SOME EXPLICIT ALMOST RUNGE – KUTTA METHODS FOR SOLVING INITIAL VALUE PROBLEMS

Ndanusa, A. and Audu, K. J.

Department of Mathematics, Federal University of Technology, Minna, Nigeria.

Corresponding Author: jameskhadeejah@yahoo.com

Abstact

Two new Explicit Almost Runge-Kutta methods for numerical integration of initial value problems are derived, by a **judicious choice** of the free parameters obtained after applying the **order** and **stability conditions** associated with Runge-Kutta methods. These methods which have the same number of stages as the order are of orders 4 and 5 i.e s$= p = 4$ and $s = p = 5$, respectively. Their convergence is established and numerical experiments with sample problems are conducted, in order to confirm their efficiency and reliability.

Keywords: Almost Runge-Kutta method, stability, Consistency, Convergence, Order

1. INTRODUCTION

Almost Runge-Kutta (ARK) methods are a special type of general linear methods whose properties closely resemble those of explicit Runge-Kutta methods. They were introduced by Butcher (1997) for the purpose of preserving the multi-stage character of Runge-Kutta scheme as well as passing many values between steps, thereby giving the method a multi-value character (Rattenbury, 2005). Several authors have developed some Almost Runge-Kutta methods. Three values are passed between steps and they are known as the starting values i.e

$$y(x_0), \quad hf(y(x_0)) \text{ and } hf(y(x_0) + hf(y(x_0))) - hf\big(y(x_0)\big) \qquad (1)$$

It uses three input and output approximations and are represented by 4 matrices:

$$A, B, U \text{ and } V \quad or \quad \begin{bmatrix} A & U \\ B & V \end{bmatrix} \qquad (2)$$

Other authors who have made their input towards the development of this method include Abraham (2010) and Alimi (2014).

2. Derivation of the Methods

2.1 Derivation of ARK4 ($S = P = 4$)

The general form of a fourth order four stage Almost Runge-Kutta scheme ($S = P = 4$) method takes the form:

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] = \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 1 & c_1 & \frac{1}{2}c_1^2 \\ a_{21} & 0 & 0 & 0 & 1 & c_2 - a_{21} & \frac{1}{2}c_2^2 - a_{21}c_1 \\ a_{31} & a_{32} & 0 & 0 & 1 & c_3 - a_{31} - a_{32} & \frac{1}{2}c_3^2 - a_{31}c_1 - a_{32}c_2 \\ b_1 & b_2 & b_3 & 0 & 1 & b_0 & 0 \\ \hline b_1 & b_2 & b_3 & 0 & 1 & b_0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & 0 & \beta_0 & 0 \end{array}\right] \tag{3}$$

Given the abscissa vector $c = [c_1, c_2, c_3, 1]^T$, $b^T = [b_1, b_2, b_3, 0]$ $\beta^T = [\beta_1, \beta_2, \beta_3, \beta_4]$. The first output approximations for order four with four stages are:

$$b_0 + b^T e = 1 \tag{4}$$

$$b^T c = \frac{1}{2} \tag{5}$$

$$b^T c^2 = \frac{1}{3} \tag{6}$$

$$b^T c^3 = \frac{1}{4} \tag{7}$$

$$b^T A c^2 = \frac{1}{12} \tag{8}$$

$$b^T \left(\frac{1}{2}c^2 - Ac\right) = 0 \tag{9}$$

Combining equations (6) and (9) we get

$$b^T A c = \frac{1}{6} \tag{10}$$

From the annihilation conditions, it follows that,

$$\beta^T e + \beta_0 = 0 \tag{11}$$

$$\beta^T c = 1 \tag{12}$$

With the Runge-Kutta stability conditions:

$$\beta^T (I + \beta_4 A) = \beta_4 e_4^T \tag{13}$$

$$c_1 = -2\frac{exp_4(-\beta_s)}{\beta_4 exp_3(-\beta_4)} \tag{14}$$

$$\left(1 + \frac{1}{2}\beta_4 c_1\right) b^T A^2 c = \frac{1}{4!} \tag{15}$$

$$c_1 = -\frac{2\left(1 - \beta_4 + \frac{1}{2}\beta_4^2 - \frac{1}{6}\beta_4^3 + \frac{1}{24}\beta_4^4\right)}{\beta_4\left(1 - \beta_4 + \frac{1}{2}\beta_4^2 - \frac{1}{6}\beta_4^3\right)} \tag{16}$$

From equations (4) - (7), we obtained

$$\left.\begin{aligned} b_0 + b_1 + b_2 + b_3 &= 1 \\ b_1 c_1 + b_2 c_2 + b_3 c_3 &= \frac{1}{2} \\ b_1 c_1^2 + b_2 c_2^2 + b_3 c_3^2 &= \frac{1}{3} \\ b_1 c_1^3 + b_2 c_2^3 + b_3 c_3^3 &= \frac{1}{4} \end{aligned}\right\} \quad (17)$$

Which gives

$$b_1 = \frac{6c_2c_3 - 4c_2 - 4c_3 + 3}{12c_1(c_1 - c_3)(c_1 - c_2)} \quad (18)$$

$$b_2 = \frac{6c_1c_3 - 4c_1 - 4c_3 + 3}{12c_2(c_1 - c_2)(c_2 - c_3)} \quad (19)$$

$$b_3 = \frac{6c_1c_2 - 4c_1 - 4c_2 + 3}{12c_3(c_2 - c_3)(c_1 - c_3)} \quad (20)$$

$$b_0 = \frac{12c_1c_2c_3 - 6c_1c_2 - 6c_2c_3 + 4c_1 + 4c_2 + 4c_3 - 3}{12c_1c_2c_3} \quad (21)$$

And

$$a_{21} = \frac{1}{12b_3a_{32}c_1(2 + \beta_4 c_1)} \quad (21)$$

$$a_{31} = \frac{\frac{1}{6} - b_3a_{32}c_2 - b_2a_{21}c_1}{b_3c_1} \quad (23)$$

$$a_{32} = \frac{1 - 2c_1}{12b_3c_2(c_2 - c_1)} \quad (24)$$

From (11) and (13), values of β^T and vector β_0 are evaluated to obtain the ARK4 method: **SCHEME 1**: ARK4 with $c^T = \left[\frac{15}{34}, \frac{1}{2}, 1, 1\right]$, $\beta_4 = 4$

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] = \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 1 & \frac{15}{34} & \frac{225}{2312} \\ \frac{289}{1920} & 0 & 0 & 0 & 1 & \frac{671}{1920} & \frac{15}{256} \\ -\frac{289}{480} & 2 & 0 & 0 & 1 & -\frac{191}{480} & -\frac{15}{64} \\ 0 & \frac{2}{3} & \frac{1}{6} & 0 & 1 & \frac{1}{6} & 0 \\ \hline 0 & \frac{2}{3} & \frac{1}{6} & 0 & 1 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{578}{45} & \frac{32}{3} & -\frac{8}{3} & 4 & 0 & -\frac{38}{45} & 0 \end{array}\right] \qquad (25)$$

2.2. Derivation of ARK5 ($\boldsymbol{S = P = 5}$)

A Fifth order, five stage ARK method takes the form:

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] = \left[\begin{array}{ccccc|ccc} 0 & 0 & 0 & 0 & 0 & 1 & c_1 & \frac{1}{2}c_1^2 \\ a_{21} & 0 & 0 & 0 & 0 & 1 & c_2 - a_{21} & \frac{1}{2}c_2^2 - a_{21}c_1 \\ a_{31} & a_{32} & 0 & 0 & 0 & 1 & c_3 - a_{31} - a_{32} & \frac{1}{2}c_3^2 - a_{31}c_1 - a_{32}c_2 \\ a_{41} & a_{42} & a_{43} & 0 & 0 & 1 & c_4 - a_{41} - a_{42} - a_{43} & \frac{1}{2}c_4^2 - a_{41}c_1 - a_{42}c_2 - a_{43}c_3 \\ b_1 & b_2 & b_3 & b_4 & 0 & 1 & b_0 & 0 \\ \hline b_1 & b_2 & b_3 & b_4 & 0 & 1 & b_0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & 0 & \beta_0 & 0 \end{array}\right] \qquad (26)$$

SCHEME 2: ($s = p = 5$) ARK5 with $c^T = \left[\frac{52}{165}, \frac{1}{3}, , \frac{3}{4}, 1, 1\right]$

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] = \left[\begin{array}{ccccc|ccc} 0 & 0 & 0 & 0 & 0 & 1 & \frac{52}{165} & \frac{1352}{27225} \\ \frac{605}{19656} & 0 & 0 & 0 & 0 & 1 & \frac{5947}{19656} & \frac{26}{567} \\ -\frac{768955}{159744} & \frac{6027}{1024} & 0 & 0 & 0 & 1 & -\frac{51449}{519744} & -\frac{377}{2304} \\ \frac{1802767635}{51845976} & -\frac{7006}{193} & \frac{231424}{166173} & 0 & 0 & 1 & \frac{1388045}{1264536} & \frac{21346}{36477} \\ \frac{5490375}{6745648} & -\frac{3}{8} & \frac{1024}{2583} & \frac{193}{2712} & 0 & 1 & \frac{175}{1872} & 0 \\ \hline \frac{5490375}{6745648} & -\frac{3}{8} & \frac{1024}{2583} & \frac{193}{2712} & 0 & 1 & \frac{175}{1872} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{34641995}{6745648} & -\frac{405}{8} & -\frac{256}{287} & -\frac{579}{904} & 3 & 0 & -\frac{457}{208} & 0 \end{array}\right] \quad (27)$$

3. CONVERGENCEANALYSIS

Stability: The matrix V must have bounded power for the method to be stable.

$$V = \begin{bmatrix} 1 & \frac{1}{6} & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{38}{45} & 0 \end{bmatrix} \quad (28)$$

The characteristic polynomial of V is given as

$$\rho(\lambda) = det(\lambda I_n - V) \quad \Longrightarrow \quad det(\lambda I_3 - V) \quad = \quad |\lambda I_3 - V| \quad (29)$$

$$= det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & \frac{1}{6} & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{38}{45} & 0 \end{bmatrix}\right) \quad (30)$$

$$= \begin{vmatrix} \lambda - 1 & -\frac{1}{6} & 0 \\ 0 & \lambda & 0 \\ 0 & \frac{38}{45} & \lambda \end{vmatrix} = \lambda^3 - \lambda^2 \quad (31)$$

To obtain the eigenvalues of the characteristic polynomial, we have

$$\rho(\lambda) = \lambda^3 - \lambda^2 = 0 \implies \lambda^2(\lambda - 1) = 0 \quad \lambda_1 = 1, \quad \lambda_2 = 0, \quad \lambda_3 = 0 \quad (32)$$

Applying Cayley-Hamilton theorem, we obtain

$$\rho(V) = V^3 - V^2 = 0 \quad => \qquad V^3 = V^2 \tag{33}$$

$$\begin{bmatrix} 1 & \frac{1}{6} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & \frac{1}{6} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{34}$$

Similarly,

$$V^4 - V^2 = 0 \quad => V^4 = V^2, \qquad V^5 - V^2 = 0 \qquad => V^5 = V^2 \tag{35}$$

$V^n = V^2$ for every n greater than 2, which shows that V^n is bounded. This implies that scheme 1 is stable.

Consistency: By definition of consistency, the method is consistent since the order of the method is $\mathrm{p} = 4 > 1$. Hence the method is convergent since it is both stable and consistent.

4. NUMERICAL EXPERIMENT

In order to validate the two developed methods, we carry out some numerical experiments and compared the obtained results with methods from Rattenbury (2005), Abraham (2010) and Alimi (2014).The results are presented in Figures 1- 6.

Problem 1:

$$\left.\begin{array}{c} y' = x + y \\ y(0) = 1 \qquad h = 0.1 \qquad 0 \leq x \leq 2 \\ Exact\ solution: \quad y_E(x) = 2e^x - x - 1 \end{array}\right\} \tag{36}$$

Problem 2:

$$\left.\begin{array}{c} y' = \frac{y}{4}\left(1 - \frac{y}{20}\right) \\ y(0) = 1 \qquad h = 0.1 \qquad 0 \leq x \leq 2 \\ Exact\ solution: \quad y_E(x) = y_E(x) = \frac{20}{1+19e^{\frac{-1}{4}x}} \end{array}\right\} \tag{37}$$

Problem 3:

$$\left.\begin{array}{c} y' = \frac{y + x}{y - x} \\ y(0) = 1 \qquad h = 0.1 \qquad 0 \leq x \leq 2 \\ Exact\ solution: \quad y_E(x) = x + \sqrt{1 + 2x^2} \end{array}\right\} \tag{38}$$

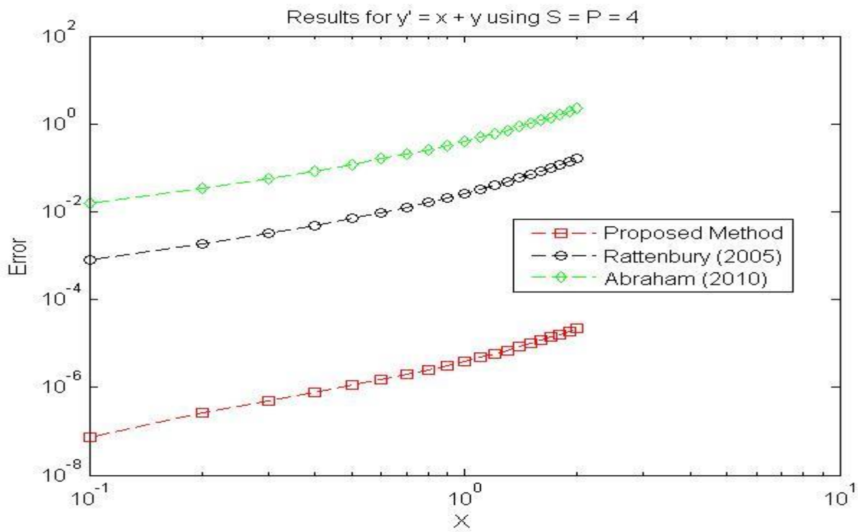

Figure 1: Comparison of Results for Scheme 1 (Problem 1, h=0.1)

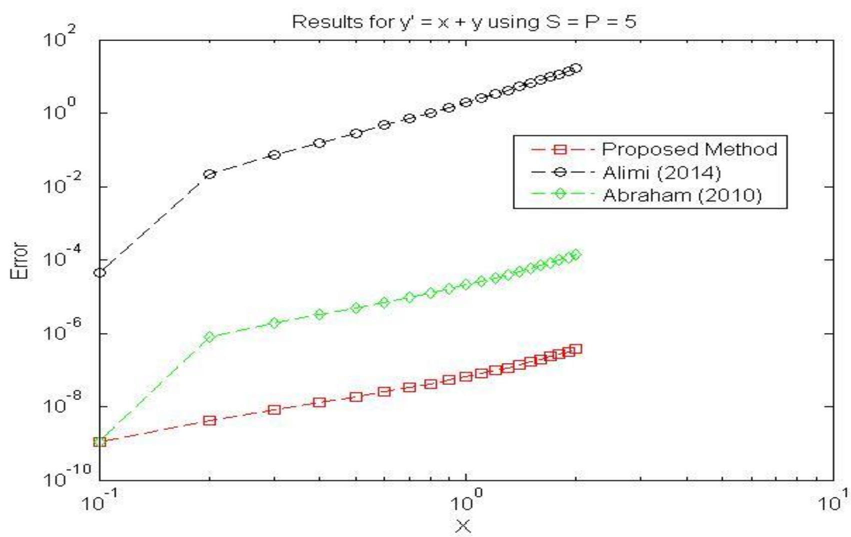

Figure 2: Comparison of Results for Scheme 2 (Problem 1, h=0.1)

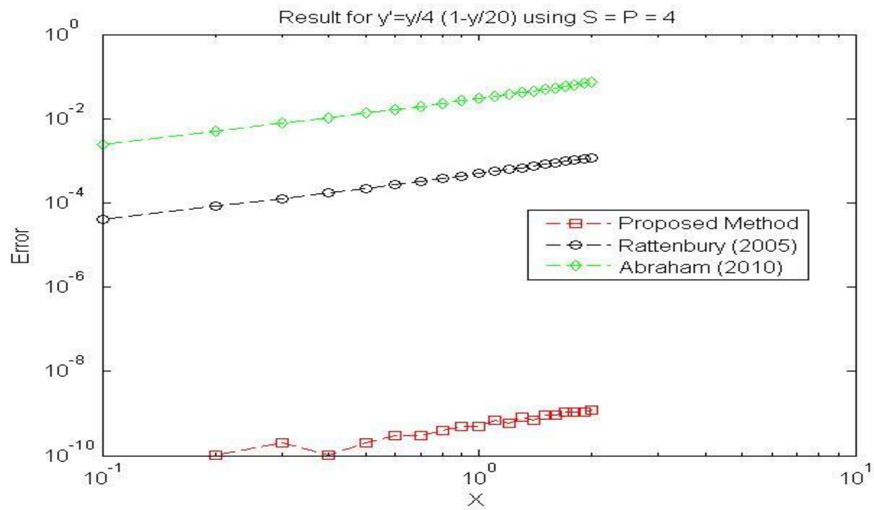

Figure 3: Comparison of Results for Scheme 1 (Problem 2, h=0.1)

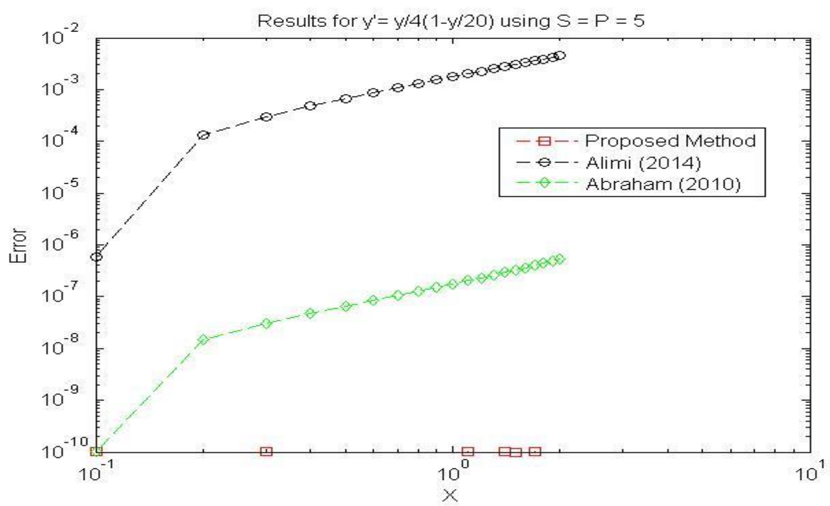

Figure 4: Comparison of Results for Scheme 2 (Problem 2, h=0.1)

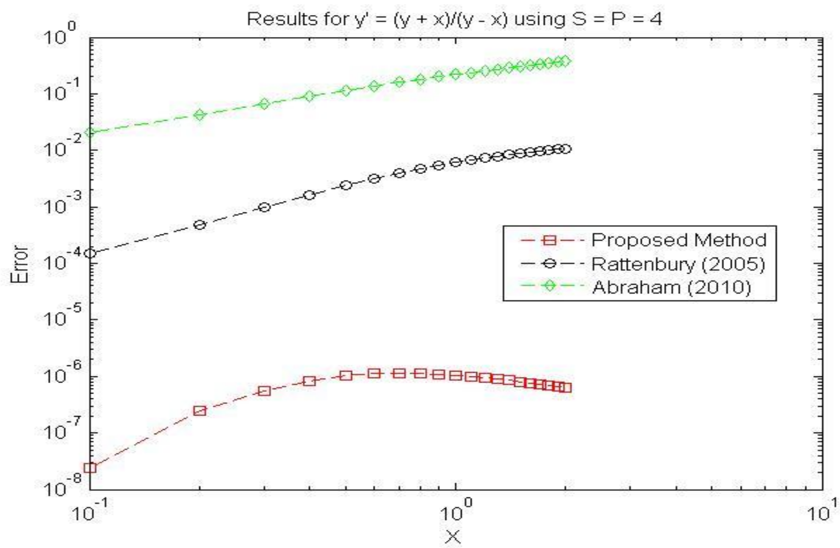

Figure 5: Comparison of Results for Scheme 1 (Problem 3, h=0.1)

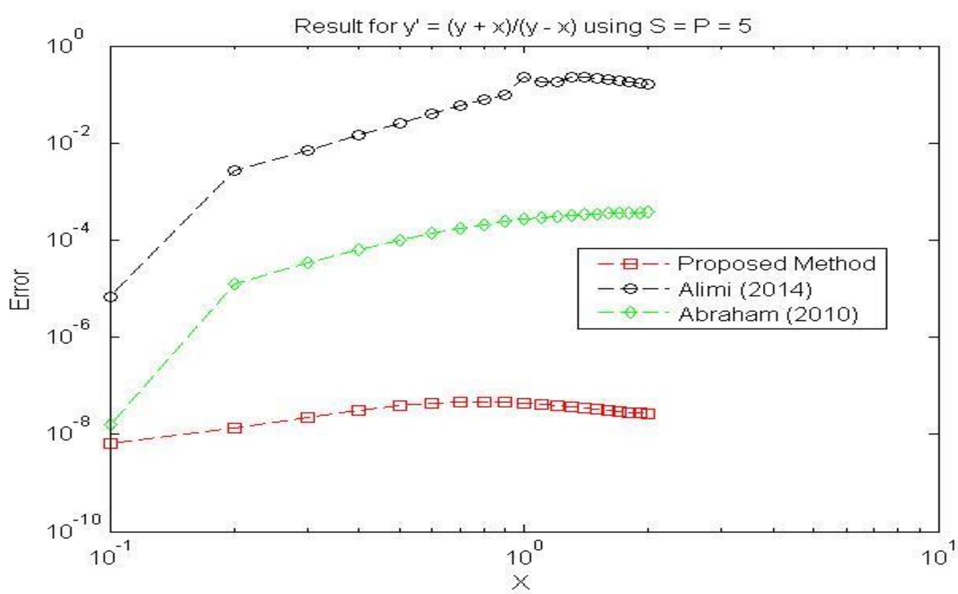

Figure 6: Results Comparison for Scheme 2 (Problem 3, h=0.1**)**

DISCUSSION OF RESULTS

From figures 1 – 6, on comparison of the obtained, we noticed that the results from scheme 1 and 2 (Proposed method) produces less errors than errors from results of Abraham (2010) and Rattenbury (2005) methods, hence we concluded that our proposed behaved adequately excellent than the schemes of Rattenbury (2005), Abraham (2010) and Alimi (2014) for problems 1-3.

CONCLUSION

We proposed two Almost Runge-Kutta (ARK) methods, ARK4 $(s = p = 4)$ and ARK5 $(s = p = 5)$. The methods are proven to be stable and consistent, thereby guaranteeing their convergence.

By the foregoing, it is instructive that the proposed ARK methods of orders 4 and 5 exhibits efficiency and reliability, as evident by their respective inconsequential errors in relation to the exact solutions. They are accurate as they produce results which compared favorably with the results obtained from existing methods. Future studies can compare the existing explicit ARK methods with variable stepsize.

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