DUALITY OF A LINEAR PROGRAM

NYOR N.[1*], RAUF K.[2] AND OMOEHIN J. O.[3]

[1]Department of Mathematics, Federal University of Technology Minna
[2]Department of Mathematics, University of Ilorin
[3]Department of Mathematics, Federal University Lokoja
[*]Corresponding author: ngutornyor@yahoo.com; 08051087384; 07081491344

Abstract.
In this work, an Integer Programming (IP) problem of an Airline Crew Formulation was considered. It was relaxed to a minimization form of Linear Programming (LP) problem. The duality of the LP in minimization form yielded the maximization form of the LP problem. The results of the IP, LP relaxation and its Dual consistently gave the same objective value of 3183, proving the consistency and the characteristic property of duality.

Key words and phrases: Linear programming (LP), Duality, Airline Crew Formulation, LP relaxation, Minimization, Maximization

1. Introduction

Associated with every linear programming problem is a corresponding "dual" linear programming problem. According to Oliveira and Carravila (1997), duality is a unifying theory that develops the relationships between a given linear program and another related linear program stated in terms of variables with this shadow-price interpretation. The dual problem is constructed from the cost and constraints of the original, or "primal," problem. Being an LP problem, the dual can be solved using the simplex method. However, as we shall see, the solution to the dual can also be obtained from the solution of the primal problem, and vice versa (Taha, 2010). Solving an LP problem through its dual may be simpler in certain cases, and also often provides further insight into the nature of the problem. In this study basic properties of duality and an interpretive example of duality is provided. Duality can be used to improve the performance of the simplex algorithm (leading to the so called "primal-dual" algorithm), as well as to develop non-simplex algorithms for solving LP problems

2. Literature Review

Duality of a Linear Programming (LP) Problem

The dual problem is an LP defined directly and systematically from the primal (or original) LP model. The two problems are so closely related that the optimal solution of one problem automatically provides the optimal solution to the other. In most LP treatments, the dual is defined for various forms of the primal depending on the sense of optimization (maximization or minimization), types of constraints ($\geq$, $\leq$ or =), and orientation of the variables (nonnegative or unrestricted). According to Wikipedia.org, in mathematical optimization theory, duality or the duality principle is the principle that optimization problems may be viewed from either of two perspectives, the primal problem or the dual problem. The solution to the dual problem provides a lower bound to the solution of the primal (minimization) problem. This type of treatment is somewhat confusing, and for this reason we offer a *single* definition that automatically subsumes *all* forms of the primal. Our definition of the dual problem requires expressing the primal problem in the *equation form* (all the constraints are equations with
nonnegative right-hand side and all the variables are nonnegative). This requirement is consistent with the format of the simplex starting tableau. Hence, any results obtained from the primal optimal solution will apply directly to the associated dual problem (Shanno, 2012).

Linear programming problems are optimization problems in which the objective function and the constraints are all linear. In the primal problem, the objective function is a linear combination of n variables. There are m

constraints, each of which places an upper bound on a linear combination of the *n* variables. The goal is to maximize the value of the objective function subject to the constraints. A *solution* is a vector (a list) of *n* values that achieves the maximum value for the objective function.

In the dual problem, the objective function is a linear combination of the *m* values that are the limits in the *m* constraints from the primal problem. There are *n* dual constraints, each of which places a lower bound on a linear combination of *m* dual variables (Vanderbei, 2007).

Relationship between the primal problem and the dual problem

In the linear case, in the primal problem, from each sub-optimal point that satisfies all the constraints, there is a direction or subspace of directions to move that increases the objective function. Moving in any such direction is said to remove slack between the candidate solution and one or more constraints. An *infeasible* value of the candidate solution is one that exceeds one or more of the constraints.

In the dual problem, the dual vector multiplies the constraints that determine the positions of the constraints in the primal. Varying the dual vector in the dual problem is equivalent to revising the upper bounds in the primal problem. The lowest upper bound is sought. That is, the dual vector is minimized in order to remove slack between the candidate positions of the constraints and the actual optimum. An infeasible value of the dual vector is one that is too low. It sets the candidate positions of one or more of the constraints in a position that excludes the actual optimum.

Economic interpretation

If we interpret our primal LP problem as a classical "Resource Allocation" problem, its dual can be interpreted as a "Resource Valuation" problem.

The weak duality lemma

The weak duality lemma states that a feasible solution to either problem yields a bound on the optimal cost of the other problem. The cost in the dual is never above the cost in the primal. In particular, the optimal cost of the dual is less than or equal to the optimal cost of the primal, that is, "maximum<minimum." Hence, if the cost of one of the problems is unbounded, then the other problem has no feasible solution. In other words, if "minimum= - ∞” or "maximum= +∞, then the feasible set in the other problem must be empty (Vanderbei, 2007).

The Duality Theorem

Duality Theorem states that, if the primal problem (either in symmetric orasymmetric form) has an optimal solution, then so does the dual, and the optimal values of their respective objective functions are equal (Vanderbei, 2007).

LP Relaxation

In mathematics, the linear programming relaxation of a 0 - 1 integer program is the problem that arises by replacing the constraint that each variable must be 0 or 1 by a weaker constraint, that each variable belong to the interval [0,1]. The resulting relaxation is a linear program, hence the name. This relaxation technique transforms an NP-hard optimization problem (integer programming) into a related problem that is solvable in polynomial time (linear programming); the solution to the relaxed linear program can be used to gain information about the solution to
the original integer program.

To formulate this as a 0 - 1 integer program, form an indicator variable x_i for each set S_i, which takes the value 1 when S_i belongs to the chosen subfamily and 0 when it does not. Then a valid cover can be described by an assignment of values to the indicator variables satisfying the constraints

$$x_i \in (0, 1),$$

that is, only the specified indicator variable values are allowed. The linear programming relaxation of the set cover problem describes a fractional cover in which the input sets are assigned weights such that the total weight of the sets containing each element is at least one and the total weight of all sets is minimized. The linear programming relaxation of an integer program may be solved using any standard linear programming technique. If the optimal solution to the linear program happens to have all variables either 0 or 1, it will also be an optimal solution to the original integer program. However, this is generally not true, except for some special cases (e.g., problems with totally uni-modular matrix specifications).

In all cases, though, the solution quality of the linear program is at least as good as that of the integer program, because any integer program solution would also be a valid linear program solution. That is, in a maximization problem, the relaxed program has a value greater than or equal to that of the original program, while in a minimization problem such as the set cover problem the relaxed program has a value smaller than or equal to that of the original program. Thus, the relaxation provides an optimistic bound on the integer program's solution (https://en.wikipedia.org).

3. Methodology

Every Primal Problem:

$$\text{Maximize } \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = 1,2,\ldots,m$$
$$x_j \geq 0 \qquad j = 1,2,\ldots,n$$

Has a dual

$$\text{Minimize } \sum_{i=1}^{m} b_i y_i$$

$$\text{subject to } \sum_{i=1}^{m} y_i a_{ij} \geq c_j \qquad j = 1,2,\ldots,n$$
$$y_i \geq 0 \qquad i = 1,2,\ldots,m$$

The dual in standard form:

$$\text{-Maximize } \sum_{i=1}^{m} -b_i y_i$$

$$\text{subject to } \sum_{i=1}^{m} -a_{ij} y_i \geq c_j \qquad j = 1,2,\ldots,n$$
$$y_i \geq 0 \qquad i = 1,2,\ldots,m$$

4. An Integer Programming Problem

Consider a formulated IP problem as below (Rauf *et al.*, 2016):

Minimize $Z = 128x_1 + 181x_2 + 245x_3 + 280x_4 + 384x_5 + 445x_6 + 146x_7 + 210x_8 + 45x_9 + 349x_{10} + 410x_{11} + 168x_{12} + 203x_{13} + 307x_{14} + 368x_{15} + 82x_{16} + 186x_{17} + 247x_{18} + 136x_{19} + 197x_{20} + 132x_{21} + 216x_{22} + 241x_{23} + 210x_{24} + 197x_{25} + 126x_{26} + 217x_{27} + 249x_{28} + 186x_{29} + 281x_{30} + 186x_{31} + 326x_{32} + 301x_{33} + 292x_{34} + 280x_{35} + 154x_{36}$

Subject to:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_{22} \geq 1$$

$$x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{23} \geq 1$$
$$x_{12} + x_{13} + x_{14} + x_{15} + x_{24} \geq 1$$
$$x_{16} + x_{17} + x_{18} + x_{31} + x_{35} \geq 1$$
$$x_{19} + x_{20} + x_{25} + x_{28} + x_{29} \geq 1$$
$$x_{21} \geq 1$$
$$x_2 + x_3 + x_4 + x_5 + x_6 + x_{22} \geq 1$$
$$x_3 + x_4 + x_5 + x_6 + x_8 + x_9 + x_{10} + x_{11} + x_{23} + x_{32} \geq 1$$
$$x_{33} \geq 1$$
$$x_{30} \geq 1$$
$$x_4 + x_5 + x_6 + x_9 + x_{10} + x_{11} + x_{13} + x_{14} + x_{15} \geq 1$$
$$x_{32} \geq 1$$
$$x_5 + x_6 + x_{10} + x_{11} + x_{14} + x_{15} + x_{17} + x_{18} + x_{31} + x_{36} \geq 1$$
$$x_6 + x_{11} + x_{15} + x_{18} + x_{20} + x_{25} + x_{26} + x_{27} + x_{32} + x_{33} + x_{34} \geq 1$$
$$x_1 + x_{22} + x_{23} + x_{35} \geq 1$$
$$x_2 + x_7 + x_{24} + x_{28} \geq 1$$
$$x_3 + x_8 + x_{12} \geq 1$$
$$x_4 + x_9 + x_{13} + x_{16} + x_{29} \geq 1$$
$$x_5 + x_{10} + x_{14} + x_{17} + x_{19} + x_{21} \geq 1$$
$$x_7 + x_{11} + x_{15} + x_{18} + x_{20} + x_{21} + x_{25} \geq 1$$
$$x_{22} + x_{35} \geq 1$$
$$x_{23} \geq 1$$
$$x_2 + x_{30} + x_{35} \geq 1$$
$$x_{28} \geq 1$$
$$x_{32} + x_{33} \geq 1$$
$$x_{35} \geq 1$$
$$x_{30} \geq 1$$
$$x_{34} \geq 1$$
$$x_{29} + x_{36} \geq 1$$
$$x_{27} \geq 1$$
$$x_{26} \geq 1$$
$$x_j = 0 \text{ or } 1 (j = 1 - 31)$$

3. LP Relaxation of Problem Formulation

The Integer Program in section 2 is relaxed to a Linear Program (LP) of the minimization form as below:

Minimize $Z = 128x_1 + 181x_2 + 245x_3 + 280x_4 + 384x_5 + 445x_6 + 146x_7 + 210x_8 + 245x_9+349x_{10}+410x_{11}+168x_{12}+203x_{13}+307x_{14}+368x_{15}+82x_{16}+186x_{17}+247x_{18}+ 136x_{19} + 197x_{20} + 132x_{21} + 216x_{22} + 241x_{23} + 210x_{24} + 197x_{25} + 126x_{26} + 217x_{27} + 249x_{28} + 186x_{29} + 281x_{30} + 186x_{31} + 326x_{32} + 301x_{33} + 292x_{34} + 280x_{35} + 154x_{36}$

Subject to:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_{22} \geq 1$$
$$x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{23} \geq 1$$
$$x_{12} + x_{13} + x_{14} + x_{15} + x_{24} \geq 1$$
$$x_{16} + x_{17} + x_{18} + x_{31} + x_{35} \geq 1$$
$$x_{19} + x_{20} + x_{25} + x_{28} + x_{29} \geq 1$$
$$x_{21} \geq 1$$
$$x_2 + x_3 + x_4 + x_5 + x_6 + x_{22} \geq 1$$
$$x_3 + x_4 + x_5 + x_6 + x_8 + x_9 + x_{10} + x_{11} + x_{23} + x_{32} \geq 1$$
$$x_{33} \geq 1$$
$$x_{30} \geq 1$$
$$x_4 + x_5 + x_6 + x_9 + x_{10} + x_{11} + x_{13} + x_{14} + x_{15} \geq 1$$
$$x_{32} \geq 1$$
$$x_5 + x_6 + x_{10} + x_{11} + x_{14} + x_{15} + x_{17} + x_{18} + x_{31} + x_{36} \geq 1$$

$$x_6 + x_{11} + x_{15} + x_{18} + x_{20} + x_{25} + x_{26} + x_{27} + x_{32} + x_{33} + x_{34} \geq 1$$
$$x_1 + x_{22} + x_{23} + x_{35} \geq 1$$
$$x_2 + x_7 + x_{24} + x_{28} \geq 1$$
$$x_3 + x_8 + x_{12} \geq 1$$
$$x_4 + x_9 + x_{13} + x_{16} + x_{29} \geq 1$$
$$x_5 + x_{10} + x_{14} + x_{17} + x_{19} + x_{21} \geq 1$$
$$x_7 + x_{11} + x_{15} + x_{18} + x_{20} + x_{21} + x_{25} \geq 1$$
$$x_{22} + x_{35} \geq 1$$
$$x_{23} \geq 1$$
$$x_2 + x_{30} + x_{35} \geq 1$$
$$x_{28} \geq 1$$
$$x_{32} + x_{33} \geq 1$$
$$x_{35} \geq 1$$
$$x_{30} \geq 1$$
$$x_{34} \geq 1$$
$$x_{29} + x_{36} \geq 1$$
$$x_{27} \geq 1$$
$$x_{26} \geq 1$$
(Non-negativity Condition) $x_j \geq 0$; ($j = 1, \ldots 31$)

4. Duality of Problem Formulation

The method of solving a standard minimization problem is called dual method. The dual method converts a standard minimization problem into a standard maximization problem. By the theory of duality, a minimization LP problem can be converted to a maximization LP problem. That is, the dual of the primal LP in minimization form is the corresponding maximization LP problem. Hence, the dual LP problem is stated as follows:

Maximize z = w1 + w2 + w3 + w4 + w5 + w6 + w7 + w8 + w9 + w10 + w11 + w12 + w13 +w14 +w15 +w16 +w17 +w18 +w19 +w20 +w21 +w22 +w23 +w24 +w25 +w26 + w27 + w28 + w29 + w30 + w31 + w32 + w33 + w34 + w35 + w36

subject to:

$$w1 + w15 \leq 128$$
$$w1 + w7 + w16 \leq 181$$
$$w1 + w7 + w8 + w17 \leq 245$$
$$w1 + w7 + w8 + w11 + w17 \leq 280$$
$$w1 + w7 + w8 + w11 + w13 + w19 \leq 384$$
$$w1 + w7 + w8 + w11 + w13 + w14 \leq 445$$
$$w2 + w16 + w20 \leq 146$$
$$w2 + w8 + w17 \leq 210$$
$$w2 + w8 + w11 + w18 \leq 245$$
$$w2 + w8 + w11 + w13 + w19 \leq 349$$
$$w2 + w8 + w11 + w13 + w14 + w20 \leq 410$$
$$w3 + w17 \leq 168$$
$$w3 + w11 + w18 \leq 203$$
$$w3 + w11 + w13 + w19 \leq 307$$
$$w3 + w11 + w13 + w14 + w20 \leq 368$$
$$w4 + w18 \leq 82$$
$$w4 + w13 + w20 \leq 186$$
$$w4 + w13 + w14 + w20 \leq 247$$
$$w15 + w19 \leq 136$$
$$w5 + w14 + w20 \leq 197$$
$$w6 + w20 \leq 132$$

$$w1 + w7 + w15 + w21 \leq 216$$
$$w2 + w8 + w15 + w22 \leq 241$$
$$w3 + w16 + w23 \leq 210$$
$$w5 + w14 + w20 \leq 197$$
$$w14 + w31 \leq 126$$
$$w14 + w30 \leq 217$$
$$w5 + w16 + w24 \leq 249$$
$$w5 + w18 + w29 \leq 186$$
$$w10 + w23 + w27 \leq 281$$
$$w4 + w13 + w19 \leq 186$$
$$w8 + w12 + w14 + w25 \leq 326$$
$$w9 + w14 + w25 \leq 301$$
$$w14 + w28 \leq 292$$
$$w4 + w15 + w21 + w23 + w26 \leq 280$$
$$w13 + w29 \leq 154$$
$$w_j \geq 0; (j = 1, \ldots 31)$$

5. Results and Discussion

Using TORA Software, the result of the IP formulation under section 2 yielded the following result:
Objective Value = 3183

$$X_{Variables} = \begin{cases} 1 \text{ for } x_5; x_{12}; x_{21}; x_{23}; x_{26}; x_{27}; x_{28}; x_{29}; x_{30}; x_{32}; x_{33}; x_{34}; x_{35} \\ 0 \text{ otherwise.} \end{cases}$$

And the LP relaxation in section 3 gives the same result as in IP in section 2 thus:
Objective value = 3183

$$X_{Variables} = \begin{cases} 1 \text{ for } x_5; x_{12}; x_{21}; x_{23}; x_{26}; x_{27}; x_{28}; x_{29}; x_{30}; x_{32}; x_{33}; x_{34}; x_{35} \\ 0 \text{ otherwise.} \end{cases}$$

This proves that our relaxed problem from IP to LP maintains its characteristic properties. Note that the problem converged at the 50th iteration.

It can be observed from the dual result below that, the objective value of the dual problem, 3183, is the same with that of its primal. This satisfies the principle of duality which states that; when a standard minimization problem and its dual have solutions, then the maximum value of the function is the same as the minimum value of the function (Saravanan, 2011; Robbert, 2007). Note however that the x combinations differ from the primal result and the dual version converged at the 33rd iteration.

$$X_{Variables} = \begin{cases} 1 \text{ for } x_1; x_3; x_4; x_7; x_9; x_{10}; x_{11}; x_{12}; x_{13}; x_{16}; x_{17}; x_{18}; x_{19}; x_{20}; x_{21}; x_{22}; \\ x_{24}; x_{26}; x_{28}; x_{29}; x_{30}; x_{31} \\ 0 \text{ otherwise} \end{cases}$$

6. Conclusion

Though both primal and dual problems have the same objective function, the dual problem has given flight combinations that look more acceptable. And since it converged faster at the 33rd iteration than it's primal at 50th iteration, We consider the dual result to be better. It is concluded that, this result is optimal.

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