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ANALYTICAL SOLUTION OF A MATHEMATICAL MODEL OF TUBERCULOSIS WITH PASSIVELY IMMUNE COMPARTMENT.

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Abstract

In this study, we proposed a Mathematical Model of tuberculosis dynamics. The model is a system of four first order ordinary differential equations. The population is partitioned into four compartments of passively immune infant class $M(t)$, Susceptible $S(t)$, Infected $I(t)$ and Recovered $R(t)$. The analytical solutions using Homotopy Perturbation method (HPM) were obtained. Graphical profiles for each of the four compartments were obtained using MAPLE computer software package. The results shows that the disease has a [redacted] *tendency of dying out with time when there is high recovery rate.*

KEYWORDS:
Tuberculosis, Immunity, Analytical Solution, Homotopy Perturbation and Numerical Simulations.

Introduction

Tuberculosis is one of the oldest disease known, yet one of the most difficult to control; Nigeria has the second highest burden of TB in Africa and the 4th highest in the world. The World Health Organization (WHO) has said TB has emerged as the leading cause of death from single infectious agent and has continued to be a major public health problem all over the world. Of more than 14 million cases reported worldwide, Nigeria ranked fourth in terms of incidence (World Health Organization,2010).

The high incidence of Tuberculosis in developing countries is as a result of poverty and underdevelopment, which lead to overcrowding, malnutrition, lack of access to good Health care services which are contributory factors to the spread of the disease.

A good reason to be apprehensive is that Nigeria has just a few MDR-TB wards. These are in Ibadan (commissioned in July 2010), managed by the Damcen foundation; in Lagos commissioned by Global fund in 2012; and another at the Zonal Reference Lab in Calabar (Onche, 2013).

Despite the availability of highly efficacious treatment for decades, TB remains a major global health problem. In 1993, the World Health Organization (WHO) declared TB a global public health emergency, at a time when an estimated 7–8 million cases and 1.3–1.6 million deaths occurred each year (World Health Organization, 2007).

Literature Review.

Research has shown that Genetic Susceptibility affect Endemic prevalence levels and alters the effect of Treatment of Tuberculosis patient. Genetic Susceptible are part of the Susceptible subpopulation that can be infected with Mycobacterium Tuberculosis, as not all people are equally Susceptible to TB (Belamy, 20004).

In a related Research work by (Koriko and Yusuf Koriko, 2008) the Dynamics of Tuberculosis Disease Population was considered using the Susceptible- Infected-Recovered but Susceptible (SIRS) Model.

Tuberculosis is an Air-borne Contagious Disease affecting about one third of the World Population, out of which two third live in developing countries. In a study on the effect of DOTS in Nigeria. Daniel and Andrei (2007) presented a Mathematical Model for Tuberculosis and its Dynamics under the implementation of DOTS in Nigeria.

The condition for the Eradication of Tuberculosis in Nigeria established by the Model was based on the fraction of detected infectious individual under the DOTS treatment. Both Numerical and Qualitative Analysis of the model were performed. The effect of the fraction of detected cases of active TB on the various Epidemiological groups was investigated. In an attempt to study the effect of Vaccination, treatment and population area size on the transmission dynamics of TB in a proportionate mixing population.

Umar (2007) proposed a Mathematical Model that incorporates the Density dependent Dynamics of Tuberculosis, the effect of Treatment and Vaccination. The Study reveals that if the Population area size is Large the Density of the

Susceptible will be small and this will reduce the size of the Basic Reproduction Number.

Yusuf (2008) proposed a Deterministic Compartmental Model but ignored the different Rates of Progression from Latent to infectious Class; this however precludes the speedy Progression of TB caused by HIV infections. By weakening the immune system of a TB patient, HIV acts as catalyst in the progression of TB from Latent Class to Infectious Class. A patient with AIDS who become infected with Mycobacterium Tuberculosis has a 50% chance of developing Active Tuberculosis within 2 months and a 5 to 10% chance of developing Active Disease thereafter. Infants and young children are also more likely to develop Active TB than older people since their Immune System are not yet well developed (World Health Organization, 2003).

MATERIALS AND METHODS

Model Development

The population was divided into Four Classes: $M(t)$ represents Infant that are Immured at Birth; $S(t)$ represents the number of Individuals that are Prone to the disease at time t, or those Susceptible to the Disease; $I(t)$ denote the number of Individuals who have been Infected with the Disease and are Capable of spreading the Disease to those in the Susceptible Class; $R(t)$ is the Compartment consisting those Individuals who have been infected and then Recovered from the Disease after treatment. This can be shown as a Flow Diagram in which the boxes represents the different Compartments and the arrows the transition between the Compartments.

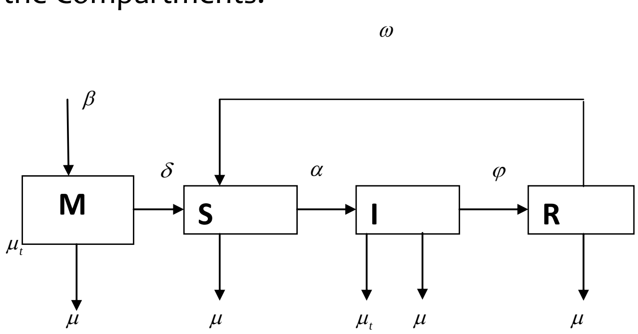

Figure 1 : Schematic presentation of the Model

Assumptions

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We assume that all New Births are immunised. Members of the Susceptible Class moved into the Infected class via interaction of the Susceptible with the Infected at the Rate α .

There is constant recruitment β into Passively Immune Infant Class via vaccination at birth. The Population of Passively Immune Class decreases due to Natural Death at the Rate μ and movement of the individual into the Susceptible Class as a Result of Waning off of Vaccine at the Rate δ the Population of the Susceptible increases due to coming in of individual from Passively immune Class and Recovered Class at the Rate δ and ω respectively. The Population of the Susceptible Class decreases due to the movement of individual into the infected Class at the Rate α and Natural Death at the Rate μ . The Population of the infected Class decreases due to Treatment against Tuberculosis at the Rate φ and Natural Death Rate μ and Death as a Result of TB infection at the Rate μ_t the Population of the Recovered Class increases due to recovery of infected individuals at the Rate φ from infected Class and decreases due to movement of individual into Susceptible Class at the Rate ω and Natural Death Rate μ

Model Equations.

The model was described using the following system of ordinary differential equations.

$$\frac{dM}{dt} = \beta - (\mu + \delta)M \qquad (1)$$

$$\frac{dS}{dt} = \delta M - (\mu + \alpha I)S + \omega S \qquad (2)$$

$$\frac{dI}{dt} = \alpha SI - (\mu + \mu_t + \varphi)I \qquad (3)$$

$$\frac{dR}{dt} = \varphi I - (\mu + \omega)R \qquad (4)$$

Analytical Solution of the Model

Homotopy Perturbation Method (HPM)

Fundamentals of Homotopy Perturbation Method (HPM) were first proposed by Ji-Huan He (2000).

To illustrate the basic ideas of this Method, the following nonlinear differential equation was considered:

$$A(u) - f(r) = 0, \quad r \in \Omega \qquad (5)$$

Subject to the boundary condition:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \qquad (6)$$

Where A is a general differential operator, B a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain ***Ω***. The operator A can be divided into two parts L and N, where L is the linear part, and N is the nonlinear component. Equation (5) may therefore be rewritten as:

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega \qquad (7)$$

The Homotopy Perturbation structure is shown as follows

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \qquad (8)$$

Where:

$$v(r,p) : \Omega \in [0,1] \to R \qquad (9)$$

In equation (7) $p \in [0,1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. It can be assumed that the solution of equation (7) can be written as power series in *p* as follows:

$$v = v_0 + pv_1 + p^2 v_2 + ... \qquad (10)$$

and the best approximation for the solution is:

$$u = \lim_{p \to 1} v = v_0 + pv_1 + p^2 v_2 + ... \qquad (11)$$

The series (11) is convergent for most cases. However, the convergence rate depends on the nonlinear operator A(v).

Solution of the Model Equation

Equations (1) to (4) can be written as

$$\frac{dM}{dt} + (\mu + \delta)M - \beta = 0 \qquad (12)$$

$$\frac{dS}{dt} - \delta M - \omega S + (\mu + \alpha I)S = 0 \qquad (13)$$

$$\frac{dI}{dt} - \alpha SI + (\mu + \mu_t + \varphi)I = 0 \qquad (14)$$

$$\frac{dR}{dt} - \varphi I + (\mu + \omega)R = 0 \qquad (15)$$

With the following initials conditions $M(0) = M_0$, S (0) = S_0, I(0)=I_0 and R(0)= R_0

Applying HPM to (12) we have

$$(1-p)\frac{dM}{dt}+p\left[\frac{dM}{dt}+(\mu+\delta)M-\beta\right]=0 \quad (16)$$

Let

$$M=w_0+pw_1+p^2w_2+.... \quad (17)$$

$$S=x_0+px_1+p^2x_2+.... \quad (18)$$

$$I=y_0+py_1+p^2y_2+.... \quad (19)$$

$$R=z_0+pz_1+p^2z_2+.... \quad (20)$$

Substituting (17) into (16)

$$(1-p)(w_0^{'}+pw_1^{'}+p^2w_2^{'}+....)+p\left[\left(w_0^{'}+pw_1^{'}+p^2w_2^{'}+....\right)+(\mu+\delta)\left(w_0+pw_1+p^2w_2+...\right)-\beta\right]=0 \quad (21)$$

$$(w_0^{'}+pw_1^{'}+p^2w_2^{'}+...)+p\left[\mu w_0+p\mu w_1+p^2w_2\mu+...+\delta w_0+\delta pw_1+\delta p^2w_2+...\right]-p\beta=0$$
(22)

Collecting the coefficient of the powers of p we have

$$p^0: w_0^{'}=0 \quad (23)$$

$$p^{'}: w_1^{'}+\mu w_0+\delta w_0-\beta=0 \quad (24)$$

$$p^{2'}: w_2^{'}+\mu w_1+\delta w_1=0 \quad (25)$$

$$p^3: w_2\mu+\delta w_2=0 \quad (26)$$

Applying (HPM) to (13)

$$\frac{dS}{dt}-\delta M-\omega S+(\mu+\alpha I)S=0 \quad (27)$$

$$(1-p)\frac{dS}{dt}+p\left[\frac{dS}{dt}+\delta M-\omega S+(\mu+\alpha I)S\right]=0 \quad (28)$$

Substituting (17), (18) and (19) into (28)

$$(1-p)(x_0^{'}+px_1^{'}+.p^2x_2^{'}+...)+p\begin{bmatrix}\left(x_0^{'}+px_1^{'}+p^2x_2^{'}+....\right)+\delta\left(w_0+pw_1+p^2w_2+...\right)\\-\omega\left(x_0+px_1+p^2x_2+.....\right)+\mu\left(x_0+px_1+p^2x_2\right)\\+\alpha\left(x_0+px_1+p^2x_2+...\right)\left(y_0+py_1+p^2y_2+...\right)\end{bmatrix}=0 \quad (29)$$

Collecting the coefficient of the powers of p we have

$$p^0: x_0^{'}=0 \quad (30)$$

$$p^1 : x_1^{'} + \delta w_0 + \omega x_0 + \mu x_0 + \alpha x_0 y_0 = 0 \quad (31)$$

$$p^2 : x_2^{'} + \delta w_1 - \omega x_1 + \mu x_1 + \alpha x_1 y_1 = 0 \quad (32)$$

$$p^3 : \delta w_2 - \omega x_2 + \mu x_2 + \alpha x_2 y_2 = 0 \quad (33)$$

Applying HPM to (14) gives

$$(1-p)\frac{dI}{dt} + p\left[\frac{dI}{dt} - \alpha SI + (\mu + \mu_t + \phi) I\right] = 0 \quad (34)$$

Substituting (18) and (19) into (34) gives

$$(1-p)(y_0^{'} + py_1^{'} + .p^2 y_2^{'} + ...) + p\begin{bmatrix} (y_0^{'} + py_1^{'} + p^2 y_2^{'} + ...) \\ -\alpha(x_0 + px_1 + p^2 x_2 + ...)(y_0 + py_1 + p^2 y_2 + ...) \\ +(\mu + \mu_t + \phi)(y_0 + py_1 + p^2 y_2 + .) \end{bmatrix} = 0 \quad (35)$$

Collecting the coefficient of the power of p

$$p^0 : y_0^{'} = 0 \quad (36)$$

$$p^1 : y_1^{'} - \alpha x_0 y_0 + (\mu + \mu_t + \phi) y_0 = 0 \quad (37a)$$

$$p^2 : y_2^{'} - \alpha x_1 y_1 + (\mu + \mu_t + \phi) y_1 = 0 \quad (337b)$$

$$p^3 : -\alpha x_2 y_2 + (\mu + \mu_t + \phi) y_2 = 0 \quad (38)$$

Applying HPM to (15) gives

$$(1-p)\frac{dR}{dt} + p\left[\frac{dR}{dt} - \phi I + (\mu + \omega) R\right] = 0 \quad (39)$$

Substituting (20) and (19) into (39) gives

$$(1-p)(z_0^{'} + pz_1^{'} + .p^2 z_2^{'} + ...) + p\begin{bmatrix} (z_0^{'} + pz_1^{'} + p^2 z_2^{'} + ...) \\ -\varphi(y_0 + py_1 + p^2 y_2 + ...) \\ +(\mu + \omega)(z_0 + pz_1 + p^2 z_2 + .) \end{bmatrix} = 0 \quad (40)$$

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Collecting the coefficient of the power of p

$$p^0 : z_0' = 0 \quad (41)$$

$$p^1 : z_1' - \phi y_0 + \mu z_0 + \omega z_0 = 0 \quad (42)$$

$$p^2 : z_2' - \phi y_1 + \mu z_1 + \omega z_1 = 0 \quad (43)$$

$$p^3 : -\varphi y_2 + \mu z_2 + \omega z_2 = 0 \quad (44)$$

From (23)

$$p^0 : w_0' = 0$$

$$\Rightarrow w_0' = 0 \quad (45)$$

Integrating both side

$$w_0 = A \quad (46)$$

Applying the initial condition

$$w_0(0) = M_0 = w_0 \quad (47)$$

Hence

$$A = M_0 \quad (48)$$

$$w_0 = M_0 \quad (49)$$

From (30)

$$p^0 : x_0' = 0 \quad (50)$$

$$x_0' = 0$$

Integrating both side we have

$$x_0 = B \quad (51)$$

Applying the initial condition

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$$x_0(0) = S_0 = x_0 \quad (52)$$

$$B = S_0 \quad (53)$$

$$x_0 = S_0 \quad (54)$$

From (36)

$$p^0 : y_0^{'} = 0 \quad (55)$$

$$y_0^{'} = 0 \quad (56)$$

Integrating both side we obtained

$\therefore$

$$y_0 = C \quad (57)$$

Applying the initial condition

$$y_0(0) = I_0 = y_0 \quad (58)$$

$$I_0 = y_0 \quad (59)$$

From (41)

$$p^0 : z_0^{'} = 0 \quad (60)$$

$$z_0^{'} = 0 \quad (61)$$

Integrating both side gives

$$z_0 = D \quad (62)$$

Applying the initial condition

$$Z_0(0) = R_0 = Z_0 \quad (63)$$

$$D = R_0 \quad (64)$$

$$R_0 = Z_0 \quad (65)$$

From (24)

$$p^{'} : w_1^{'} + \mu w_0 + \delta w_0 - \beta = 0 \quad (66)$$

$$\Rightarrow w_1^{'} = \beta - \mu w_0 - \delta w_0 = 0 \quad (67)$$

$$\int w_1^{'} = \int (\beta - \mu w_0 - \delta w_0) dt \quad (68)$$

$$w_1(t) = (\beta - \mu w_0 - \delta w_0)t + E \qquad (69)$$

Where E is the constant of integration. Applying the initial condition

$$w_1(0) = 0 \qquad (70)$$

$$w_1 = 0, t = 0 \qquad (71)$$

$\Rightarrow$

$$E = 0 \qquad (72)$$

Hence

$$w_1 = (\beta - \mu w_0 - \delta w_0)t \qquad (73)$$

Substituting (49) into (73) gives

$$w_1 = (\beta - \mu M_0 - \delta M_0)t \qquad (74)$$

From (3.74)

$$p^1 : x_1' + \delta w_0 + \omega x_0 + \mu x_0 + \alpha x_0 y_0 = 0 \qquad (75)$$

$\Rightarrow$

$$x_1' = -\alpha x_0 y_0 - \mu x_0 - \omega x_0 - \delta w_0 \qquad (76)$$

Integrating (76) we obtained

$$x_1(t) = (-\alpha x_0 y_0 - \mu x_0 - \omega x_0 - \delta w_0)t + F \qquad (77)$$

Where F is constant of integration. Applying the initial condition

$$x_1(0) = 0 \qquad (78)$$

$\Rightarrow$

$$F = 0 \qquad (79)$$

$$x_1 = (-\alpha x_0 y_0 - \mu x_0 - \omega x_0 - \delta w_0)t \qquad (80)$$

Substituting (49),(54) and (59) into (80)

$$x_1 = (-\alpha S_0 I_0 - \mu S_0 - \omega S_0 - \delta M_0)t \qquad (81)$$

From (37)

$$p^1 : y_1' - \alpha x_0 y_0 + (\mu + \mu_t + \varphi) y_0 = 0 \qquad (82)$$

$$y_1^{'} = \alpha x_0 y_0 - (\mu + \mu_t + \varphi) y_0 \quad (83)$$

Integrating (83) gives,

$$y_1(t) = (\ \alpha x_0 y_0 - (\mu + \mu_t + \varphi) y_0\ )t + G \quad (84)$$

Where G is constant of integration. Applying the initial condition

$$y_1(0) = 0 \quad (85)$$

$\Rightarrow$

$$G = 0 \quad (86)$$

Hence

$$y_1 = (\ \alpha x_0 y_0 - (\mu + \mu_t + \varphi) y_0\ )t \quad (87)$$

Substituting (54) and (59) into (87) gives,

$$y_1 = (\ \alpha S_0 I_0 - (\mu + \mu_t + \varphi) I_0\ )t \quad (89)$$

From (42)

$$p^1 : z_1^{'} - \phi y_0 + \mu z_0 + \omega z_0 = 0 \quad (90)$$

$\Rightarrow$

$$z_1^{'} = \varphi y_0 - \mu z_0 - \omega z_0 \quad (91)$$

Integrating (91) gives

$$z_1(t) = (\ \varphi y_0 - \mu z_0 - \omega z_0\ )t + H \quad (93)$$

Where H is constant of integration. Applying the initial condition

$$Z_1(0) = 0$$

$$\Rightarrow H = 0$$

hence

$$z_1 = (\ \varphi y_0 - \mu z_0 - \omega z_0\ )t \quad (94)$$

Substituting (59) and (65) into (94) gives,

$$z_1 = (\ \varphi I_0 - \mu R_0 - \omega R_0\ )t \quad (95)$$

Substituting (74) into (25) gives

$$w_2^{'} + (\mu + \delta)(\ \beta - \mu w_0 - \delta w_0\ )t = 0 \quad (96)$$

$$w_2' = -(\mu+\delta)(\beta - \mu w_0 - \delta w_0)t = 0 \qquad (97)$$

Integrating (97) gives

$$w_2 = -(\mu+\delta)(\beta - \mu w_0 - \delta w_0)\frac{t^2}{2} + I(t) \qquad (98)$$

Applying the initial condition

$I(0) = 0$

Hence

$$w_2 = -(\mu+\delta)(\beta - \mu w_0 - \delta w_0)\frac{t^2}{2} \qquad (99)$$

Substituting (49) into (99) we obtained,

$$w_2 = -(\mu+\delta)(\beta - \mu M_0 - \delta M_0)\frac{t^2}{2} \qquad (100)$$

Applying (11) we have

$$M(t) = \lim_{p\to 1} w_0 + pw_1 + p^2 w_2 + \ldots$$
$$= \lim_{p\to 1} M_0 + p(\beta + \mu M_0 - \delta M_0)t$$
$$-p^2(\mu+\delta)(\beta - \mu M_0 - \delta M_0)\frac{t^2}{2} \qquad (101)$$

Hence

$$M(t) = M_0 + (\beta + \mu M_0 - \delta M_0)t$$
$$-(\mu+\delta)(\beta - \mu M_0 - \delta M_0)\frac{t^2}{2} \qquad (102)$$

From (32)

$$x_2' + \delta w_1 - \omega x_1 + \mu x_1 + \alpha x_1 y_1 = 0 \qquad (103)$$
$$x_2' = -\delta w_1 + \omega x_1 - \mu x_1 - \alpha x_1 y_1 \qquad (104)$$
$$x_2' = x_1(\omega - \mu) - \delta w_1 - \alpha x_1 y_1 \qquad (105)$$

Substituting (81), (74) and (89) into (105) gives

$$x_2' = (\omega-\mu)(-\alpha S_0 I_0 - \mu S_0 - \omega S_0 - \delta M_0)t - \delta(\beta - \mu M_0 - \delta M_0)t$$
$$-\alpha(-\alpha S_0 I_0 - \mu S_0 - \omega S_0 - \delta M_0)(\alpha S_0 I_0 - (\mu + \mu_t + \phi) I_0)t^2 \qquad (106)$$

Integrating (106) gives

$$x_2 = (\omega-\mu)(-\alpha S_0 I_0 - \mu S_0 - \omega S_0 - \delta M_0)\frac{t^2}{2}$$
$$-\delta(\beta - \mu M_0 - \delta M_0)\frac{t^2}{2}$$
$$-\alpha(-\alpha S_0 I_0 - \mu S_0 - \omega S_0 - \delta M_0)(\alpha S_0 I_0 - (\mu+\mu_t+\phi)I_0)\frac{t^3}{3} \quad (107)$$

Applying (11) we have

$$S(t) = \lim_{p\to 1} \Rightarrow x_0 + px_1 + p^2 x_2 + ... \quad (109)$$

Hence

$$S(t) = S_0 - (\alpha S_0 I_0 + \mu S_0 + \omega S_0 + \delta M_0)t + \left\{\begin{array}{l}(\omega-\mu)(-\alpha S_0 I_0 - \mu S_0 - \omega S_0 - \delta M_0)\frac{t^2}{2} \\ -\delta(\beta - \mu M_0 - \delta M_0)\frac{t^2}{2} \\ -\alpha(-\alpha S_0 I_0 - \mu S_0 - \omega S_0 - \delta M_0)(\alpha S_0 I_0 - (\mu+\mu_t+\phi)I_0)\frac{t^3}{3}\end{array}\right\} \quad (111)$$

From (37b)

$$y_2' - \alpha x_1 y_1(\mu + \mu_t + \varphi)y_1 = 0 \quad (113)$$

Substituting (81) and (89) into (113) gives

$$y_2' = \alpha(-\alpha S_0 I_0 - \mu S_0 - \omega S_0 - \delta M_0)t(\alpha S_0 I_0 - (\mu+\mu_t+\phi)I_0)t$$
$$-(\mu+\mu_t+\phi)(\alpha S_0 I_0 - (\mu+\mu_t+\phi)I_0)t \quad (114)$$

$\Rightarrow$

$$y_2' = \alpha(-\alpha S_0 I_0 - \mu S_0 - \omega S_0 - \delta M_0)(-\alpha S_0 I_0 - (\mu+\mu_t+\phi)I_0)t^2$$
$$-(\mu+\mu_t+\phi)(\alpha S_0 I_0 - (\mu+\mu_t+\phi)I_0)t \quad (115)$$

Integrating (115) gives

$$y_2 = \alpha(-\alpha S_0 I_0 - \mu S_0 - \omega S_0 - \delta M_0)(-\alpha S_0 I_0 - (\mu+\mu_t+\phi)I_0)\frac{t^3}{3}$$
$$-(\mu+\mu_t+\phi)(\alpha S_0 I_0 - (\mu+\mu_t+\phi)I_0)\frac{t^2}{2} \quad (116)$$

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Applying (11) we have

$$I(t) = \lim_{p \to 1} \Rightarrow y_0 + py_1 + p^2 y_2 + ... \qquad (117)$$

Substituting (59),(89), (116) into (117) gives

$$I(t) = I_0 + \left(\alpha S_0 I_0 - \left(\mu + \mu_t + \varphi\right) I_0\right) t$$
$$+\alpha\left(-\alpha S_0 I_0 - \mu S_0 - \omega S_0 - \delta M_0\right)\left(-\alpha S_0 I_0 - \left(\mu + \mu_t + \varphi\right) I_0\right)\frac{t^3}{3}$$
$$-\left(\mu + \mu_t + \phi\right)\left(\alpha S_0 I_0 - \left(\mu + \mu_t + \phi\right) I_0\right)\frac{t^2}{2} \qquad (118)$$

From (43)

$$z_2^{'} - \phi y_1 + \mu z_1 + w z_1 = 0 \qquad (119)$$

$\Rightarrow$

$$z_2^{'} = \varphi y_1 - (\ \mu + w\ ) z_1 \qquad (120)$$

Substituting (89) and (95) into (120) gives

$$z_2^{'} = \varphi\left(\alpha S_0 I_0 - \left(\mu + \mu_t + \varphi\right) I_0\right) t$$
$$-(\mu + w)\left(\varphi I_0 - \mu R_0 - \omega R_0\right) t \qquad (121)$$

Integrating (3.174) gives

$$z_2 = \varphi\left(\alpha S_0 I_0 - \left(\mu + \mu_t + \varphi\right) I_0\right)\frac{t^2}{2}$$
$$-(\mu + \omega)\left(\varphi I_0 - \mu R_0 - \omega R_0\right)\frac{t^2}{2} \qquad (122)$$

Applying (11) we have

$$R(t) = \lim_{p \to 1} \Rightarrow z_0 + pz_1 + p^2 z_2 + ... \qquad (123)$$

Substituting (65), (95) and (122) into (123) gives

$$R(t) = R_0 + (\varphi I_0 - \mu R_0 - \omega R_0)t$$
$$+\varphi(\alpha S_0 I_0 - (\mu + \mu_t + \phi) I_0)\frac{t^2}{2}$$
$$-(\mu + \omega)(\phi I_0 - \mu R_0 - \omega R_0)\frac{t^2}{2} \quad (124)$$

Hence the general solution of the model is given by (102), (111), (118), and (124) i.e.

$$M(t) = M_0 + (\beta + \mu M_0 - \delta M_0)t$$
$$-(\mu + \delta)(\beta - \mu M_0 - \delta M_0)\frac{t^2}{2} \quad (125)$$

$$S(t) = S_0 - (\alpha S_0 I_0 + \mu S_0 + \omega S_0 + \delta M_0)t + \left\{ \begin{aligned} &(\omega - \mu)(-\alpha S_0 I_0 - \mu S_0 - \omega S_0 - \delta M_0)\frac{t^2}{2} \\ &-\delta(\beta - \mu M_0 - \delta M_0)\frac{t^2}{2} \\ &-\alpha(-\alpha S_0 I_0 - \mu S_0 - \omega S_0 - \delta M_0)\begin{pmatrix} \alpha S_0 I_0 \\ -(\mu + \mu_t + \phi) I_0 \end{pmatrix}\frac{t^3}{3} \end{aligned} \right\} \quad (126)$$

$$I(t) = I_0 + (\alpha S_0 I_0 - (\mu + \mu_t + \varphi) I_0)t + \alpha(-\alpha S_0 I_0 - \mu S_0 - \omega S_0 - \delta M_0)\begin{pmatrix} -\alpha S_0 I_0 \\ -(\mu + \mu_t + \varphi) I_0 \end{pmatrix}\frac{t^3}{3}$$
$$-(\mu + \mu_t + \phi)(\alpha S_0 I_0 - (\mu + \mu_t + \phi) I_0)\frac{t^2}{2} \quad (127)$$

$$R(t) = R_0 + (\varphi I_0 - \mu R_0 - \omega R_0)t + \varphi(\alpha S_0 I_0 - (\mu + \mu_t + \phi) I_0)\frac{t^2}{2} - (\mu + \omega)(\phi I_0 - \mu R_0 - \omega R_0)\frac{t^2}{2} \quad (128)$$

Graphical Presentation of the Model Using Maple

This section shows the Graphs generated from the general solution of our Model using MAPLE.

We use Hypothetical Values to generate Graphs for

- Low Contracting Rate and High Recovery Rate respectively.
- High Contracting Rate and High Recovery Rate**.**
- Low Contracting Rate and Low Recovery Rate.
- High Contracting Rate and Low Recovery Rate.

Where $\beta = 0.2, \omega = 0.6, \mu = 0.016, \delta = 0.1, \mu_t = 0.015$

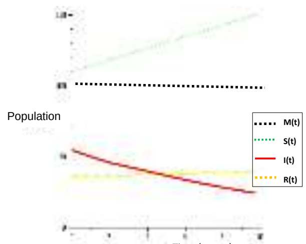

Figure 1: Graphical Profile of Each Compartment for Low Contracting Rate and High Recovery Rate $(\alpha = 0.0001, \varphi = 0.16)$

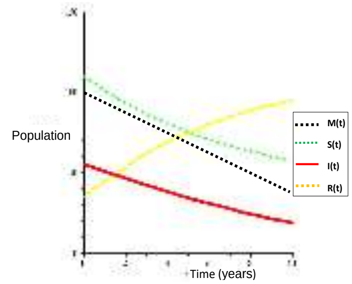

Figure 2: Graphical profile of each compartment for High Contracting Rate and High Recovery Rate $(\alpha = 0.001, \varphi = 0.16)$

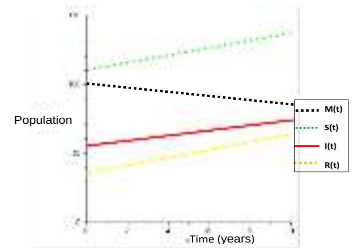

Figure 3: Graphical Profile of Each Compartment for Low Contracting Rate and Low Recovery Rate $(\alpha = 0.0001, \varphi = 0.1)$

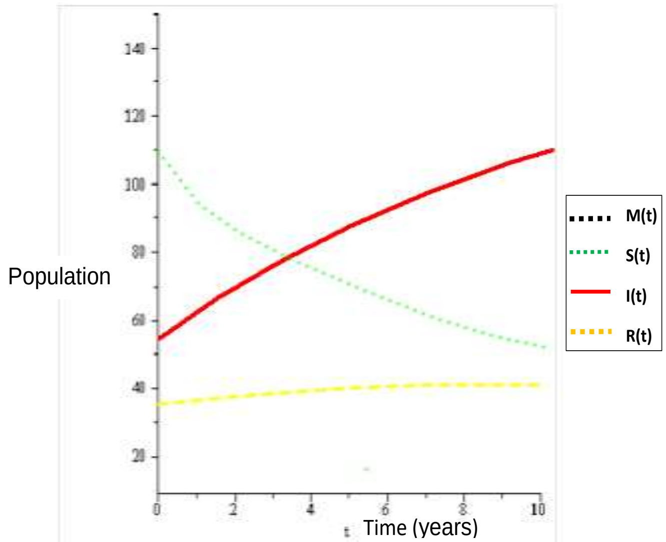

Figure 4: Graphical Profile of Each Compartment for High Contracting Rate and Low Recovery Rate $(\alpha = 0.001, \varphi = 0.1)$

Discussion of Results

In this study, we proposed a mathematical model to study Tuberculosis Disease Dynamics. The general solutions of the Model equation were obtained using Homotopy Perturbation Method (HPM). The Graphical Profiles of the compartments are presented using MAPPLE computer software package.

Figure 1 shows graphs for Low Contracting Rate and High Recovery Rate $(\alpha = 0.0001, \varphi = 0.16)$ we can see from the graph that the population of the Susceptible was increasing, while the population of the infected was decreasing since the Recovery Rate is High we also noticed that the Population of the Recovered rise a bit, also the Population of the Passively Immune Infant decreases we can conclude here that the disease is under control.

Figure 2 shows graphical profiles for the High Contracting Rate and High Recovery Rate. $(\alpha = 0.001, \varphi = 0.16)$ From the graph the Susceptible Population decreases more than when the Contacting Rate was Low and the Population of the Infected decreases more than when the Contracting Rate was Low also the Population of the Passively Immune Infant decreases more than when the Contracting Rate was Low. The Population of the Recovered decreases more than when the Contacting Rate was Low we can conclude here that the more People are Infected the lesser the Population of the Susceptible and more effort is needed to Eradicate the Disease from the Population.

Figure 3 shows the graphs for Low Contracting Rate and Low Recovery Rate $(\alpha = 0.0001, \varphi = 0.1)$ we noticed from the Graph that the Susceptible Population increased more than when the Contracting Rate was High.

Figure 4 is for High Contracting Rate and Low Recovery Rate $(\alpha = 0.001, \varphi = 0.1)$. The Graph show us that the Population of the Susceptible decreased more than when the Recovery Rate was High; this shows that more people are Infected.

Conclusion

The Graphical Profile gave us a vivid understanding of the Disease Dynamics. Fig 2 and 4 show that increase in Contact Rate drops the Population of the Susceptible and the Infected grows exponentially. Prevention is therefore necessary. Figure 1 and 3 shows us that if the Contracting Rate is Low and there is a High Recovery Rate the Infected Population will drop drastically. Hence the Diseases die out and the Population of the Susceptible will Grows exponentially. We can conclude here that the Disease have the tendency of Dying out.

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