THE GENERALIZED WEIBULL PARETO DISTRIBUTION; ITS PROPERTIES AND APPLICATION

By

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ABSTRACT

Modeling and analysis of lifetimes is an important aspect of statistical work in a wide variety of scientific and technological fields. In this study, a four-parameter lifetime model called the Generalized Weibull-Pareto (GWPD) distribution was proposed using the concept of exponentiated distributions. Some properties of the proposed distribution have been studied including explicit expressions for the moments, moments generating function, Survival and Hazard function. The method of Maximum Likelihood Estimation was used for estimating the model parameters. The proposed distribution was applied to two real data sets to prove its fit over some other baseline distributions.

Keywords: Moments, Moments Generating Function, Survival and Hazard function, Maximum likelihood Estimation.

INTRODUCTION

The Pareto distribution was originally proposed to model the unequal distribution of wealth since Pareto observed the way that, a larger portion of the wealth of any society was owned by a smaller percentage of the people. Pareto distribution has many applications in economics. Since it is a heavy tailed distribution, it is a good candidate for modeling income above a theoretical value and the distribution of insurance claims above a threshold value (Klugman *et al.* 2004). Another application of this distribution is in On-Line Analytical Processing (OLAP). Nadeau *et al.* (2003) used Pareto distribution for OLAP aimed at gaining useful information quickly from large amounts of data residing in a data warehouse. There are several forms and extensions of the Pareto distribution in the literature. Pickands (1975) was the first to propose an extension of the Pareto distribution with the generalized Pareto (GP) distribution when analyzing the upper tail of a distribution function. One major problem in the development and application of probability distribution today is that a lot of available data sets do not follow any of the existing and well known standard probability distributions (models) and hence cannot be modeled appropriately. This creates room for developing new distributions which could better describe some of these phenomena and provide greater flexibility in modeling of lifetime data. Vifredo (1896) defined Pareto distribution using the cumulative distribution function (cdf) as in equation (1) below:

$$G\left(x; {}^{-}, k\right) = 1 - \left(\frac{-}{x}\right)^{k} \quad - \le x < \infty; {}^{-}, k > 0 \qquad (1)$$

with the probability density function (pdf) given as:

$$g\left(x;¯,k\right)=\frac{k¯^{k}}{x^{k+1}}\ ¯\leq x<\infty;¯,k>0 \tag{2}$$

Where $¯ > 0$ is a scale parameter, $k > 0$ is a shape parameter and x is a variable.

This study seeks to increase the flexibility of the Pareto distribution using the generalized weibull-G family of distribution proposed by Codeiro *etal.* (2015). The derived distribution will be used to identify some basic properties which will be used to describe the new distribution (GWPD).

The Generalized Weibull–G Family of Distribution

According to Codeiro *et al.*, (2015), the cumulative distribution function (cdf) of the Generalized Weibull-G family is given by

$$F(x)=\int_{0}^{-\log[1-G(x;¦)]} \sim\ t^{\sim-1}e^{-\sim t}\ dt \tag{3}$$

Integrating equation (3) yield,

$$F(x)=1-\exp\left\{-\sim\left(-\log[1-G(x;¦)]\right)\ \right\} \tag{4}$$

Where $G(x;¦)$ is the cdf of the baseline distribution which depends on a parameter vector ¦ **.**

The pdf of the corresponding Generalized Weibull-G family is obtained by differentiating equation (4) with respect to x given by

$$f(x)=\sim\ \frac{g(x;¦)}{[1-G(x;¦)]}\left(-\log[1-G(x;¦)]\right)\ ^{-1}\exp\left\{-\sim\left(-\log[1-G(x;¦)]\right)\ \right\} \tag{5}$$

Where $g(x;¦)$ and $G(x;¦)$ are the pdf and cdf of the baseline distribution respectively which depend on parameter vector ¦ **,** $\sim > 0$ and $\ > 0$ are the scale and shape parameters.

The Generalized Weibull Pareto Distribution

Let $G(x;¦)$ be the Pareto distribution function from equation (1) with parameters ¯ and k, then equation (4) yields the Generalized Weibull-Pareto (GWPD) distribution function for $x \geq ¯$;

$$F(x;\sim,\ ,¯,k)=1-\exp\left\{-\sim\left(-\log\left[1-\left(1-\left(\frac{¯}{x}\right)^{k}\right)\right]\right)\ \right\} \tag{6}$$

Where $¯ > 0$ is a scale parameter and the other positive parameters k and are the shape parameters. The corresponding density function is obtained by substituting equation (1) and (2) into (5) as:

$$f(x)=\frac{\sim\ k¯^{k}}{\left[1-\left(1-\left(\frac{¯}{x}\right)^{k}\right)\right]x^{k+1}}\left(-\log\left[1-\left(1-\left(\frac{¯}{x}\right)^{k}\right)\right]\right)^{\ -1}\times\exp\left\{-\sim\left(-\log\left[1-\left(1-\left(\frac{¯}{x}\right)^{k}\right)\right]\right)\ \right\} \tag{7}$$

Below are the possible plots for the *pdf* and *cdf* of the *GWPD* at different parameter values.

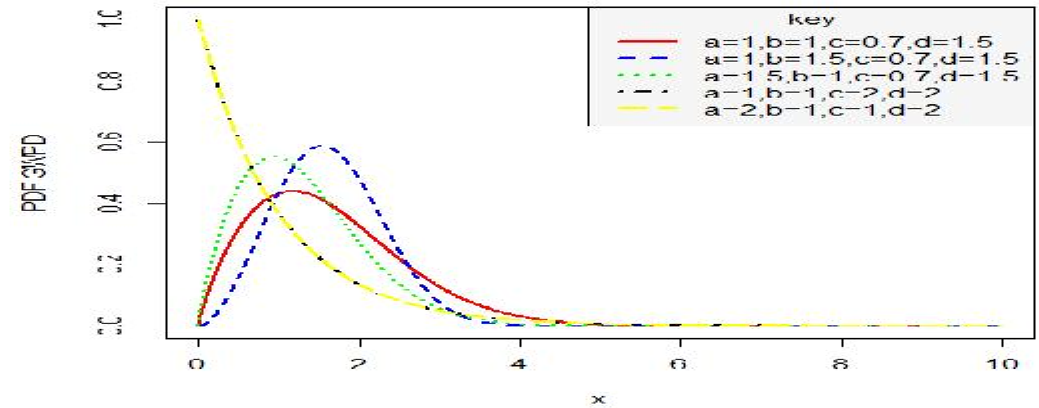

Figure 1: Plot of the *pdf* of the *GWPD* where $\sim = a, \quad = b, ^{-} = c, andk = d$

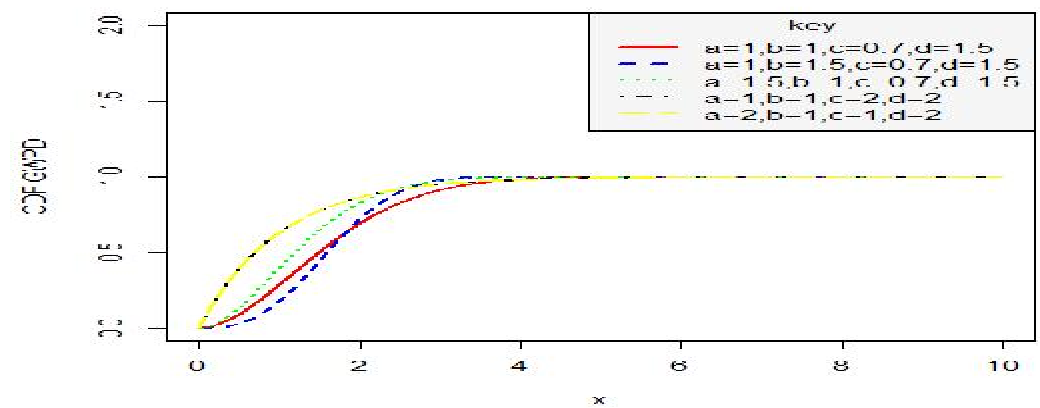

Figure 2: Plot of the *cdf* of the *GWPD* where $\sim = a, \quad = b, ^{-} = c, andk = d$

Distribution and Density Function

The cdf $F(x;\alpha,\beta,\theta,k)$ andthe pdf $f(x;\alpha,\beta,\theta,k)$of GWPD are expressed as a mixture of a linear combination of exponentiated-G density (Codeiro *et al*, 2015). The expressions are $h_a(x) = aG(x)^{a-1}g(x)$ and $H_a(x) = G(x)^a$, Gupta *et al*, (1998) and Ali *et al*. (2007).

In this sub-section, some representations of the distribution and density functions of the Generalized Weibull-Pareto distribution *(GWPD)* are presented using the concept of exponentiated distributions proposed by Ali *et al*. (2007). The mathematical relation given below will be used. Expanding the exponential function in equation (6) using the power series expansion gives:

$$\exp\left\{-\sim\left(-\log\left[1-\left(1-\left(\frac{-}{x}\right)^{k}\right)\right]\right)\right\} = \sum_{k=0}^{\infty}\frac{(-1)^{k}\sim^{k}}{k!}\left(-\log\left[1-\left(1-\left(\frac{-}{x}\right)^{k}\right)\right]\right)^{k}$$

using the power series expansion and substituting it back into equation (6) gives:

$$F(x) = 1 - \sum_{k=0}^{\infty} \frac{(-1)^k \sim^k}{k!} \left(-\log \left[1 - \left(1 - \left(\frac{-}{x} \right)^k \right) \right] \right)^k \tag{8}$$

Recall that for any real parameter c and $z \in (0,1)$ it can be shown that;

$$[-\log(1-z)]^c = z^c + c\sum_{i=0}^{\infty} pi(c+i) z^{i+c+1} \tag{9}$$

Following the above methodology by Codeiro *et al.* (2015) in equation (9), it can be shown that the last term of equation (8) becomes;

$$\left(-\log \left[1 - \left(1 - \left(\frac{-}{x} \right)^k \right) \right] \right)^k = \left(1 - \left(\frac{-}{x} \right)^k \right)^k + k \sum_{i=1}^{\infty} p_i (k \quad +1) \left(1 - \left(\frac{-}{x} \right)^k \right)^{k \quad +i+1} \tag{10}$$

Also recall that

$$\left(1 - \left(\frac{-}{x} \right)^k \right) = G(x)$$

And for any positive non integer , $G(x)$ can be expand as follows;

$$(-\log[1-G(x)])^k = (G(x))^k + k \sum_{i=1}^{\infty} p_i (k \quad +i) (G(x))^{K \quad +i+1}$$

$$(G(x))^k = [1-(1-G(x))]^k = \sum_{p=0}^{\infty} (-1)^p \binom{k}{p} [1-G(x)]^p$$

And then

$$[1-G(x)]^p = \sum_{r=0}^{p} (-1)^r \binom{p}{r} G(x)^r \qquad \text{(Using binomial expansion)}$$

Hence;

$$(G(x))^k = \sum_{p=0}^{\infty} \sum_{r=0}^{p} (-1)^{p+r} \binom{k}{r} \binom{p}{r} G(x)^r$$

Replacing $\sum_{p=0}^{\infty} \sum_{r=o}^{p}$ by $\sum_{r=0}^{\infty} \sum_{p=r}^{\infty}$, will give

$$(G(x))^k = \sum_{r=0}^{\infty} \sum_{p=r}^{\infty} (-1)^{p+r} \binom{k}{p} \binom{p}{r} G(x)^r ,$$

And this can be expressed as

$$(G(x))^k = \sum_{r=0}^{\infty} S_{r(k \ )} G(x)^r \tag{11}$$

Where the coefficients are

$$S_{r(k \ )} = \sum_{r=0}^{\infty} (-1)^{p+r} \binom{k}{p} \binom{p}{r}$$

Making use of equation (11) in (10) and simplifying the resulting expression, will give;

$$\left(-\log[1-(1-(\tfrac{-}{x})^{k})]\right)^{k} = \sum_{r=0}^{\infty} S_{r(k\ )}G(x)^{r} + k \sum_{i=1}^{\infty}\sum_{r=0}^{\infty} p_{i}(k \quad +i)S_{r(k \quad +i+1)}G(x)^{r}$$

$$= \left[\sum_{r=0}^{\infty} S_{r(k\ )} + k \sum_{i=1}^{\infty}\sum_{r=0}^{\infty} p_{i}(k \quad +i)S_{r(k \quad +i+1)}\right]G(x)^{r}$$

$$= ¦_{r}G(x)^{r} = ¦_{r}\left(1-\left(\frac{-}{x}\right)^{k}\right)^{r} \qquad (12)$$

Where $¦_{r=} \left[\sum_{r=0}^{\infty} S_{r(k\ )} + k \sum_{i=0}^{\infty}\sum_{r=0}^{\infty} p_{i}(k \quad +i)S_{r(k \quad +i+1)}\right]$

Substituting equation (12) in (8) and rearranging, will give the cdf, $F(x)$ of the *GWPD* as;

$$F(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k} \sim^{k}}{k!} ¦_{r}\left(1-\left(\frac{-}{x}\right)^{k}\right)^{r}, \quad x>0, r, ¯, k>0$$

$$F(x) = ¦_{r}\sum_{k=1}^{\infty} \frac{(-1)^{k} \sim^{k}}{k!}\left(1-\left(\frac{-}{x}\right)^{k}\right)^{r} \qquad (13)$$

Where *r* is a power parameter, *x* is a random variable and $¦_{r}$ .is a coefficient also $\sum_{k=1}^{\infty} ¦_{r} = 1$

Also, by differentiating equation (13) with respect to *x* and changing indices, the corresponding pdf of the (*GWPD*) is given as;

$$f(x) = \frac{dF(x)}{dx} = \sum_{k=1}^{\infty} \frac{(-1)^{k} \sim^{k}}{k!} ¦_{r} r \frac{k^{-k}}{x^{k+1}}\left(1-\left(\frac{-}{x}\right)^{k}\right)^{r-1}, \quad x>0, r, ¯, k>0 \qquad (14)$$

Where *r* is a power parameter, *x* is a random variable and $¦_{r}$ .is a coefficient also $\sum_{k=1}^{\infty} ¦_{r} = 1$

The result in equation (14) ensures that some Mathematical properties of the *GWPD* such as ordinary and incomplete moments, generating function and mean deviations can be derived from those quantities of the exponentiated Pareto distribution as in Ali *et al* (2007)..

Moments and Moments Generating Function

In this section, the moments and generating function of the *(GWPD)* were derived.

Moments

Let $f(x/\sim, ¯, k)$ be a continuous random variable from the (*GWPD*) in equation (7), using (14) it is easy to obtain the n^{th} moment of *x* defined as:

$$E(x^{n}) = «_{n}^{'} = \int_{0}^{\infty} x^{n} f(x)dx$$

The n^{th} Moments of the (GWPD) is given as;

$$E(x^n) = \text{«}_n^{'} = \int_0^{\infty} x^n \left(\sum_{k=1}^{\infty} \frac{(-1)^k \sim^k}{k!} \text{¦}_r \, \frac{k\text{—}^k}{x^{k+1}} \left(1 - \left(\frac{\text{—}}{x}\right)^k\right)^{r-1} \right) dx$$

$$\text{«}_n^{'} = \text{¦}_r \sum_{k=1}^{\infty} \frac{(-1)^k \sim^k}{k!} r \int_0^{\infty} x^n \, \frac{k\text{—}^k}{x^{k+1}} \left(1 - \left(\frac{\text{—}}{x}\right)^k\right)^{r-1} dx \qquad (15)$$

Using binomial expansion on the last term in equation (15), will give;

$$\left(1 - \left(\frac{\text{—}}{x}\right)^k\right)^{r-1} = \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \left(\frac{\text{—}}{x}\right)^{jk}$$

Substituting the expansion above into equation (15), will give;

$$\text{«}_n^{'} = r\,\text{¦}_r \sum_{j=0}^{r-1} \sum_{k=1}^{\infty} \frac{(-1)^{j+k} \sim^k}{k!} \binom{r-1}{j} \int_0^{\infty} x^n \, \frac{k\text{—}^k}{x^{k+1}} \left(\frac{\text{—}}{x}\right)^{jk} dx$$

$$= r\,\text{¦}_r \sum_{j=0}^{r-1} \sum_{k=1}^{\infty} \frac{(-1)^{j+k} \sim^k}{k!} \binom{r-1}{j} k\text{—}^{k(j+1)} \int_0^{\infty} x^{n-k(j+1)-1} dx$$

Considering the concept and properties of the exponentiated Pareto distribution, the *nth* moment of *x* a random variable for the generalized Weibull-Pareto distribution is given as;

$$E(x^n) = \text{«}_n^{'} = r\text{—}^n k \sum_{j=0}^{r-1} \sum_{k=1}^{\infty} \frac{(-1)^{j+k} \sim^k}{k!} \binom{r-1}{j} \frac{\text{¦}_r}{k(j+1)-n} \qquad (16)$$

If n<k and $r \in$ N

Setting n=1 in equation (16), gives the first moment of *f*(*x/α, ϑ, k) as*

$$Mean = E(x) = \text{«}_1^{'} = r\text{—} k \sum_{j=0}^{r-1} \sum_{k=1}^{\infty} \frac{(-1)^{j+k} \sim^k}{k!} \binom{r-1}{j} \frac{\text{¦}_r}{k(j+1)-1} \qquad (17)$$

If $k > 1$ and $r \in N$

Moment Generating Function (mgf)

Let *x* be a continuous random variable, the moment generating function of *x* is defined as;

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x) dx \qquad (18)$$

By expanding e^{tx} using power series expansion, will give;

$$e^{tx} = \sum_{k=0}^{\infty} \frac{t^k}{k!} x^k$$ Substitute into equation (18)

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} \sum_{k=0}^{\infty} \frac{t^k}{k!} x^k f(x) dx$$

$$M_x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mu_k^{'} \tag{19}$$

Where

$\mu_k^{'} = \int_0^{\infty} x^k f(x) dx$ is the k^{th} moment obtained from equation (16)

Characteristics Function $\phi_x(t)$

Let x be a continuous random variable, the characteristics function of x is defined as;

$$\phi_x(t) = E[e^{itx}] = E[\cos tx + i \sin tx] \tag{20}$$

Note;

$$\cos tx = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} x^{2n} \qquad i \sin tx = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} x^{2n+1}$$

$$E[\cos tx] = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} E(x^{2n}) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \mu_{2n}^{'}$$

$$E[i \sin tx] = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} E(x^{2n+1}) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \mu_{2n}^{'}$$

$$\therefore \phi_x(t) = E(\cos tx) + E(i \sin tx)$$

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \mu_{2n}^{'} + \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \mu_{2n+1}^{'} \right] \tag{21}$$

The Central Moments and Variance

The nth moment about the mean or central moment can be obtained as;

$$\mu_n = E(X - \mu_1^{'})^n = \sum_{i=1}^{n} (-1)^i \binom{n}{i} E(X^{n-i}) \mu'' \tag{22}$$

The variance is defined as the central moment of order 2 (i.e n=2)

$$Var(x) = \sigma^2 = \mu_2^{'} - \mu_1^{'2}$$

Where $\mu_2^{'}$ and $\mu_1^{'2}$ are the first and second moment and moment squared which could be obtain using equation (16) given above.

Quantile Function

According to Codeiro *et al.* (2015), let $Q(\mu) = F^{-1}(\mu)$ be the quantile function (qf) of $F^{-1}(x)$ for $0 < u < 1$. Inverting $F(x) = \mu$ in equation (6), the qf of x is given as;

$$x = f^{-1}(\mu) = Q\left(1 - \exp\left\{-\left[-\sim^{-1} \log(1-\mu)\right]^{1/}\right\}\right) \tag{23}$$

Simplifying equation (6), the qf for the *GWPD* is given as follow;

$F(x) = 1 - \exp\{-\sim(-\log[1 - G(x;\eta)])\} = \mu$, where $G(x;\boldsymbol{\eta})$ is the cdf of the baseline distribution and $x = Q(u)$, $F(Q(u)) = u$, $Q(u) = F^{-1}(u)$

$$\therefore 1 - \exp\left\{ - \sim \left(- \log\left[1 - \left(1 - \left(\frac{-}{Q(u)} \right)^{k} \right) \right] \right) \right\} = u$$

$$\left\{ - \sim \left(- \log\left[1 - \left(1 - \left(\frac{-}{Q(u)} \right)^{k} \right) \right] \right) \right\} = \ln\left(\frac{1}{1-u} \right)$$

Simplify the above equation and make $Q(u)$ the subject, the qf for the *GWPD* is given as,

$$Q(«) = ^{-}\left(\exp\left\{ - \left(\frac{1}{\sim} \ln\left(\frac{1}{1-«} \right) \right)^{\frac{1}{\,}} \right\} \right)^{\frac{-1}{k}} \tag{24}$$

In particular, the distribution of the median is obtained by substituting u=0.5 as follow;

$$Q(0.5) = ^{-}\left(\exp\left\{ - \left(\frac{1}{\sim} \ln(2) \right)^{\frac{1}{\,}} \right\} \right)^{\frac{-1}{k}} \tag{25}$$

Skewness and Kurtosis

In this study, the quantile based measures of skewness and kurtosis was employed due to non-existence of the classical measures in some cases. The Bowley's measure of skewness (Kennedy and Keeping, 1962.) based on quartiles is given by;

$$SK = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{26}$$

And the Moores' (1998) kurtosis is on octiles and is given by;

$$KT = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + \left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{1}{4}\right)} \tag{27}$$

Order Statistics

Suppose $x_1, \ldots\ldots\ldots, x_n$ is a random sample from a distribution with pdf $f(x)$, and let $x_{1:n} <, \ldots\ldots, < x_{i:n}$ denote the corresponding order statistic obtained from this sample. The pdf, $f_{i:n}(x)$ of the i^{th} order statistic is given by;

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} \left[1 - F(x)\right]^{n-i} \tag{28}$$

The *i*th order statistics of the *GWPD* is given by

$$f_{i:n}(x) = n\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\left[\sum_{k=1}^{\infty}\frac{(-1)^k \sim^k}{k!}\text{¦}_r r\frac{k^{-k}}{x^{k+1}}\left(1-\left(\frac{-}{x}\right)^k\right)^{r-1}\right]\left[\sum_{k+1}^{\infty}\frac{(-1)^k \sim^k}{k!}\text{¦}_r\left(1-\left(\frac{-}{k}\right)^k\right)^r\right]^{j+i-1} \quad (29)$$

Where $n! = n/(i-1)!(n-1)!$ and expanding $[1-F(x)]^{n-i}$ in equation (28) using binomial expansion i.e

$$[1-F(x)]^{n-i} = \sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}F(x)^j$$

The *pdf* of the smallest order statistics x_1 of the *GWPD* is therefore

$$f_{1:n}(x) = n\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\left[\sum_{k=1}^{\infty}\frac{(-1)^k \sim^k}{k!}\text{¦}_r r\frac{k^{-k}}{x^{k+1}}\left(1-\left(\frac{-}{x}\right)^k\right)^{r-1}\right]\left[\sum_{k+1}^{\infty}\frac{(-1)^k \sim^k}{k!}\text{¦}_r\left(1-\left(\frac{-}{k}\right)^k\right)^r\right]^{j} \quad (30)$$

And the pdf of largest order statistics x_n of the GWPD is;

$$f(n:n) = n\left[\sum_{k=1}^{\infty}\frac{(-1)^k \sim^k}{k!}\text{¦}_r r\frac{k^{-k}}{x^{k+1}}\left(1-\left(\frac{-}{x}\right)^k\right)^{r-1}\right]\left[\sum_{k=1}^{\infty}\frac{(-1)^k r^k}{k!}\text{¦}_r\left(1-\left(\frac{-}{x}\right)^k\right)^r\right]^{n-1} \quad (31)$$

RELIABILITY ANALYSIS

Survival Function

Survival function is the probability that a system or an individual will survive beyond a given time. Mathematically, the survival function is defined by:

$$S(x) = 1 - F(x) \quad (32)$$

where $F(x)$ is *cdf* of the *GWPD*

The survival function of the *GWPD* is given by:

$$S(x) = 1-\left(1-\exp\left\{-\sim\left(-\log\left[1-\left(1-\left(\frac{-}{x}\right)^k\right)\right]\right)\right\}\right)$$

$$S(x) = \exp\left\{-\sim\left(-\log\left[1-\left(1-\left(\frac{-}{x}\right)^k\right)\right]\right)\right\} \quad (33)$$

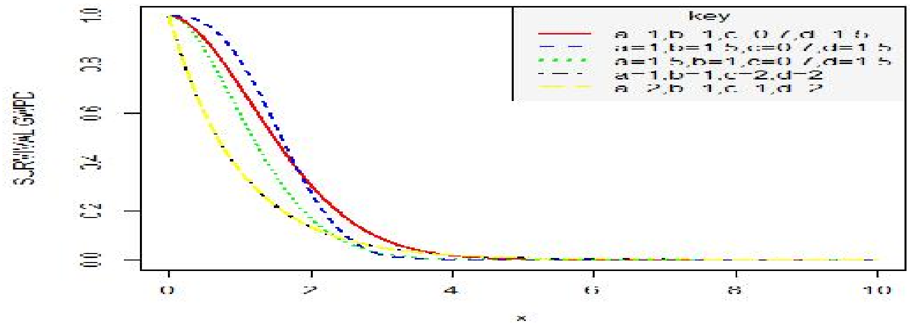

Figure 3: the plot of the survival function of the GWPD for different parameter values where $a = \sim, b = \quad, c = {}^{-} and k = d$

The survival function indicates the probability that a component or individual (y) will survive at time (x) for the GWPD.

Hazard Function

Hazard function is also called the failure or risk function and is the probability that a component will fail or die for an interval of time. The hazard function is defined as;

$$h(x) = \frac{f(x)}{1 - F(x)} \tag{34}$$

Where $f(x)$ and $F(x)$ are the pdf and cdf of the *GWPD*

The hazard function of the *GWPD* is given by:

$$h(x) = \frac{\sim}{\left[1 - \left(\frac{-}{x}\right)^{k}\right]} \frac{k^{-k}}{x^{k+1}} \left\{ -\log\left[1 - \left(1 - \left(\frac{-}{x}\right)^{k}\right)\right]\right\}^{-1} \tag{35}$$

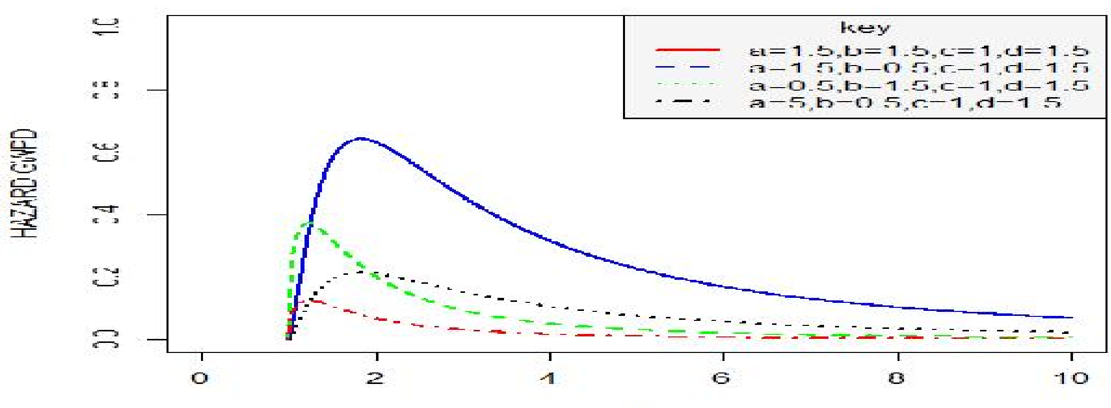

Figure 4: The plot of the hazard function of the *GWPD* for different parameter values where $a = \sim, b = \quad, c = {}^{-} and k = d$

The hazard plot displays the instant failure rate for component (y) at time (x). From the hazard function the following can be observed;

i. if $\ =1$, the failure rate is a function of x and is given by; $h(x)=\frac{\sim\ \bar{\ }}{x^2}$ This makes the *GWPD* not suitable for modeling components or systems with a constant failure rate.

ii. if $\ >1$, the hazard function is an increasing function of x, which makes the distribution *GWPD* suitable for modeling components that wears faster with time.

iii. if $\ <1$, the hazard function is a decreasing function of x, which makes the *GWPD* suitable for modeling components that wears slower with time.

MAXIMUM LIKELIHOOD ESTIMATION

In this section, the non-linear equations for finding the Maximum Likelihood Estimation (MLE) and inference of the parameters for the (*GWPD)* distribution was derived. The Maximum Likelihood Estimation is one of the most widely used estimation methods for finding the unknown parameters. Let $x_1,\ldots\ldots\ldots,x_n$ be a sample from equation (7) and $\ddot{\ }=(\sim,\ \ ,\bar{\ },k)^T$. The log-likelihood function for $\ddot{\ }$ is given by;

Let log likelihood $=\log L(x/\sim,\ \ ,k,\bar{\ })=L(\ddot{\ })$

$$L(\ddot{\ })=n\log\sim+n\log\ \ +n\log k+nk\log\bar{\ }-(k+1)\sum_{i=1}^{n}\log(x_i)-\sum_{i=1}^{n}\log\left[1-\left(1-\left(\frac{\bar{\ }}{x_i}\right)^k\right)\right]+$$

$$(\ \ -1)\sum_{i=1}^{n}\log\left\{-\log\left[1-\left(1-\left(\frac{\bar{\ }}{x_i}\right)^k\right)\right]\right\}-\sim\sum_{i=1}^{n}\left(-\log\left[1-\left(1-\left(\frac{\bar{\ }}{x_i}\right)^k\right)\right]\right) \quad (36)$$

From (36) the components of the score vectors $U(\ddot{\ })=(U_{\sim},U\ \ ,U_K,U_{-})$ are;

$$\frac{\partial l(\ddot{\ })}{\partial\sim}=\frac{n}{\sim}-\sum_{i=1}^{n}\left(-\log\left[1-\left(1-\left(\frac{\bar{\ }}{x_i}\right)^k\right)\right]\right) \quad (37)$$

$$\frac{\partial l(\ddot{\ })}{\partial\ \ }=\frac{n}{\ \ }+\sum_{i=1}^{n}\log\left\{-\log\left[1-\left(1-\left(\frac{\bar{\ }}{x_i}\right)^k\right)\right]\right\}-\sim\sum_{i=1}^{n}\left(-\log\left[1-\left(1-\left(\frac{\bar{\ }}{x_i}\right)^k\right)\right]\right)\ \ \log\left(-\log\left[1-\left(1-\left(\frac{\bar{\ }}{x_i}\right)^k\right)\right]\right) \quad (38)$$

$$\frac{\partial l(\ddot{\ })}{\partial k}=\frac{n}{k}+n\log\bar{\ }-\sum_{i=1}^{n}\log(x_i)-\sum_{i=1}^{n}\left\{\frac{\left(\frac{\bar{\ }}{x_i}\right)^k\log\left(\frac{\bar{\ }}{x_i}\right)}{\left[1-\left(1-\left(\frac{\bar{\ }}{x_i}\right)^k\right)\right]}\right\}-(\ \ -1)\sum_{i-1}^{n}\left\{\frac{\left(\frac{\bar{\ }}{x_i}\right)^k\log\left(\frac{\bar{\ }}{x_i}\right)}{\left(-\log\left[1-\left(1-\left(\frac{\bar{\ }}{x}\right)^k\right)\right]\right)\left[1-\left(1-\left(\frac{\bar{\ }}{x_i}\right)^k\right)\right]}\right\}$$

$$+\sim \sum_{i=1}^{n}\left\{\frac{\left(-\log\left[1-\left(1-\left(\frac{-}{x_i}\right)^{k}\right)\right]\right)^{-1}\left(\frac{-}{x_i}\right)^{k}\log\left(\frac{-}{x_i}\right)}{\left[1-\left(1-\left(\frac{-}{x_i}\right)^{k}\right)\right]}\right\} \qquad (39)$$

$$\frac{\partial l(\cdot\cdot)}{\partial -}=\frac{nk}{-}-\sum_{i=1}^{n}\left\{\frac{\left(\frac{k}{-}\right)\left(\frac{-}{x_i}\right)^{k}}{\left[1-\left(1-\left(\frac{-}{x_i}\right)^{k}\right)\right]}\right\}-(\quad -1)\sum_{i=1}^{n}\left\{\frac{\left(\frac{k}{-}\right)\left(\frac{-}{x_i}\right)^{k}}{\left(-\log\left[1-\left(1-\left(\frac{-}{x_i}\right)^{k}\right)\right]\right)\left[1-\left(1-\left(\frac{-}{x_i}\right)^{k}\right)\right]}\right\}+$$

$$\sim \sum_{i=1}^{n}\left\{\frac{\left(-\log\left[1-\left(1-\left(\frac{-}{x_i}\right)^{k}\right)\right]\right)^{-1}\left(\frac{k}{x_i}\right)\left(\frac{-}{x_i}\right)^{k}}{\left[1-\left(1-\left(\frac{-}{x_i}\right)^{k}\right)\right]}\right\} \qquad (40)$$

The MLE $\hat{\varphi}$ of φ is obtained by solving the nonlinear likelihood equations $U_{\sim}(\cdot\cdot)=0$, $U\ (\cdot\cdot)=0$ and $U_{\mid}(\cdot\cdot)=0$.

APPLICATION

In this section, the potentiality of the *GWPD* is illustrated in two applications to real data. The fits of the Generalized Weibull-Pareto distribution (*GWPD)* was compared with those of five baseline distributions namely: The Exponential Pareto Distribution *(EPD)*, the Exponentiated Weibull Pareto distribution (*EWPD*), the Pareto Distribution *(PD)*, the Weibull Distribution *(WD)* and the Exponentiated Pareto Distribution (*EDPD).* The two data sets are:

Data set I: The data set is on the strength of 1.5cm glass fibers. The data was originally obtained by workers at the *UK* National physical laboratory and it has been used by Smith and Naylor (1987), Barreto-souza *et al.* (2011), Bourguignon *et al.* (2014), Oguntunde e*t al.* (2015) and Afify and Aryal (2016).

Data set II: The data set represents observations on the remission times (in months) of a random sample of 128 bladder cancer patients. The data set was previously studied by Lemonte and Cordeiro (2013) and Zea *et al.* (2012). The summary of the two data sets are provided as follows;

Data 1: Strength of 1.5cm Glass Fibers

0.55 1.28, 1.51, 1.61, 1.70, 2.00, 0.74, 1.29, 1.52, 1.62, 1.70, 2.01, 0.77, 1.30, 1.53, 1.62, 1.73, 2.24, 0.81, 1.36, 1.54, 1.63, 1.76, 0.84, 1.39, 1.55, 1.64, 1.76, 0.93, 1.42, 1.55, 1.66, 1.77, 1.04, 1.48, 1.58, 1.66, 1.78, 1.11, 1.48, 1.59, 1.66, 1.81, 1.13, 1.49, 1.60, 1.67, 1.82, 1.24, 1.49, 1.61, 1.68, 1.84, 1.25, 1.50, 1.61, 1.68, 1.84, 1.27, 1.50, 1.61, 1.69, 1.89

Bourguignon *et al.* (2014).

Data 2: The Remission times (in months) of a random sample of 128 Bladder Cancer Patients

0.080,0.200,0.400,0.500,0.510,0.810,0.900,1.050,1.190,1.260,1.350,1.400,1.460,1.760,2.020,2.020,2.070,2.090,2.230,2.260,2.460,2.540,2.620,2.640,2.690,2.690,2.750,2.830,2.870,3.020,3.250,3.310,3.360,3.360,3.480,3.520,3.570,3.640,3.700,3.820,3.880,4.180,4.230,4.260,4.330,4.340,4.400,4.500,4.510,4.870,4.980,5.060,5.090,5.170,5.320,5.320,5.340,5.410,5.410,5.490,5.620,5.710,5.850,6.250,6.540,6.760,6.930,6.940,6.970,7.090,7.260,7.280,7.320,7.390,7.590,7.620,7.630,7.660,7.870,7.930,8.260,8.370,8.530,8.650,8.660,9.020,9.220,9.470,9.740,10.06,10.34,10.66,10.75,11.25,11.64,11.79,11.98,12.02,12.03,12.07,12.63,13.11,13.29,13.80,14.24,14.76,14.77,14.83,15.96,16.62,17.12,17.14,17.36,18.10,19.13,20.28,21.73,22.69,23.63,25.74,25.82,26.31,32.15,34.26,36.66,43.01,46.12,79.05

Lemonte and Cordeiro (2013).

Table 1: Summary of the two data sets.

Parameters	***N***	**Minimum**	***Q1***	**Median**	***Q3***	**Mean**	**Maximum**	**Variance**	**Skewness**	**Kurtosis**
Data set I	63	0.550	1.375	1.590	1.685	1.507	2.240	0.105	0.8786	3.9238
Data set II	128	0.080	3.348	6.395	11.84	9.366	79.050	0.795	3.286	18.483

The summary of the two data sets on table one (1) above reveals that the two data sets are positively skewed data. Data set one (1) has a skewness of 0.8786 and kurtosis of 3.9238 which indicates that the data sets is lightly skewed, while data sets two (2) has a skewess of 3.286 and kurtosis of 18.483 which indicates that the data set is highly skewed

In order to compare these distributions, the AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion) and BIC (Bayesian Information Criterion) will be used. These statistics are given as:

$AIC = -2ll + 2k,$

$CAIC = -2ll + 2kn/(n-k-1)$ and

$BIC = -2ll + k\log(n)$

Where ll denotes -log-likelihood value evaluated at the Maximum Likelihood Estimation k is the number of model parameters and n is the sample size.

Table 2: Performance of the distribution using the AIC, CAIC and BIC values of the models based on the data set I (strength of 1.5cm glass fibres).

Models	Parameter estimates	- loglikelihood	AIC	CAIC	BIC
GWPD	[illegible] $\hat{\beta}$=2.3949 $\hat{\theta}$=1.1020 $\hat{\lambda}$=0.8118 [illegible]	31.30	39.30	39.99	38.50
EDPD	$\hat{\lambda}$=0.8118 $\hat{\theta}$=19.0716 $\hat{\lambda}$=3.6803	98.62	102.62	102.82	102.22

EWPD	$\hat{\beta}$=8.1327 $\hat{\theta}$=2.6577 $\hat{\lambda}$=3.0991	171.06	179.06	179.75	178.26
EPD	$\hat{\beta}$=1.5264 $\hat{\theta}$=5.7496 $\hat{\lambda}$=0.6932	30.42	36.42	36.83	35.82
PD	$\hat{\lambda}$=0.6932 $\hat{\alpha}$=1.0662 $\hat{\beta}$=1.7857	31.18	35.18	35.42	34.78
WD	$\hat{\alpha}$=1.0336 $\hat{\beta}$=3.0341	33.81	41.81	42.05	41.41

Table 3: Performance of the distribution using the AIC, CAIC and BIC values of the models based on the data set II (the remission times (in months) of a random sample of 128 bladder cancer patients).

Models	Parameter estimates	- loglikelihood	AIC	CAIC	BIC
GWPD	$\hat{\beta}$=3.3025 $\hat{\theta}$=0.2779 $\hat{\lambda}$=0.0587	38.90	46.90	46.72	47.26
EDPD	$\hat{\theta}$=0.5647 $\hat{\lambda}$=0.0061 $\hat{\beta}$=0.5700	111.08	117.08	116.99	117.40
EWPD	$\hat{\beta}$=4.1020 $\hat{\theta}$=2.6007 $\hat{\lambda}$=2.0911	50.04	58.04	57.79	58.46
EPD	$\hat{\beta}$=1.0004 $\hat{\theta}$=0.3786 $\hat{\lambda}$=0.0059	128	134.88	134.79	135.20
PD	$\hat{\lambda}$=0.0059 $\hat{\alpha}$=0.5173 $\hat{\beta}$=0.0034	96.48	100.48	100.45	100.69
WD	$\hat{\beta}$=0.0034 $\hat{\alpha}$=0.2485 $\hat{\beta}$=0.2146	69.56	73.56	73.53	73.77

DISCUSSION OF RESULTS

Table 2 show the Maximum Likelihood Estimates (MLEs) to each one of the six fitted distributions for the data set I. The table also show the corresponding values of minus log-likelihood, AIC, BIC and CAIC for each model. The values in Table 2 show that the Exponential Pareto distribution performed better than the other five models considered in the analysis and could be chosen as the best model compared to the other baseline distributions used here for fitting the same data set. Hence, the Generalized Weibull-Pareto Distribution is still far better than the Exponentiated Pareto distribution, Weibull distribution and the Exponentiated Weibull-Pareto distribution.

The values in Table 3 show that the Generalized Weibull-Pareto distribution *(GWPD)* performed better than the other five models considered in the analysis and could be chosen as the best model compared to the other five baseline distributions used here for fitting the same data set. Hence, the proposed model which has more parameters can only be used comfortably when there are many population characteristics to be captured in the model and also for modeling a highly skewed data set.

CONCLUSION

A new four-parameter model called the Generalized Weibull-Pareto Distribution (*GWPD)* was introduced. Some Mathematical and Statistical properties of the propose distribution were also studied appropriately. The study derived explicit expressions for its Moments, Moment generating function, Survival and Hazard function. The implications of the plots for the survival and hazard functions indicate that, the Generalized Weilbull-Pareto distribution would be appropriate in modeling time or age-dependent events, where survival and failure rate decreases or increase with time or age. Some plots of the distribution revealed that it is a highly skewed Probability distribution. The model parameters were estimated using the method of Maximum Likelihood Estimation. Finally, the performance of the new model tested and result shows that the *GWPD* distribution is more flexible and appropriate for modeling highly skewed or asymmetric data sets better than some extensions of the Pareto distributions considered here using the same data set.

RECOMMENDATIONS

Based on the findings in this research, the study recommend that the proposed distribution should be used for modeling if the data set in question is highly skewed most especially if it is positively skewed. The distribution should not be use if the data set is lightly skewed that is, if it is not highly skewed since the distribution is an extension of a highly skewed probability distribution. Therefore, it follows a highly skew dada set. It is also recommended that this distribution can be used confidently in modeling time or age dependent events, systems, components or random variables.

REFERENCE

Afify, A. Z., Yousof, H. M., Hamedani, G. G. and Aryal, G. (2016). The Exponentiated Weibull Pareto Distribution with Application. *Journal of Data Science,* 6: 1-19.

Ali, M. M., Manisha, P. and Jungsoo, W. (2007). Some Exponentiated Distributions, *the Korean Communications in Statistics*, 14: 93-109

Barreto-Souza, W. M., and Simas, A. B, (2011). Some result of Beta Frechet Distribution. *Communication in Statistics-theory and method.* 40: 798-811

Bourguignon, M. R., Silva, L. M, and Cordeiro, G. M, (2012). The Kumaraswamy Pareto Distribution, *Journal of Statistical Theory and Application,* 12:129-144.

Codeiro, G.M., Edwin, M. M., Ortega, M, and Thiago G. R. (2015). A new Generalized Weibull family of

Distributions: Mathematical properties and applications; *Journal of Statistical Distributions and Applications* (2015) 2:13001 10.1186/s40488-015-0036-6

Gupta R. C., Gupta R. D, and Gupta P. L. (1998); Modeling failure time data by Lehman alternatives. *Communications in Statistics-Theory and Methods,* 27, 887–904.

Klugman, S. A., Panjer, H. H., Wilmot, G. E. Loss models from data to decisions, second Edition., Wiley-inter-science, a John Wiley and sons, inc., New York, (2004).

Lemonte, A. J. and Codeiro, G. M. (2013). An Extended Lomax distribution. *Journal of Theoretical and Applied Statistics*, 47(4): 800-816.

Nadeau T. P. and Teorey, T. J. 2003; A Pareto Model for OLAP View Size Estimation. *Information Systems Frontiers*, 5, 137–147

Oguntunde, P. E., Balogun, O. S., Okagbue, H. I, and Bishop, S. A. (2015): The Weibull- Exponential Distribution: Its Properties and Applications, *Journal of Applied Sciences.* 15(11): 1305-1311.

Pickands, J. (1975). Statistical inference using extreme order Statistics, *Annals of Statistics,* 3: 119–131.

Smith, R. L. and Naylor, J. C. (1987), A Comparison of Maximum Likelihood and Bayesian Estimators for the Three-Parameter Weibull Distribution, *Applied Statistics,* 36: 358-369.

Wikipedia, Pareto Distribution. Available at https://en.wikipedia.org/wiki/paretodistribution. Retrieved on 12th June, 2016.

Zea, L. M., Silva, R. B., Bourguinon, M., Santos, A. M., and Codeiro, G. M. (2012). The Beta Exponentiated Pareto Distribution with Application to Bladder Cancer Susceptibility, *International Journal of Statistics and Probability* 1: 8-19.

