

Mathematical Study of Contaminant Transport with Time-Dependent Dispersion Coefficient and Source Concentration in an Aquifer

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Abstract

This paper presents two-dimensional mathematical model describing the transport of a conservative contaminant through a homogeneous finite aquifer under transient flow. We assume the aquifer is subjected to contamination due to the time-dependent source concentration. Both the sinusoidally varying and exponentially decreasing forms of seepage velocity are considered for the purposes of studying seasonal variation problems. The model is solved analytically using parameter-expanding method and direct eigenfunctions expansion technique. The results are presented graphically and discussed. Our results showed that the contaminant concentration decreases along longitudinal and lateral directions as initial dispersion coefficients and initial groundwater velocities increases. This concentration decreases as time increases in the domain.

Keywords and phrases: Contaminant transport, Seepage velocity, Aquifer, Advection-dispersion equation, Source concentration

1.0 Introduction

Many constituents present in the surface water eventually find their way into the ground water through unsaturated zones. The movement of water and solutes through the unsaturated zone has been of importance in traditional applications of ground water hydrology, soil physics, and agronomy. In recent years, the need to understand the behavior of hazardous waste and toxic chemicals in soils has resulted in a renewed interest in this subject. One of the primary concerns is that dissolved contaminants may migrate through the unsaturated zone, reach the saturated zone, and contaminate the ground water.

Contaminant transport in aquifers has become of arising interest in the last few years for scientist working in environmental engineering, hydrology and chemical engineering. These include Winter et al. [1] who defined one- and two-dimensional formation analytically, relate the dispersion parameter to the statistics of the hydraulic conductivity spatial distribution. Batu [2] discussed time-dependent linearized two-dimensional infiltration and evaporation from non-uniform and non-periodic strip source. Latinopoulos et al. [3] studied the chemical transport in two-dimensional aquifer. Aral and Liao [4] examined solutions to two-dimensional advection-dispersion equation with time-dependent dispersion coefficients. In particular, they developed instantaneous and continuous point source solutions for constant, linear, asymptotic, and exponentially varying dispersion coefficients. Stenbacka et al. [5] employed a two-dimensional analytical model for estimating the first-order degradation rate constant of hydrophobic organic compounds (HOCs) in contaminated groundwater under steady-state conditions. Massabo et al. [6] gave some analytical solutions for a two-dimensional advection equation with anisotropic dispersion. Chemical decay or adsorption-like reaction inside the liquid phase is considered. Essa et al. [7] investigated the dispersion of pollutants from a point source, analytically taking into consideration the vertical variation of both wind speed and eddy diffusivity. Shapiro and Bedrikovetsky [8] proposed a new approach to transport of the suspensions and tracers in porous media. In this paper, two-dimensional analytical solution for prediction of concentration distribution in shallow aquifer is presented. Aquifer is considered homogeneous, isotropic, finite and non-reactive. Both (longitudinal and lateral) dispersion coefficients and flow velocities are considered as time-dependent. Seepage velocities, which are the average fluid velocities within the pores, are function of time. Time-dependent source concentration is considered at origin. Initially the domain is not solute free. Dispersion is proportional to seepage velocity. First order decay term which is proportional to dispersion coefficient and retardation factor are also considered. To simulate the flow analytically using

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Parameter-expanding Method and Eigenfunctions Expansion Technique, we assume there is no solute flux at end of both boundaries.

2.0 Model Formulation

Let the contaminant invades the groundwater level from point source in a homogeneous finite aquifer of length L and depth H . The contaminant being of a significantly higher density than the groundwater moves towards the bottom of the shallow aquifer along vertically downward, from its each point the contaminant is bound to spread in the horizontal plane along the transient groundwater flow. It is assumed that initially (i.e., at time $t = 0$), the aquifer is not clean (i.e., the domain is not solute free). Let c_i be the initial contaminant concentration in the aquifer describe the distribution of the concentration at all points of the flow domain. The time-dependent source concentration is assumed at the origin (i.e., $x = 0, y = 0$) of the aquifer. At the end of both boundaries (i.e., $x = L, y = H$), we assumed there is no solute flux. Let $c(x, y, t)$ be the contaminant concentration in the aquifer at position (x, y) and time t , u and v the component of horizontal and lateral (transverse) flow velocity of the medium transporting the contaminants, and D_x and D_y the dispersion coefficients along longitudinal and lateral direction respectively. Then, a two-dimensional problem with first order decay can be mathematically formulated as follows:

$$R \frac{\partial c}{\partial t} = D_x \frac{\partial^2 c}{\partial x^2} + D_y \frac{\partial^2 c}{\partial y^2} - u(t) \frac{\partial c}{\partial x} - v(t) \frac{\partial c}{\partial y} - R\sigma c \tag{1}$$

where R the retardation factor, which is defined as

$$R = 1 + \frac{\rho_d k_d}{n} \tag{2}$$

k_d is distribution coefficient which is defined as ratio of the adsorbed contaminant concentration to the dissolved contaminants, ρ_d is dry unit weight of soil, n is porosity, σ is first-order decay term or first-order chemical transformation term.

Here, we made following assumptions:

1. Fluid is of constant density and viscosity.
2. Solute is subject to first-order chemical transformation (i.e., $\sigma \neq 0$).
3. No adsorption, $k_d = 0$.
4. σ is time-dependent.

Let

$$u(t) = u_0 f(t), \quad v(t) = v_0 f(t), \tag{3}$$

where u_0 and v_0 are initial velocity components along x and y axes respectively.

Based on the above assumptions, (1) reduces to

$$\frac{\partial c}{\partial t} = D_x \frac{\partial^2 c}{\partial x^2} + D_y \frac{\partial^2 c}{\partial y^2} - u(t) \frac{\partial c}{\partial x} - v(t) \frac{\partial c}{\partial y} - \sigma(t)c \tag{4}$$

As initial and boundary conditions, we choose

$$\left. \begin{aligned} c(x, y, t) &= c_i; & x \geq 0, & y \geq 0, & t = 0 \\ c(x, y, t) &= c_0 (1 + \exp(-qt)); & x = 0, & y = 0, & t > 0 \\ \frac{\partial c}{\partial x} &= 0, & \frac{\partial c}{\partial y} &= 0; & x = L, & y = H, & t \geq 0 \end{aligned} \right\} \tag{5}$$

where

c_i is the initial contaminant concentration in the aquifer, c_0 is the solute concentration and q is the parameter like flow resistance coefficient.

3.0 Method of Solution

Ebach and White [9], have established that the dispersion coefficient vary approximately directly to flow velocity, for different types of porous medium. Here, we let $D_x = au(t)$ and $D_y = av(t)$ in (4), where a is the dispersivity that

depends upon the pore geometry. Also, first order decay term which is proportional to dispersion coefficient and retardation factor is considered. Using (3), we get

$$D_x = D_{x0}f(t), \quad D_y(t) = D_{y0}f(t), \quad \sigma(t) = \sigma_0f(t) \quad (6)$$

where $D_{x0} = au_0$ and $D_{y0} = av_0$ are initial dispersion coefficient components along the two respective directions and σ_0 is the first order decay constant.

Using (6) and combining (3) and (4), we obtain

$$\frac{\partial c}{\partial t} = D_{x0}f(t)\frac{\partial^2 c}{\partial x^2} + D_{y0}f(t)\frac{\partial^2 c}{\partial y^2} - u_0f(t)\frac{\partial c}{\partial x} - v_0f(t)\frac{\partial c}{\partial y} - \sigma_0f(t)c \quad (7)$$

Consider the temporally dependent forms of solute dispersion. Let $f(t) = v(t)$, $v(t)$ is the seepage velocity. Then, (7) becomes

$$\frac{1}{v(t)}\frac{\partial c}{\partial t} = D_{x0}\frac{\partial^2 c}{\partial x^2} + D_{y0}\frac{\partial^2 c}{\partial y^2} - u_0\frac{\partial c}{\partial x} - v_0\frac{\partial c}{\partial y} - \sigma_0c \quad (8)$$

Here, in order to account for the seasonal variation in a year on tropical regions $v(t)$ will be considered in two forms:

1. A sinusoidal varying form, $v(t) = 1 - \sin mt$ and
2. An exponentially decreasing form, $v(t) = \exp(-mt)$, $mt < 1$,

where m is the flow resistance coefficient.

We introduce a new time variable [10]:

$$\tau = \int_0^t v(s)ds \quad (9)$$

such that

$$\frac{d\tau}{dt} = v(t) \quad \text{and} \quad \frac{dt}{d\tau} = \frac{1}{v(t)} \quad (10)$$

Then, (8) and the corresponding initial and boundary conditions (5) become

$$\frac{\partial c}{\partial \tau} = D_{x0}\frac{\partial^2 c}{\partial x^2} + D_{y0}\frac{\partial^2 c}{\partial y^2} - u_0\frac{\partial c}{\partial x} - v_0\frac{\partial c}{\partial y} - \sigma_0c \quad (11)$$

$$\left. \begin{aligned} c(x, y, \tau) &= c_i; \quad x \geq 0, \quad y \geq 0, \quad \tau = 0 \\ c(x, y, \tau) &= c_0(2 - q\tau); \quad x = 0, \quad y = 0, \quad \tau > 0 \\ \frac{\partial c}{\partial x} &= 0, \quad \frac{\partial c}{\partial y} = 0; \quad x = L, \quad y = H, \quad \tau \geq 0 \end{aligned} \right\} \quad (12)$$

Let us introduce a new space variable as:

$$z = x + y\sqrt{\frac{D_{y0}}{D_{x0}}} \quad (13)$$

then, (11) and the corresponding initial and boundary conditions (12) become

$$\frac{\partial c}{\partial \tau} = D\frac{\partial^2 c}{\partial z^2} - U\frac{\partial c}{\partial z} - \sigma_0c \quad (14)$$

$$\left. \begin{aligned} c(z, \tau) &= c_i; \quad z \geq 0, \quad \tau = 0 \\ c(z, \tau) &= c_0(2 - q\tau); \quad z = 0, \quad \tau > 0 \\ \frac{\partial c}{\partial z} &= 0; \quad z = L + H\sqrt{\frac{D_{y0}}{D_{x0}}} = l, \quad \tau \geq 0 \end{aligned} \right\} \quad (15)$$

where $D = D_{x0} \left(1 + \frac{D_{y0}^2}{D_{x0}^2} \right)$ and $U = \left(u_0 + v_0 \sqrt{\frac{D_{y0}}{D_{x0}}} \right)$

3.1 Non-dimensionalisation

We non-dimensionalised (14) and (15) using the following set of dimensionless variables:

$$z' = \frac{z}{l}, \quad c' = \frac{c}{c_0}, \quad \tau' = \frac{D\tau}{l^2}, \quad U' = \frac{Ul}{D}, \quad q' = \frac{ql^2}{D}, \quad \sigma'_0 = \frac{\sigma_0 l^2}{D} \quad (16)$$

to obtain (after dropping prime)

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial z^2} - U \frac{\partial c}{\partial z} - \sigma_0 c \quad (17)$$

$$\left. \begin{aligned} c(z, \tau) &= \frac{c_i}{c_0}; & z \geq 0, \quad \tau = 0 \\ c(z, \tau) &= (2 - q\tau); & z = 0, \quad \tau > 0 \\ \frac{\partial c}{\partial z} &= 0; & z = 1, \quad \tau \geq 0 \end{aligned} \right\} \quad (18)$$

For both the expressions of $v(t)$, the non-dimensional time variable τ may be written as:

$$\tau = \frac{D}{l^2} \int_0^t v(s) ds \quad (19)$$

So that for

1. A sinusoidal varying form, $\tau = \frac{D}{ml^2} (mt - (1 - \cos mt))$ (20)

2. An exponentially decreasing form, $\tau = \frac{D}{ml^2} (1 - \exp(-mt))$, $mt < 1$ (21)

3.2 Solution by Parameter-expanding Method

Suppose the solution $c(z, \tau)$ and the constant U in (17) can be expressed as

$$c = c_0 + \sigma_0 c_1 + \sigma_0^2 c_2 + h.o.t. \quad (22)$$

$$U = \sigma_0 p_0 + \sigma_0^2 p_1 + h.o.t. \quad (23)$$

where *h.o.t.* read "higher order terms in σ_0 ". In our analysis we are interested only in the first two terms. Substituting (22) and (23) into (17) and (18), and processing, we obtain:

$$\frac{\partial c_0}{\partial \tau} = \frac{\partial^2 c_0}{\partial z^2} \quad (24)$$

$$c_0(z, 0) = \frac{c_i}{c_0}, \quad c_0(0, \tau) = (2 - q\tau), \quad \left. \frac{\partial c_0}{\partial z} \right|_{z=1} = 0$$

$$\frac{\partial c_1}{\partial \tau} = \frac{\partial^2 c_1}{\partial z^2} - p_0 \frac{\partial c_0}{\partial z} - c_0 \quad (25)$$

$$c_1(z, 0) = 0, \quad c_1(0, \tau) = 0, \quad \left. \frac{\partial c_1}{\partial z} \right|_{z=1} = 0$$

Transform (24) to an inhomogeneous equation with homogeneous boundary conditions and seek a direct eigenfunctions expansion, we obtain

$$c_0(z, \tau) = 2 - q\tau + \sum_{n=1}^{\infty} \left(\frac{4 \left(\frac{c_i}{c_0} - 2 \right) e^{-\left(\frac{2n-1}{2}\right)^2 \pi^2 \tau} - 16q \left(1 - e^{-\left(\frac{2n-1}{2}\right)^2 \pi^2 \tau} \right)}{(2n-1)\pi} - \frac{16q \left(1 - e^{-\left(\frac{2n-1}{2}\right)^2 \pi^2 \tau} \right)}{(2n-1)^3 \pi^3} \right) \sin\left(\frac{2n-1}{2}\right)\pi z \quad (26)$$

$$c_1(z, \tau) = \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{4 \left(\frac{c_i}{c_0} - 2 \right) \left(4p_0 + (2n-1)^2 \pi^2 \right) \tau e^{-\left(\frac{2n-1}{2}\right)^2 \pi^2 \tau}}{(2n-1)^3 \pi^3} \sin\left(\frac{2n-1}{2}\right)\pi z - \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{64q \left(4p_0 + (2n-1)^2 \pi^2 \right) \left(1 - \left(\left(\frac{2n-1}{2} \right)^2 \pi^2 \tau + 1 \right) e^{-\left(\frac{2n-1}{2}\right)^2 \pi^2 \tau} \right)}{(2n-1)^7 \pi^7} \sin\left(\frac{2n-1}{2}\right)\pi z \quad (27)$$

$$+ \sum_{n=1}^{\infty} \frac{64 \left(2 \left(\frac{2n-1}{2} \right)^2 \pi^2 + q \right) \left(1 - e^{-\left(\frac{2n-1}{2}\right)^2 \pi^2 \tau} \right) - \left(\frac{2n-1}{2} \right)^2 \pi^2 q \tau}{(2n-1)^5 \pi^5} \sin\left(\frac{2n-1}{2}\right)\pi z$$

For the sinusoidally varying velocity, we substitute (20) into (26) and (27) while for the exponentially decreasing velocity, we substitute (21) into (26) and (27).

The computations were done using computer symbolic algebraic package MAPLE.

4.0 Results and Discussion

Analytical solutions given by (26) and (27) are computed for the values of $c_i = 200$, $c_0 = 1.0$, $q = 0.2$ (/day), $u_0 = 1, 2, 4$ (km / day), $v_0 = 0.1, 0.2, 0.4$ (km / day),

$D_{x0} = 1.5, 3.0, 4.5$ (km² / day), $D_{y0} = 0.15, 0.30, 0.45$ (km² / day), $l = 1$ km, $m = 2$ (/day) (for sinusoidally varying velocity) and $m = 0.9$ (/day) (for exponentially decreasing velocity). The concentration values are depicted graphically in Figures 1 – 10.

The contaminant concentration distribution behaviors along transient groundwater flow for sinusoidally varying velocity are shown in Figures 1 – 5. Figure 1 depicts the graph of $c(x, y, t)$ against x and y for different values of D_{x0} . It is observed that the contaminant concentration decreases along longitudinal and lateral directions as initial dispersion coefficient along longitudinal direction increases. Figure 2 depicts the graph of $c(x, y, t)$ against x and y for different values of D_{y0} . It is observed that the contaminant concentration increases and later decreases along longitudinal and lateral directions as initial dispersion coefficient along lateral direction increases.

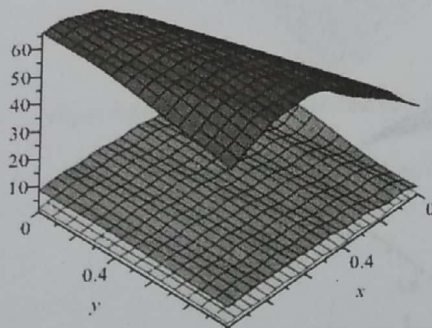


Figure 1. Plots of $c(x, y, \tau)$ against x and y for different values of D_{x0} and $c_i = 200, c_0 = 1.0, q = 0.2, m = 2, u_0 = 1, v_0 = 0.1, D_{y0} = 0.15, \sigma_0 = 0.0004, \tau = 1$

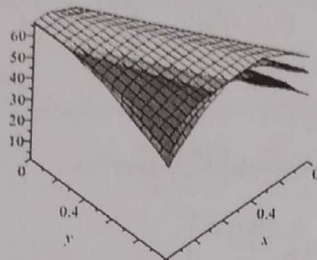


Figure 2. Plots of $c(x, y, t)$ against x and y for different values of D_{y0} and
 $c_i = 200, c_0 = 1.0, q = 0.2, m = 2, u_0 = 1, v_0 = 0.1, D_{x0} = 1.5, D_{y0} = 0.0004, \tau = 1$

Figure 3 depicts the graph of $c(x, y, t)$ against x and y for different values of u_0 . It is observed that the contaminant concentration increases along longitudinal and lateral directions as initial velocity along longitudinal direction increases. Figure 4 depicts the graph of $c(x, y, t)$ against x and y for different values of v_0 . It is observed that the contaminant concentration increases along longitudinal and lateral directions as initial velocity along lateral direction increases.

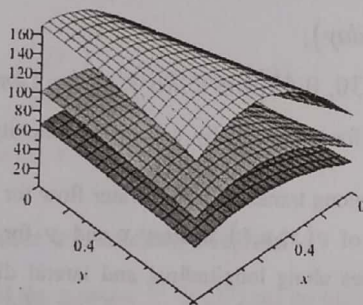


Figure 3. Plots of $c(x, y, t)$ against x and y for different values of u_0 and
 $c_i = 200, c_0 = 1.0, q = 0.2, m = 2, v_0 = 0.1, D_{x0} = 1.5, D_{y0} = 0.15, \sigma_0 = 0.0004, \tau = 1$

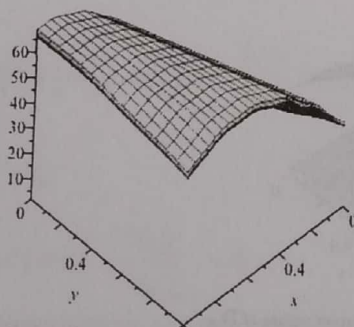


Figure 4. Plots of $c(x, y, t)$ against x and y for different values of v_0 and
 $c_i = 200, c_0 = 1.0, q = 0.2, m = 2, u_0 = 1, D_{x0} = 1.5, D_{y0} = 0.15, \sigma_0 = 0.0004, \tau = 1$

Figure 5 depicts the graph of $c(x, y, t)$ against x and y for different values of t . It is observed that the contaminant concentration decreases along longitudinal and lateral directions with increasing time.

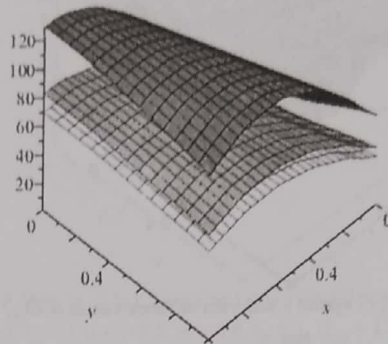


Figure 5. Plots of $c(x, y, t)$ against x and y for different values of t and $c_i = 200, c_0 = 1.0, q = 0.2, m = 2, u_0 = 1, v_0 = 0.1, D_{x0} = 1.5, D_{y0} = 0.15, \sigma_0 = 0.0004$

The contaminant concentration distribution behaviors along transient groundwater flow for exponentially decreasing velocity are shown in Figures 6 – 10. Figure 6 depicts the graph of $c(x, y, t)$ against x and y for different values of D_{x0} . It is observed that the contaminant concentration decreases along longitudinal and lateral directions as initial dispersion coefficient along longitudinal direction increases. Figure 7 depicts the graph of $c(x, y, t)$ against x and y for different values of D_{y0} . It is observed that the contaminant concentration increases and later decreases along longitudinal and lateral directions as initial dispersion coefficient along lateral direction increases.

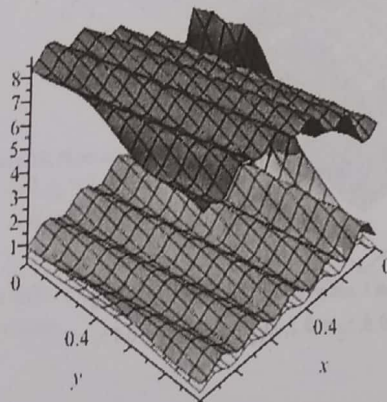


Figure 6. Plots of $c(x, y, t)$ against x and y for different values of D_{x0} and $c_i = 200, c_0 = 1.0, q = 0.2, m = 0.9, u_0 = 1, v_0 = 0.1, D_{y0} = 0.15, \sigma_0 = 0.0004, t = 1$

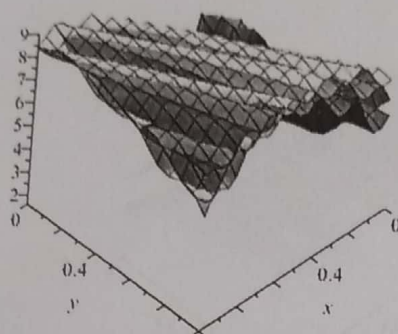


Figure 7. Plots of $c(x, y, t)$ against x and y for different values of D_{y0} and $c_i = 200, c_0 = 1.0, q = 0.2, m = 0.9, u_0 = 1, v_0 = 0.1, D_{x0} = 1.5, \sigma_0 = 0.004, \tau = 1$

Figure 8 depicts the graph of $c(x, y, t)$ against x and y for different values of u_0 . It is observed that the contaminant concentration increases along longitudinal and lateral directions as initial velocity along longitudinal direction increases. Figure 9 depicts the graph of $c(x, y, t)$ against x and y for different values of v_0 . It is observed that the contaminant concentration increases along longitudinal and lateral directions as initial velocity along lateral direction increases.

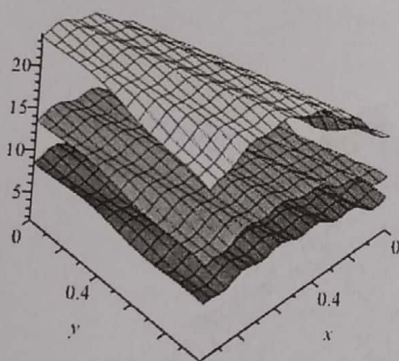


Figure 8. Plots of $c(x, y, t)$ against x and y for different values of u_0 and $c_i = 200, c_0 = 1.0, q = 0.2, m = 0.9, v_0 = 0.1, D_{x0} = 1.5, D_{y0} = 0.15, \sigma_0 = 0.0004, \tau = 1$

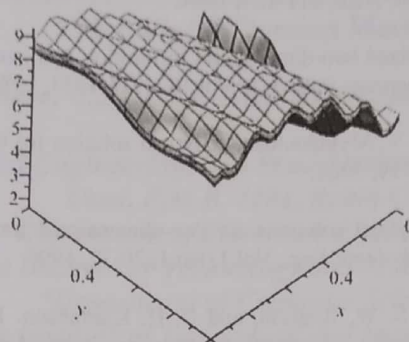


Figure 9. Plots of $c(x, y, t)$ against x and y for different values of v_0 and $c_i = 200, c_0 = 1.0, g = 0.2, m = 0.9, u_0 = 1, D_{x0} = 1.5, D_{y0} = 0.15, \sigma_0 = 0.0004, \tau = 1$

Figure 10 depicts the graph of $c(x, y, t)$ against x and y for different values of t . It is observed that the contaminant concentration decreases along longitudinal and lateral directions with increasing time.

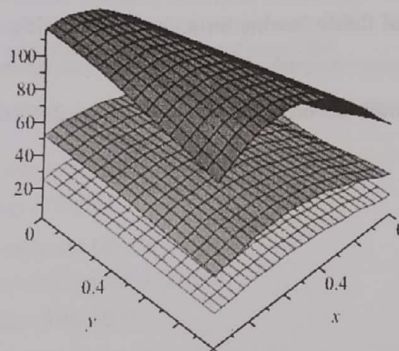


Figure 10. Plots of $c(x, y, t)$ against x and y for different values of t and $c_i = 200, c_0 = 1.0, g = 0.2, m = 0.9, u_0 = 1, v_0 = 0.1, D_{x0} = 1.5, D_{y0} = 0.15, \sigma_0 = 0.0004$.

It is worth pointing out that the effect observed in Figures 5 and 10, is an indication that as time increases in an aquifer, contaminant concentration decreases.

5.0 Conclusion

A two-dimensional solute transport model with time dependent source concentration formulated to predict contaminant concentration along transient groundwater flow in a homogeneous finite shallow aquifer is solved analytically using parameter expanding method and direct eigenfunctions expansion technique. The governing parameters of the problem are the initial dispersion coefficient along longitudinal direction (D_{x0}), initial dispersion coefficient along lateral direction (D_{y0}), initial groundwater velocity along longitudinal direction (u_0) and initial groundwater velocity along lateral direction (v_0). It is discovered that the contaminant concentration distribution is significantly influenced by the parameters involved. This concentration decreases as time increases in the domain.

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