. Also, $\tilde{D}$ is an $M-$matrix and $\omega\tilde{E} \geq 0$. Thus, $J = \tilde{D} - \omega\tilde{E}$ is an $M-$splitting. Now, $\omega\tilde{D}^{-1}\tilde{E}$ is a strictly lower triangular matrix, and by implication its eigenvalues lie on its main diagonal; in this case they are all zeros. Therefore, $\rho\left(\omega\tilde{D}^{-1}\tilde{E}\right) = 0$. since $\rho\left(\omega\tilde{D}^{-1}\tilde{E}\right) < 1$, $J = \tilde{D} - \omega\tilde{E}$ is a convergent splitting. By the foregoing, $J = \tilde{D} - \omega\tilde{E}$ is an $M-$splitting and $\rho\left(\omega\tilde{D}^{-1}\tilde{E}\right) < 1$, we employ Lemma 3 to establish that J is an $M-$matrix. Since J is an $M-$matrix, by definition, $J^{-1} = (\tilde{D} - \omega\tilde{E})^{-1} \geq 0$. Thus, $V \geq 0$ and $T \geq 0$.

(i) If $\lambda < 1$, then $\mathcal{L}_{\omega(2)}x - \lambda x \leq 0$ but not equal to 0. Therefore, $\mathcal{L}_{\omega(2)}x \leq \lambda x$. From Lemma 2, we have $\rho\left(\mathcal{L}_{\omega(2)}\right) < \lambda = \rho(\mathcal{L}_\omega)$.

(ii) If $\gamma = 1$, then $\mathcal{L}_{\omega(2)}x - \gamma y = 0$. Therefore, $\mathcal{L}_{\omega(2)}x = \lambda x$. From Lemma 2, we have $\rho\left(\mathcal{L}_{\omega(2)}\right) = \lambda = \rho(\mathcal{L}_\omega)$.

(iii) If $\lambda > 1$, then $\mathcal{L}_{\omega(2)}x - \lambda x \geq 0$ but not equal to 0. Therefore, $\mathcal{L}_{\omega(2)}x \geq \lambda x$. From Lemma 2, we have $\rho\left(\mathcal{L}_{\omega(2)}\right) > \lambda = \rho(\mathcal{L}_\omega)$.

(iv)

Numerical Experiments

Sample problems are presented in order to further validate the convergence analysis established by the theorems advanced.

***Problem* 1** Let the coefficient matrix A of the linear system (1) be given by the following 5×5 matrix. This problem is a numerical example that can found in Huang *et al.* (2016).

$$A = \begin{pmatrix} 1 & -0.2 & -0.3 & -0.1 & -0.2 \\ -0.1 & 1 & -0.1 & -0.3 & -0.1 \\ -0.2 & -0.1 & 1 & -0.1 & -0.2 \\ -0.2 & -0.1 & -0.1 & 1 & -0.3 \\ -0.1 & -0.2 & -0.2 & -0.1 & 1 \end{pmatrix}$$

Problem 2 (Huang *et al.* (2016) Let the coefficient matrix A of the linear system (1) be given by the 7×7 matrix

$$A = \begin{pmatrix} 1 & q & r & s & q & r & s \\ s & 1 & q & r & s & q & r \\ q & s & 1 & q & r & s & q \\ r & q & s & 1 & q & r & s \\ s & r & q & s & 1 & q & r \\ q & s & r & q & s & 1 & q \\ r & q & s & r & q & s & 1 \end{pmatrix}$$

where $q = -p/n$, $r = -p/(n+1)$, $s = -p/(n+2)$, $n = 7$ and $p = 1$.

Results and Discussion

The results of problems 1 and 2 are computed with the aid of Maple 2019 software package and presented in Tables I and II respectively. In the tables, the notations $\rho(\mathcal{L}_\omega)$, $\rho\left(\mathcal{L}_{\omega(1)}\right)$, $\rho\left(\mathcal{L}_{\omega(2)}\right)$ and $\rho\left(\mathcal{L}_{\omega(N)}\right)$ represent the spectral radius of SOR, preconditioned SOR (11), preconditioned SOR (12) and preconditioned SOR method of Ndanusa and Adeboye (2012), respectively.

Table 1: Comparison of results for Problem 1

ω	$\rho(\mathcal{L}_{\omega})$	$\rho(\mathcal{L}_{\omega(N)})$	$\rho(\mathcal{L}_{\omega(1)})$	$\rho(\mathcal{L}_{\omega(2)})$
0.1	0.9644090640	0.9539352950	0.9513093108	0.9487706750
0.2	0.9264180370	0.9050606920	0.8995894909	0.8948089750
0.3	0.8856946400	0.8529979890	0.8444253633	0.8377503510
0.4	0.8418228960	0.7972703120	0.7852897456	0.7771301540
0.5	0.7942684577	0.7372580538	0.7214907367	0.7123358942
0.6	0.7423219616	0.6721237843	0.6520793817	0.6425234885
0.7	0.6849998071	0.6006731629	0.5756717219	0.5664557928
0.8	0.6208537069	0.5210652978	0.4900562182	0.4821474331
0.9	0.5475543628	0.4300944329	0.3911239201	0.3859049667

Table 2: Comparison of results for Problem 2

ω	$\rho(\mathcal{L}_{\omega})$	$\rho(\mathcal{L}_{\omega(N)})$	$\rho(\mathcal{L}_{\omega(1)})$	$\rho(\mathcal{L}_{\omega(2)})$
0.1	0.9735960110	0.9652295940	0.9650865648	0.9637454210
0.2	0.9450680960	0.9280487730	0.9274330393	0.9248632890
0.3	0.9141011790	0.8881203950	0.8866219554	0.8829676410
0.4	0.8803021930	0.8450226883	0.8421239306	0.8375700830
0.5	0.8431699784	0.7982145211	0.7932486414	0.7880350069
0.6	0.8020477532	0.7469789371	0.7390626349	0.7335046327
0.7	0.7560437506	0.6903249539	0.6782402510	0.6727634288
0.8	0.7038884351	0.6268006095	0.6087628289	0.6039659495
0.9	0.6436497250	0.5540893195	0.5272084388	0.5240003219

Tables 1 and 2 display comparison of spectral radii of four different iterative schemes of the SOR method for two problems, Problem 1 and Problem 2 respectively; these are the classical SOR method with iteration matrix $\mathcal{L}_{\omega}$, the preconditioned SOR method with iteration matrix $\mathcal{L}_{\omega(1)}$, the preconditioned SOR method with iteration matrix $\mathcal{L}_{\omega(2)}$ and the preconditioned SOR method of Ndanusa and Adeboye (2012) with iteration matrix $\mathcal{L}_{\omega(N)}$. As is revealed from the two tables, $\rho(\mathcal{L}_{\omega}) < \rho(\mathcal{L}_{\omega(N)}) < \rho(\mathcal{L}_{\omega(1)}) < \rho(\mathcal{L}_{\omega(2)})$, which indicates the superiority of the new preconditioned SOR iterations $\mathcal{L}_{\omega(1)}$ and $\mathcal{L}_{\omega(2)}$ over the un-preconditioned SOR and another preconditioned SOR iteration in literature. It is further observed that convergence tend to be faster when the value of the relaxation parameter ω moves closer to 1 than when it moves closer to 0.

Conclusion

A new approach of constructing preconditioners for linear system is adopted; it entails the introduction of a preconditioner that does not eliminate any of the entries of the coefficient matrix of the linear system; rather, it scales down the entries. The result is a combination preconditioner whose performance far outweighs the performance of the un-preconditioned system as well as some other preconditioned system in literature.

Acknowledgements

The authors express their utmost gratitude and appreciation to the anonymous reviewers for their positive criticisms of the work.

References

Abdullahi, I. and Ndanusa, A. (2020). A new modified preconditioned accelerated overrelaxation (AOR) iterative method for $L-$matrix linear algebraic systems. Science World Journal, 15(2): 45-50.

Ames, W. F. (1977). Numerical methods for partial differential equations (2nd ed.). Great Britain: Thomas Nelson & Sons Ltd. p. 365.

Bai, Y. and Wang, A. (2015). An acceleration of preconditioned generalized accelerated over relaxation methods for systems of linear equations. Int. J. Comput. Math., 92: 1012–1024.

Dehghan, M. and Hajarian, M. (2009). Improving preconditioned SOR-type iterative methods for $L-$matrices. International Journal for Numerical Methods in Biomedical Engineering, 27:774-784.

Evans, D. J., Martins, M. M., Trigo, M. E. (2001). The AOR iterative method for new preconditioned linear systems. Journal of Computational and Applied Mathematics, 132: 461-466.

Faruk, A. I. and Ndanusa, A. (2019). Improvements of successive overrelaxation iterative (SOR) method for L-Matrices. Savanna Journal of Basic and Applied Sciences (SJBAS), 1(2): 218-223.

Gunawardena, A. D., Jain, S. K. and Snyder, L. (1991). Modified iterative methods for consistent linear systems. Linear Algebra and its Applications, 154-156: 123-143.

Huang, Z., Wang, L., Xu, Z. and Cui, J. (2016). Convergence analysis of some new preconditioned AOR iterative methods for L-matrices. IAENG International Journal of Applied Mathematics, 46(2): 202-209.

Kotakemori, H., Harada, K., Morimoto, M. and Niki, H. (2002). A comparison theorem for the iterative method with the preconditioner $(I + S_{max})$. Journal of Computational and Applied Mathematics, 145: 373-378.

Kotakemori, H., Niki, H. and Okamoto, N. (1996). Accelerated iteration method for $Z-$matrices. Journal of Computational and Applied Mathematics, 75: 87-97.

Kohno, T., Kotakemori, H., Niki, H. and Usui, M. (1997). Improving modified Gauss–Seidel method for Z-matrices. Linear Algebra Appl., 267: 113-123.

Li C. and Evans, D. J. (1994). Improving the SOR method. Int. J. Comput. Math., 54: 207–213.

Li, W. and Sun, W. (2000). Modified Gauss–Seidel type methods and Jacobi type methods for $Z-$ matrices. Linear Algebra and its Applications, 317: 227-240.

Mayaki, Z. and Ndanusa, A. (2019). Modified successive overrelaxation (SOR) type methods for M-matrices. Science World Journal, 14, 4, pp. 1-5.

Milaszewicz, J. P. (1987). Improving Jacobi and Gauss-Seidel iterations. Linear Algebra and its Applications, 93: 161-170.

Morimoto, M., Harada, K., Sakakihara, M. and Sawami, H. (2004). The Gauss–Seidel iterative method with the preconditioning matrix $(I + S_{max} + S_m)$. Japan J. Indust. Appl. Math., 21: 25-34.

Morimoto, M., Kotakemori, H., Kohno, T. and Niki, H. (2003). The Gauss–Seidel method with preconditioner *(I + R)*. Trans. Japan Soc. Indust. Appl. Math., 13: 439–445.

Ndanusa, A. (2020). Convergence of preconditioned Gauss-Seidel iterative method for $L-$matrices. Communication in Physical Sciences, 6(1): 803 – 808.

Ndanusa, A. and Adeboye, K. R. (2012). Preconditioned SOR iterative methods for $L-$matrices. American Journal of Computational and Applied Mathematics, 2(6): 300-305.

Ndanusa, A., David, B. E., Ayantola, B. B. and Abubakar, A. W. (2020). Improving preconditioned Gauss-Seidel iterative method for $L-$matrices. FUDMA Journal of Science (FJS), 4(1): 453 – 459.

Song, Y. (2020). On preconditioned AOR method for solving linear systems. Retrieved from: arXiv:2020.00690[math.NA]

Varga, R. S. (1981). Matrix iterative analysis (2nd ed.). Englewood Cliffs, New Jersey: Prentice-Hall, p. 358.

Young, D. M. (1950). Iterative methods for solving partial difference equations of elliptic type. Doctoral Thesis, Harvard University, Cambridge, MA. p. 74.