REFINEMENT OF EXTENDED ACCELERATED OVER-RELAXATION METHOD FOR SOLUTION OF LINEAR SYSTEMS

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Abstract:

In this paper, a Refinement of Extended Accelerated Over-Relaxation (REAOR) iterative method for solving linear systems is presented. The method is designed to solve problems of partial differential equations that results into linear systems having coefficient matrices such as weak irreducible diagonally dominant matrix and $L-$matrix (or $M-$matrix). Sufficient criterion for convergence are examined and few numerical illustrations are considered to ascertain efficiency of the new method. Outcome of the numerical results reveals that the REAOR iterative method is more efficient when compared with Extended Accelerated Over-Relaxation iterative method in terms of computational time, level of accuracy and required number of iterations for convergence.

Keywords: sufficient, Refinement, REAOR, Extended, diagonally dominant matrix, convergence, efficiency, partial differential equation, linear systems.

1. INTRODUCTION

Solutions of linear systems is basically one of the most important aspect of solving physical problems encountered in fields like science, engineering, biological sciences and so on. This is due to the fact that, it is virtually impossible to perform any sort of numerical analysis without coming across linear algebraic equations (Laskar and Behera, 2014). Moreso, sets of linear equations generated from partial differential equations are mostly sparse and large, consumes computation time and resources. The Direct and iterative methods are usually employed to solve such linear systems and utilizing some properties of the coefficient matrices like sparseness for example usually makes it possible for storage reduction, run time and cost (Kiusalaas, 2005). Iterative methods are quite effective and usually preferable over direct methods in finding solutions to sparse/large linear systems. Modification of existing iterative methods in order to achieve higher rate of convergence led to several refinement methods. Dafchahi (2008) introduced the refinement of Jacobi (RJ) method and proved that the method is superior to Jacobi method. Vatti and Eneyew (2011) and Genanew (2016) enhanced the convergence rate of the Gauss-Seidel method by developing the refinement of Gauss-Seidel (RGS) method. Kyurkhiev and Iliev (2013) modified the SOR and SSOR schemes by proposing the refinement of Successive Over-Relaxation (RSOR) and refinement of Symmetric Successive Over-Relaxation (RSSOR) methods. Vatti *et al.* (2015) improved on the SOR scheme and proposed the RSOR method. Vatti *et al.* (2018) attempted to magnify convergence rate of the AOR method by the method called refinement of Accelerated Over-Relaxation (RAOR) method. Muleta and Gofe (2018) developed a Refinement of generalized Accelerated Over Relaxation (RGAOR) method. Recently, Audu *et al.* (2021) introduced the Extended Accelerated Over-Relaxation method, an efficient three-parameter method that generalizes the AOR, SOR, Jacobi and Gauss-Seidel methods and superior to the above mentioned methods.

This research work is aimed at increasing the rate of convergence of the Extended Accelerated Over-Relaxation (EAOR) method by refining the method into a more efficient method that would hasten the numerical solution of linear systems.

We consider the linear system expressed as

$$Az = b \tag{1}$$

where $\tilde{A} = [a_{ij}]$ is the coefficient matrix, z is the unknowns and b is the constant at the right hand side. Many methods are employed in solving (1) through the splitting of

$$A = \widetilde{D} - \tilde{L} - \widetilde{U} \tag{2}$$

where $-\overline{U}, -\overline{L}$ and $\widetilde{D}$ are strictly upper, strictly lower and diagonal part of A respectively. Furthermore, (1) can be written as

$$\left(\widetilde{D} - \tilde{L} - \widetilde{U}\right)z = b \tag{3}$$

Or

$$(I - L - U)z = b \tag{4}$$

Where $\widetilde{D}^{-1}\tilde{L} = L$, $\widetilde{D}^{-1}\widetilde{U} = U$, $\widetilde{D}^{-1}\widetilde{D} = I$ and $\widetilde{D}^{-1}b = \tilde{b}$. Hence the system of equation in (1) is transformed into

$$\tilde{A}z = \tilde{b} \tag{5}$$

where $\tilde{A} = \widetilde{D}^{-1}A$. The popular AOR method by Hadjidimos (1978) for numerical solution of (1) is expressed as

$$z^{(i+1)} = E_{\omega,r}z^{(i)} + [D - rL]^{-1}\omega\tilde{b}, \qquad i = 0, 1, 2 \tag{6}$$

Here $E_{\omega,r} = [I - rL]^{-1}[(1-\omega)I + (\omega - r)L + \omega U]$ is the iteration matrix of the AOR method. The Extended Accelerated Over-Relaxation method for numerical solution of (1) is given as

$$z^{(i+1)} = E_{\omega,v,r}z^{(i)} + [I - (r+v)L]^{-1}\omega\tilde{b}, \qquad i = 0, 1, 2, \ldots \tag{7}$$

Where $E_{\omega,v,r} = [I - (r+v)L]^{-1}[(1-\omega)I + [\omega - (r+v)]L + \omega U]$ represents the EAOR iterative matrix and the spectral radius of the iterative matrix is denoted as $\rho\left(E_{\omega,v,r}\right)$. We shall consider weak irreducible diagonally dominant and L (or M) matrices for the coefficient matrix A in (1).

2. Development of the REAOR Method

Considering the regular splitting of

$$\tilde{A} = \frac{1}{\omega}(P - Q) \xrightarrow{yields} \omega\tilde{A} = P - Q \tag{8}$$

Where the choices for P and Q of the EAOR method are represented as

$$\left.\begin{aligned} P &= [I - (v+r)L] \\ Q &= (1-\omega)I + [\omega - (v+r)]L + \omega U \end{aligned}\right\} \tag{9}$$

to solve (1) by a refinement iterative method, we multiply the linear system $\tilde{A}z = \tilde{b}$ by the over-relaxation parameter ω to obtain

$$\omega\left(\tilde{A}z\right) = \omega\left(\tilde{b}\right) \tag{10}$$

Or

$$(P-Q)z=\omega\tilde{b} \tag{11}$$

Which gives

$$\big[[I-(v+r)L]-[(1-\omega)I\ +[\omega-(v+r)]L+\omega U]\big]z=\omega\tilde{b} \tag{12}$$

$$[I-(v+r)L]z=[(1-\omega)I+[\omega-(v+r)]L+\omega U]z+\omega\tilde{b} \tag{13}$$

$$[I-(v+r)L]z=[I-(v+r)L]z+\omega\big(\tilde{b}-\tilde{A}z\big) \tag{14}$$

$$z=z+\omega[I-(v+r)L]^{-1}\big(\tilde{b}-\tilde{A}z\big) \tag{15}$$

From (15), the **REAOR formula** takes the form;

$$\bar{z}^{(i+1)}=z^{(i+1)}+\omega[I-(v+r)L]^{-1}\big(\tilde{b}-\tilde{A}z^{(i+1)}\big) \tag{16}$$

Where $z^{(i+1)}$ appearing in the right hand side, is $(i+1)^{th}$ estimation of the EAOR iterative method. We simplify further to obtain

$$\begin{aligned}\bar{z}^{(k+1)}&=z^{(k+1)}+\omega[P-(v+r)L_A]^{-1}\big(b-Az^{(k+1)}\big)\\&=z^{(k+1)}+\omega P^{-1}\big(b-Az^{(k+1)}\big)\\&=z^{(k+1)}+\omega P^{-1}b-P^{-1}\omega Az^{(k+1)}\\&=z^{(k+1)}+P^{-1}\omega b-P^{-1}(P-Q)z^{(k+1)}\\&=z^{(k+1)}+P^{-1}\omega b-P^{-1}Pz^{(k+1)}+P^{-1}Qz^{(k+1)}\\&=z^{(k+1)}+P^{-1}\omega b-z^{(k+1)}+P^{-1}Qz^{(k+1)}\\&=P^{-1}\omega b+P^{-1}Qz^{(k+1)}\end{aligned} \tag{17}$$

But $z^{(k+1)}$ is the EAOR method in (7) given as $z^{(k+1)}=P^{-1}Qz^{(k+1)}+P^{-1}\omega b$, so substituting it in the last equation above gives

$$\begin{aligned}\bar{z}^{(k+1)}&=P^{-1}\omega b+P^{-1}Q\big(P^{-1}Qz^{(k)}+P^{-1}\omega b\big)\\&=P^{-1}\omega b+(P^{-1}Q)^2z^{(k)}+(P^{-1})^2Q\omega b\\&=(P^{-1}Q)^2z^{(k)}+(I+P^{-1}Q)P^{-1}\omega b\end{aligned} \tag{18}$$

Substituting the values of P and Q into the last equation gives

$$\begin{aligned}\bar{z}^{(i+1)}&=([I-(v+r)L]^{-1}[(1-\omega)I+[\omega-(v+r)]L+\omega U])^2z^{(i)}\\&+(I+[(1-\omega)I+[\omega-(v+r)]L+\omega U])[I-(v+r)L]^{-1}\omega\tilde{b}\end{aligned} \tag{19}$$

The method in (19) is called the REAOR method, or can be represented as

$$\bar{z}^{(i+1)}=Hz^{(i)}+K, \qquad i=0,1,2 \tag{20}$$

Where $H=\Big((I-(r+v)L)^{-1}\big((1-\omega)I+[\omega-(r+v)]L+\omega U\big)\Big)^2$ is the iteration matrix of the REAOR method. The spectral radius of the REAOR method is the modulus of the largest eigenvalue of its iteration matrix represented as $\rho\big(RE_{v,\omega,r}\big)$. A stationary iterative method $z^{(i+1)}=Hz^{(i)}+K$ converges if the spectral radius of its iteration matrix (H) is less than 1, that is to say if $\rho(H)<1$.

3. Convergence Analysis of the Method

Lemma 1. Suppose A is an L or M −matrix with the range of $0<v+r\le\omega\le 1$ for the parameters, then the EAOR method is convergent for any arbitrary initial approximation $z^{(0)}$.

Proof: *see* Audu *et al.* (2021).

Lemma 2. If A is an irreducible weak diagonally dominant matrix for $0 < v + r < 1$ and $0 < \omega \leq 1$. Then the EAOR method is convergent for any arbitrary initial estimation $z^{(0)}$.

Proof: *see* Audu *et al.* (2021).

Theorem 1*:* If a square matrix $\left(A_{ij}\right)_{m\times m}$ is considered an $L-$ matrix or an $M-$ matrix

$L-$matrix: a matrix where $a_{ij} \leq 0\ (i \neq j)$ and $a_{ii} > 0,\ i = 1,2,\cdots,n.$ Or

$M-$matrix: a matrix where $a_{ij} \leq 0\ (i \neq j),\ a_{ii} > 0,\ A$ is non-singular and $A^{-1} \geq 0,$ then

The Refinement of the Extended Accelerated Over-Relaxation (REAOR) method converges to the exact solution for any initial guess $z^{(0)}$.

Proof:

If z is the true solution of $Az = b$, since the coefficient matrix A is an $L-$matrix, it follows from corollary 1 that the EAOR method is convergent. This makes it possible to arbitrarily close $z^{(k+1)}$ to z, hence we get

$$\left.\begin{aligned}\left\|\bar{z}^{(k+1)} - z\right\|_\infty &= \left\|z^{(k+1)} + \omega[I - (v+r)L]^{-1}\left(b - Az^{(k+1)}\right) - z\right\|_\infty \\ &\leq \left\|z^{(k+1)} - z\right\|_\infty + |\omega|\|[I - (v+r)L]^{-1}\|_\infty \left\|\left(\tilde{b} - Az^{(k+1)}\right)\right\|_\infty\end{aligned}\right\} \tag{21}$$

From the point that $\left\|z^{(k+1)} - z\right\|_\infty \to 0$, we get $\left\|\left(b - Az^{(i+1)}\right)\right\|_\infty \to 0$. This signifies that $\bar{z}^{(i+1)}$ tends to zero as $k \to \infty$

$$\left\|\bar{z}^{(i+1)} - z\right\|_\infty \to \quad 0 \tag{22}$$

Thus $\rho\left(RE_{v,r,\omega}\right) < 1$ or equivalently $\rho\left((I - (v+r)L)^{-1}\left((1-\omega)I + [\omega - (v+r)]L + \omega U\right)\right)^2 < 1$. This shows that the REAOR is a convergent method and the proof is completed.

Theorem 2: Let A be weak irreducible diagonally dominant. A square matrix A is considered as a weak diagonally dominant if and only if $|a_{ii}| \geq \sum_{j=1,j\neq i}^{n}\left|a_{ij}\right|,\ i = 1, 2, \cdots, n$ Then for any choice of initial guess $z^{(0)}$, the REAOR method converges to the true solution .

Proof:

Let z be the true solution of (1) and $A = \left(a_{ij}\right)$ is weak irreducible diagonally dominant. Then in view of lemma 2, the EAOR method is convergent and applying similar procedure of theorem 1, it is obvious that the REAOR method is convergent for irreducible weak diagonally dominant matrix and this completes the proof.

Theorem 3: For any initial estimation $z^{(0)}$, the refinement of Extended Accelerated Over Relaxation method converges to the true solution twice as fast as the EAOR method.

Proof:

Let $\|J\| < 1$, we consider the EAOR method $z^{(i+1)} = Jz^{(i)} + E$ and the REAOR method $\bar{z}^{(i+1)} = J^2 z^{(i)} + K$ where

$$\begin{gathered} J = [I - rL - vL]^{-1}[(1-\omega)I + (\omega - r - v)L + \omega U] \\ E = [I - (r+v)L]^{-1}\omega\tilde{b} \\ K = (I + J)[I - (r+v)L]^{-1}\omega\tilde{b} \end{gathered} \tag{23}$$

Let Z be the true solution of (1) that satisfies $z^{(i+1)} = Jz^{(i)} + E$, then it implies that $Z = JZ + E$ with respect to EAOR method and similarly $Z = J^2Z + K$ also satisfies the equation $\bar{z}^{(i+1)} = J^2 z^{(i)} + K$ with respect to REAOR method.

Considering the EAOR method, we have $z^{(i+1)} = Jz^{(i)} + E$ and $Z = JZ + E$, we obtain

$$\begin{aligned} z^{(i+1)} - Z &= Jz^{(i)} + E - Z \\ &= J(z^{(i)} - Z) + JZ + E - Z \\ &= J(z^{(i)} - Z) + JZ + E - (JZ + E) \\ z^{(i+1)} - Z &= J(z^{(i)} - Z) \end{aligned} \tag{24}$$

Taking norm of $z^{(i+1)} - Z = J\left(z^{(i)} - Z\right)$ gives

$$\begin{aligned} \left\|z^{(i+1)} - Z\right\|_\infty &= \left\|J\left(z^{(i)} - Z\right)\right\| \\ \|z^{(i+1)} - Z\|_\infty &\leq \|J\|\|(z^{(i)} - Z)\| \\ \|z^{(i+1)} - Z\|_\infty &\leq \|J^2\|\|(z^{(i-1)} - Z)\| \\ \|z^{(i+1)} - Z\|_\infty &\leq \|J\|^i\|(z^{(0)} - Z)\| \end{aligned} \tag{25}$$

Then $z^{(i+1)} \to Z$ as $i \to \infty$ if $\|J\|^i < 1$. Now considering the REAOR method, we have

$$\begin{aligned} \bar{z}^{(i+1)} - Z &= J^2 z^{(i)} - Z + K \\ &= J^2(z^{(i)} - Z) + J^2 Z + K - Z \\ &= J^2(z^{(i)} - Z) + JZ + E - (JZ + E) \\ \bar{z}^{(i+1)} - Z &= J^2(z^{(i)} - Z) \end{aligned} \tag{26}$$

Taking norm of $\bar{z}^{(i+1)} - Z = J^2\left(z^{(i)} - Z\right)$ gives;

$$\begin{aligned} \left\|\bar{z}^{(i+1)} - Z\right\|_\infty &= \left\|J^2\left(z^{(i)} - Z\right)\right\|_\infty \\ \|\bar{z}^{(i+1)} - Z\|_\infty &\leq \|J^4\|\|(z^{(i-1)} - Z)\|_\infty \\ &\vdots \\ \|\bar{z}^{(i+1)} - Z\|_\infty &\leq \|J^{2i}\|\|(z^{(0)} - Z)\|_\infty \\ \|\bar{z}^{(i+1)} - Z\|_\infty &\leq \|J\|^{2i}\|(z^{(0)} - Z)\|_\infty \end{aligned} \tag{27}$$

Comparing coefficients of $\|J\|_\infty^i\left\|\left(z^{(1)} - Z\right)\right\|_\infty$ and $\|J\|_\infty^{2i}\left\|\left(z^{(1)} - Z\right)\right\|_\infty$, clearly $\left\|J^{2i}\right\| < \left\|J^i\right\|$ since $\|J\| < 1$ and this tells us that the REAOR method converges faster than the EAOR method and the proof is completed.

4.1 Numerical Experiment

In this section we present some numerical tests with respect to the new method. Specifically, we computed the spectral radius of the REAOR iteration matrix and further obtained its convergence results. We compared the results obtained with those of Accelerated Over-Relaxation method, Refinement of Accelerated Over-Relaxation method and Extended Accelerated Over-Relaxation to test the efficiency of the new method. The computations were carried out using Maple 2017 software with accuracy of 10 decimal places and the results are presented in tables 1 to 4.

Problem 1: Solve the linear system A_1 whose coefficient matrix is an Irreducible diagonally dominant matrix by EAOR and REAOR methods.

$$A_1 = \begin{pmatrix} 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 1 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \\ z_9 \end{pmatrix} = \begin{pmatrix} 375 \\ 250 \\ 250 \\ 500 \\ 0 \\ 0 \\ 375 \\ 250 \\ 250 \end{pmatrix}$$

Problem 2: Solve the linear system A_2, whose coefficient matrix is an L matrix by EAOR and REAOR methods.

$$A_2 = \begin{pmatrix} 1 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & 0 \\ -\frac{1}{7} & 1 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 \\ 0 & -\frac{1}{7} & 1 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} \\ -\frac{1}{7} & 0 & -\frac{1}{7} & 1 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 \\ 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 1 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} \\ -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 1 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 \\ 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 1 & -\frac{1}{7} & 0 & -\frac{1}{7} \\ -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 1 & -\frac{1}{7} & 0 \\ 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 1 & -\frac{1}{7} \\ 0 & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 0 & -\frac{1}{7} & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \\ z_9 \\ z_{10} \end{pmatrix} = \begin{pmatrix} 11.90 \\ 9.32 \\ 8.09 \\ 9.32 \\ 8.09 \\ 8.32 \\ 8.09 \\ 8.32 \\ 8.09 \\ 8.32 \end{pmatrix}$$

Table 1: Comparison Results for Problem 1

ω	r	$\rho(AOR)$	$\rho(RAOR)$	v	$\rho(EAOR)$	$\rho(REAOR)$
0.1	0.04	0.9702892488	0.9414612263	0.05	0.9697412031	0.9403980010
0.2	0.08	0.9397055636	0.8830465463	0.10	0.9373783471	0.8786781656
0.3	0.12	0.9082009036	0.8248288813	0.15	0.9026234150	0.8147290293
0.4	0.16	0.8757229293	0.7668906489	0.20	0.8651200559	0.7484327111
0.5	0.20	0.8422144385	0.7093251604	0.25	0.8244179744	0.6796649965
0.6	0.24	0.8076127016	0.6522382758	0.30	0.7799331029	0.6082956450
0.7	0.28	0.7718486727	0.5957503735	0.35	0.7308822855	0.5341889152
0.8	0.32	0.7348460470	0.5399987128	0.40	0.6761686319	0.4572040187
0.9	0.36	0.6965201257	0.4851402855	0.45	0.6141612268	0.3771940125

Table 2: Comparison Results for Problem 2

ω	r	$\rho(AOR)$	$\rho(RAOR)$	v	$\rho(EAOR)$	$\rho(REAOR)$
0.1	0.04	0.9695659299	0.9400580924	0.05	0.9690220033	0.9390036428
0.2	0.08	0.9382651880	0.8803415630	0.10	0.9359663473	0.8760330033
0.3	0.12	0.9060538059	0.8209334992	0.15	0.9005738634	0.8110332834
0.4	0.16	0.8728842724	0.7619269530	0.20	0.8625309526	0.7439596442
0.5	0.20	0.8387051160	0.7034262716	0.25	0.8214516534	0.6747828189
0.6	0.24	0.8034604202	0.6455486468	0.30	0.7768509636	0.6034974196
0.7	0.28	0.7670892570	0.5884259282	0.35	0.7281033573	0.5301344989
0.8	0.32	0.7295250213	0.5322067567	0.40	0.6743746009	0.4547811023
0.9	0.36	0.6906946434	0.4770590904	0.45	0.6145013535	0.3776119135

Table 3: Convergence Result for Problem 1

Numerical Methods	**Number of Iterations**	**CPU Time** (seconds)
EAOR	56	0.296
EAOR	56	0.266
RAOR	41	0.125
REAOR	29	0.094

Table 4: Convergence Result for Problem 2

Numerical Methods	**Number of Iterations**	**CPU Time** (seconds)
AOR	60	0.109
EAOR	43	0.031
RAOR	31	0.062
REAOR	22	0.016

4.2 Discussion of Results

The rate of convergence is strongly connected to the spectral radius. The closer the spectral radius is to zero the faster the convergence. Comparison results of spectral radius of the new REAOR method and other existing methods for problem 1 and 2 are displayed in tables 1 and 2 respectively. It is observed that the spectral radius of the new method is lower than the spectral radii of EAOR, RAOR and AOR methods due to the fact that $\rho(J_{REAOR}) < \rho(J_{RAOR}) < \rho(J_{EAOR}) < \rho(J_{AOR}) < 1$. This implies that the REAOR method will converge faster to the true solution than EAOR, AOR and RAOR methods.

To validate the spectral radius result, we checked the convergence results for problem 1 and 2 given in tables 3 and 4. The REAOR method converges at 29^{th} and 22^{nd} iterations in comparison to 56^{th} and 43^{rd} iterations of the EAOR method, 41^{st} and 31^{st} iterations of the RAOR method and 81^{st} and 60^{th} iterations of the AOR method with respect to problems 1 and 2 respectively. The result shows that the new REAOR method converges faster to the true solution and took shorter time to achieve convergence when compared to similar existing methods.

5. Conclusion

Iterative methods are often employed in solving sparsely large linear systems. Although convergence is a major requirement of such methods, but how fast they converge is equally important. In this study, a new numerical iterative method for solving linear systems Refinement of Extended Accelerated Over-Relaxation method has been proposed. Convergence criteria for some special matrices were examined and two numerical examples of 9×9 and 10×10 linear systems were presented. Comparison of results of the new method with other existing methods in literature by their spectral radii and number of iterations, indicates that the REAOR method surpasses the other methods. Also, the study has shown that the Refinement method increases the convergence rate of the Extended Accelerated Over-Relaxation method.

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