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A REFORMULATION OF IMPLICIT SIX STEP ADAMS MOULTON METHOD IN CONTINUOUS FORM FOR SOLUTION OF FIRST ORDER INITIAL VALUE PROBLEMS

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ABSTRACT

In this paper we reformulate the six step Adams mouton method into the continuous form. The process produces some schemes which are combined together to form the block method for parallel or sequential solution of ODE's. The suggested approach eliminates requirement for a starting value and there is anticipated speed up of computations. The method which is of order eight is shown to be consistent and zero stable, hence convergent.

1. Introduction

It is well known that initial valued problems of ordinary differential equations often arise in many practical applications, such as chemical reactor, theory of fluid mechanics, automatic control and combustion etc. Aikeu (1985). The traditional methods for solving ODEs generally fall into two main classes: linear multistep (multi-value) and Runge-Kutta (multi-stage) methods Wright (2002). A linear multistep method with continuous coefficients is considered and applied to solve (ivps). The traditional multistep methods including the hybrid ones can be made continuous through the idea of multistep collocation Lie and Norsett (1989) and Onumanyi etal (1994:1999). Following Onumanyi etal (1994:1999), we identify a continuous formula (CF). The CF is evaluated at some distinct points involving step and off-step points along with its first and second derivatives, where necessary, to obtain multiple discrete formulae for a simultaneous application to the ODEs with initial conditions. This approach of using simultaneous discrete formulae (linked to a CF) both as corrector formula circumvent the requirement for special predictor in the use of single discrete formula as corrector formula.

2. The Method

Let us consider the first order system of ODEs

$$y' = f(x, y), \quad a \leq x \leq b \tag{1}$$

Where y satisfies some additional two-points or multi-point boundary conditions which can involve

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derivative values either the point a, b or other points in between as well. The function f is sufficiently smooth, $f: R^m+1$ ---- R^m. where y is an m-dimensional vector and x is a scalar variable.

A particular useful class of methods for (1) is the k-step linear multistep methods (1mms) of the form.

$$\sum_{j=0}^{k}\phi_j y_{n+j} = h\sum_{j=0}^{k}\varphi_j f_{n+j} \tag{2}$$

The idea of the k-step MC, following Onumanyi et al (1994:1999), is to find a polynomial U of the form

$$U = \sum_{j=0}^{t-1}\phi_j(x)y_{n+j} + h\sum_{j=0}^{m-1}\varphi_j(x)f\left(x_j, u(\overline{x}_j)\right), \qquad x_n \le x \le x_{n+k} \tag{3}$$

Where t denotes the number of interpolation points x_{n+1}, $i = 0, 1, \ldots, t-1$, m denotes the distinct collocation points $x_i \in [x_n, x_{n+k}]$ $i = 0, 1, \ldots, m-1$

The points x_i, are chosen from the steps x_{n+1} as well as one more off-step points.

We make the following assumption:

(1) For a given mesh

$x_n : x_n = a + nh$, $n = 0(1)N$ where $h = x_{n+1} - x_n$ $N = (b-a)/h$ is a constant step size (2)

That (1) has a unique solution and the coefficients $\phi_j(x)$ and $\varphi_j(x)$ in (3) can be represented by polynomials of the form.

$$\phi_j(x) = \sum_{i=0}^{t+m-1}\varphi_{j,i+1}x^i; \qquad j \in \{0, 1, \ldots, t-1\} \tag{4}$$

$$h\varphi_j(x) = \sum_{i=0}^{t+m-1}\varphi_{j,i+1}x^i; \qquad j \in \{0, 1, \ldots, m-1\} \tag{5}$$

With constant coefficients $\phi_{j,j+1}$, $h\varphi_{j,j+1}$ to be determined using interpolation and collocation conditions:

$$U(x_{n+1}) = y_{n+1}, \quad i \in \{0, 1, \ldots, t-1\} \tag{6}$$

$$U'(\overline{x}_j) = f(\overline{x}_j, u(\overline{x}_j)); \quad j \in \{0, 1, \ldots, m-1\} \tag{7}$$

With these assumptions we obtain an MC polynomial see Onumanyi etal (1994:1999) in the form:

$$U(x) = \sum_{i=0}^{t+m-1}a_i x^i, \; a_i = \sum_{i=0}^{t-1}c_{i+1,j+1}y_{n+j} + \sum_{j=0}^{m-1}c_{i+1,j+1}f_{n+j} \tag{8}$$

Where $x_n \le x \le x_{n+k}$ and $c_{i,j}$, $i, j = 1, 2, \ldots, t+m$ are constants given by the elements of the inverse matrix $C = D^{-1}$ the multistep collocation matrix D is an $m+1$ square matrix of the type

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{r+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{r+m-1} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & x_{n+r-1} & x_{n+r-1}^2 & \cdots & x_{n+r-1}^{r+m-1} \\ 0 & 1 & 2x_0 & \cdots & (r+m-1)x_0^{r+m-2} \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ 0 & 1 & 2x_{m-1} & \cdots & (r+m-1)x_{m-1}^{r+m-2} \end{bmatrix} \tag{9}$$

with exact solution $y(x) = e^{-x}$ and Taking $h = 0.1$

Example 2

Consider the initial value problem

$$y' = -9y,\ y(0) = e^{x},\quad 0 \le x \le 1$$

with exact solution $y(x) = e^{1-9x}$ and Taking $h = 0.1$

Firstly we transform the schemes by substitution to get a recurrence relation and solving simultaneously at each step, we obtain values of $y(x)$ and the results are summarized in Table 1 and Table 2

Table 1 : Example 1

N	x	Numerical Solution	Exact Solution	Absolute Error
0	0	1.000000000	1.000000000	0
1	0.1	0.9048374175	0.9048374180	4.999999303E-10
2	0.2	0.8187307519	0.8187307531	1.199999988E-09
3	0.3	0.7408182199	0.7408182207	8.000000662E-10
4	0.4	0.6703200460	0.6703200460	0
5	0.5	0.6065306597	0.6065306597	0
6	0.6	0.5488116365	0.5488116361	3.999999221E-10
7	0.7	0.4965853039	0.4965853038	1.000000083E-10
8	0.8	0.4493289642	0.4493289641	1.000000083E-10
9	0.9	0.4065696594	0.4065696597	2.999999693E-10
10	1.0	0.3678794410	0.3678794412	2.000000165E-10

Table 2 : Example 2

N	x	Numerical Solution	Exact Solution	Absolute Error
0	0	2.718281828	2.718281828	0
1	0.1	1.104467461	1.105170918	7.034570000E-04
2	0.2	0.4492293732	0.4493289640	9.959080000E-05
3	0.3	0.1825388112	0.1826835240	1.447128000E-04
4	0.4	0.07432156949	0.07427357800	4.799149000E-05
5	0.5	0.03001528010	0.03019738300	1.821029000E-04
6	0.6	0.01302217282	0.01227734000	7.448328200E-04
7	0.7	0.005291050123	0.004991594000	2.994561230E-04
8	0.8	0.002152073475	0.002029431000	1.226424750E-04
9	0.9	0.000874468496	0.0008251050000	4.936349600E-05
10	1.0	0.000356044123	0.0003335463000	2.249782300E-05

6. Conclusions

We have converted a six-step implicit Adams mouton method into the continuous form. The

$$y_{n+1} = y_{n+5} + \frac{8}{495}hf_n - \frac{38}{105}hf_{n+1} - \frac{136}{105}hf_{n+2} - \frac{664}{945}hf_{n+3} - \frac{136}{105}hf_{n+4}$$
$$-\frac{38}{105}hf_{n+5} + \frac{8}{945}hf_{n+6}$$

order 8 $C_{p+1} = -\frac{13}{14175}$

$$y_n = y_{n+5} - \frac{3715}{12096}hf_n - \frac{725}{504}hf_{n+1} - \frac{2125}{4032}hf_{n+2} - \frac{250}{189}hf_{n+3} - \frac{3875}{4032}hf_{n+4}$$
$$-\frac{235}{504}hf_{n+5} + \frac{275}{12096}hf_{n+6}$$

order 8 $C_{p+1} = -\frac{325}{10368}$ (11)

Hence (11) constitute the five member block methods which are later solved simultaneously along with the initial values of (1). This is an initial value approach and its main advantage is the elimination of the use of special predictors in the application (11).

4. Discussion Of Results

Using the matlab package, we were able to plot the stability region of the proposed block method. This is done by reformulating the block method as general linear method to obtain the values of the matrices A, B, U, V which are then substituted into stability matrix and stability function. Then the utilized maple package yielded the stability polynomial of the block method. Using a matlab program we plot the absolute stability region of proposed 6-step block Adams Mouton Method as:

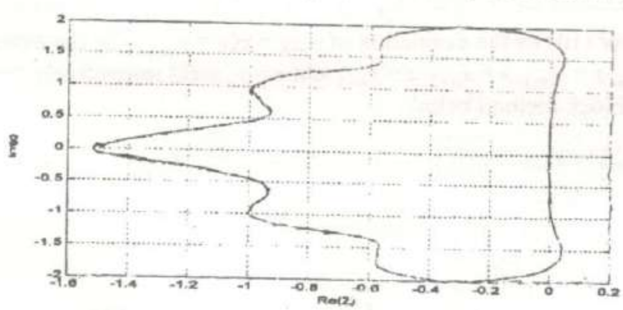

Figure 1 : Region Of Absolute Stability

5. Numerical Experiments

In this section we have tested the performance of our method on two examples and for each example we compared the result with the exact solution.

Example 1

Consider the initial value problem

$y' = -y, y(0) = 1, \quad 0 \le x \le 1$

Is the multistep collocation matrix of dimension $(r + m) \times (r + m)$. Then it follows from (9) that the column of matrix $C = D^{-1}$ give the continuous coefficients $\alpha_j(x)$ and $\beta_j(x)$

3. Derivation Of The Continuous Method

Let us consider six step Adams mouton method Butcher (2003)

$$y_{n+6} = y_{n+5} - \frac{863}{60480}hf_n + \frac{263}{2520}hf_{n+1} - \frac{6737}{20160}hf_{n+2} + \frac{586}{945}hf_{n+3} - \frac{15487}{20160}hf_{n+4} + \frac{2713}{2520}hf_{n+5} + \frac{19087}{60480}hf_{n+6} \quad (10)$$

The general approach is to consider the data for the matrix D in $\{x_m, x_{n,+6}\}$. With $m = 7$, $t = 1$ and $p = m + t$ also

$$D = \begin{bmatrix} 1 & x_{n+5} & x_{n+5}^2 & x_{n+5}^3 & x_{n+5}^4 & x_{n+5}^5 & x_{n+5}^6 & x_{n+5}^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & 7x_{n+4}^6 \\ 0 & 1 & 2x_{n+5} & 3x_{n+5}^2 & 4x_{n+5}^3 & 5x_{n+5}^4 & 6x_{n+5}^5 & 7x_{n+5}^6 \\ 0 & 1 & 2x_{n+6} & 3x_{n+6}^2 & 4x_{n+6}^3 & 5x_{n+6}^4 & 6x_{n+6}^5 & 7x_{n+6}^6 \end{bmatrix}$$

And the element of D^{-1} yield C_q $i, j = 1, 2, 3, 4, 5, 6, 7, 8$ in (8) to give the continuous form.

We recover (10) by the evaluation of $y_{n+6} = u(x = x_{n+6})$. On the evaluating of the continuous form at $x = x_{n+4}$, $x = x_{n+3}$, $x = x_{n+2}$, $x = x_{n+1}$ and $x = x_n$ yield respectively the five discrete formulae which form the block method below.

$$y_{n+4} = y_{n+5} - \frac{271}{60480}hf_n + \frac{29}{840}hf_{n+1} - \frac{811}{6720}hf_{n+2} + \frac{254}{945}hf_{n+3} - \frac{5221}{6720}hf_{n+4} - \frac{349}{840}hf_{n+5} + \frac{863}{60480}hf_{n+6}$$

order 8 $C_{p+1} = -\frac{16811}{1814400}$

$$y_{n+3} = y_{n+5} - \frac{1}{756}hf_n + \frac{1}{126}hf_{n+1} - \frac{11}{1260}hf_{n+2} - \frac{322}{945}hf_{n+3} - \frac{1621}{1260}hf_{n+4} - \frac{233}{630}hf_{n+5} + \frac{37}{3780}hf_{n+6}$$

order 8 $C_{p+1} = -\frac{251}{56700}$

$$y_{n+2} = y_{n+5} - \frac{13}{2240}hf_n + \frac{3}{56}hf_{n+1} - \frac{1161}{2240}hf_{n+2} - \frac{34}{35}hf_{n+3} - \frac{2631}{2240}hf_{n+4} - \frac{111}{280}hf_{n+5} + \frac{29}{2240}hf_{n+6}$$

order 8 $C_{p+1} = -\frac{29}{3200}$

continuous formulae are immediately employed as block method for direct solution of six point IVPs. The direct solutions are in discrete form which can be substituted into the continuous formula for dense output. The proposed method is self starting, convergent and $A(\alpha)$ stable as shown by the plotted region of absolute stability (Figure 1). The method demonstrated satisfactory performance when applied to solve a simple ODE, without recourse to any other method to provide the starting values.

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