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Temperature Dependent Poiseuille Fluid Flow between Parallel Plates

Odutayo R. Rufai[1*], Ademola M. Rabiu[2], Ismail B. Adefeso[3], Kazeem O. Sanusi[4] and Sarafa O. Azeez[5]

Abstract — *We investigated the flow of slowly reacting non-Newtonian fluids between two parallel horizontal plates. The viscosity of the fluid is an exponential function of the fluid mean temperature. The reaction rate is assumed to take the form of a power function according to the Arrhenius law and the system is characterized by very large activation energy. We non-dimensionalize both the momentum and energy equations and the steady state equations were solved analytically and numerically. We proved that the steady state problem has a solution using the shooting method technique. It is shown that the system parameters have significant influence impact on the solutions. A major result of the paper is the existence of two solutions for the velocity equation.*

Key Words — Non-Newtonian fluid, Poiseuille flow, Heat and Mass transfer, Steady flow, Shooting method.

I. INTRODUCTION

Much research efforts have been devoted to the study of heat transfer and thermal stability of reacting non-Newtonian fluids [for instance 1-5]. Poiseuille flow constitutes a class of parallel flows in fluid mechanics with many applications in modeling of several biological and engineering systems. This type of flow normally occurs between two parallel planes due to an imposed constant pressure or flow uniformity on both planes [1-2]. The flow behaviour of non-Newtonian fluids has wide applications in many branches of science and engineering.

Of particular interest is the thermal behaviour of fluids whose viscosity changes with temperature and the flow is accompanied by a simultaneous transfer of mass, energy and momentum in the system due to reaction occurring between the fluids. The ability to adequately describe such systems is necessary for the prediction of its thermal stability among others. This is of extreme importance not to compromise on safety of life and materials during handling and processing of such fluids [6,7]; and for quality control purposes in many manufacturing and processing industries [1,3]. An improvement in thermal recovery and utilization during the convective flow in any fluid is one of the fundamental problems of the engineering processes. An improved thermal integration of such systems will provide for better material processing, energy conservation and more environmentally benign process [8].

The first published work in modeling of flow of chemically reactive fluids was credited to Liljenroth [9]. He investigated how autocatalyticity led to ignitions and multiple steady states in an autothermal reactor. He also studied the balance between heat production by an exothermic reaction and its removal by convective flow of the process streams.

Following up on this ground breaking work, Adler [10] using numerical techniques, studied the temperature and radius of the hot gas bubble in a chemically reactive flow system consisting of viscous, incompressible fluids to obtain the criteria for the initiation thermal explosion. In this work however, the work failed to account for the viscosity dependence as well as the gravity effects. The possibility of the existence of a considerable resistance to heat transfer between the reacting fluids and system as a result of low conducting fluids or highly conductive vessel wall, resulting in significant temperature gradient, was reported by Frank-Kamenetskii [11].

Various constitutive models have been proposed to describe the properties of non-Newtonian fluids. The major problem however is that none of these models can adequately describe the peculiarity of these class of fluids. In recent time, the mathematical formulation of thermally critical systems mainly focuses on the determination of the critical regimes separating the regions of explosivity and non-explosivity of chemical reactions. Adesanya *et al* [12] reported the existence of a secondary flow for a temperature dependent viscous Couette flow. A detail review of various works on stability of flows was reported by Billingham [14]. Yihao Zheng *et al.* [15] investigated the kinetic behavior and hydrodynamics of pressure-driven Poiseuille flow. Makinde [5] studied the thermal stability of a reactive third-grade liquid flowing steadily between two parallel plates with symmetrical convective cooling at the walls. Shonhiwa [16] succeeded in obtaining transitional values for reactive plane-Poiseuille flow with approximation to the Arrhenius-rate term.

The effect of the dimensionless non-Newtonian coefficient on the thermal stability of a reactive viscous liquid flowing between two parallel heated plates was investigated by Okoya [17]. In the work, values for transition (that is, where criticality disappeared) ranging from n=0 to 2 were obtained. For a) bimolecular temperature dependence, n was reported to be ½; b) n = 0 for Arrhenius or zero-order reaction and c) for sensitized temperature dependence, n = -2. Extending the work to steady ~~flow of~~ reactive incompressible third-grade homogeneous fluids between two parallel plates with the lower plate at rest and the upper plate in uniform motion, Okoya [18] employed numerical methods to obtained the critical and transitional values of the flow parameters for the above three cases. For generalized Couette flow Okoya [19] investigated the thermal transition of a reactive flow of a third-grade fluid with viscous heating and chemical reaction.

Also, extensive work has been carried out on the subject for various shapes of the cross-section of the thermal explosion, [22-27]. Hence the study of hydrodynamics and thermal explosion within a channel is very important for practical purposes.

The goal of this paper is to investigate the variations of velocity profile for a steady, fully developed, incompressible; fluid whose viscosity depends strongly on temperature.

II. MATHEMATICAL MODEL

We consider the flow of an incompressible viscous fluid between two parallel plates. The equations governing the motion of the fluid are:

Momentum equation,

$$\rho\left(\frac{\partial u}{\partial t}+V_0\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)-\frac{\partial p}{\partial x} \qquad (1)$$

Energy equation,

$$\rho C_p\left(\frac{\partial T}{\partial t}+V_0\frac{\partial T}{\partial y}\right)=\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+Q\exp\left(-\frac{E}{RT}\right) \qquad (2)$$

where ρ is the density, μ is the viscosity, C_p is heat capacity, u and V_o are velocity components along x and y axis respectively, T is the temperature, P is pressure, k is thermal conductivity, x is the co-ordinate in the direction of flow, E is the activation energy, R is the universal gas constant and Q is heat released per unit mass during reactions.

The boundary and the initial condition of the flow are:

$$u(h,t)=u(-h,t)=0, u(y,0)=0,$$
$$T(h,t)=T_0, T(-h,t)=T_0,$$
$$T(y,0)=T_0 \qquad (3)$$

We assume a temperature dependent viscosity

$$\mu=\mu_0\exp[(\alpha(T-T_0)] \qquad (4)$$

III. NON-DIMENSIONALIZATION

Let,

$$\theta=(T-T_o)\frac{E}{RT_o^2},\quad \varphi=\frac{u}{V_o},\quad \bar{y}=\frac{y}{h},\quad \bar{t}=\frac{t}{t_0} \qquad (5)$$

where t_0 is a reference time

Then, equations (1) and (2) becomes

$$\frac{\partial\phi}{\partial t}+a\frac{\partial\phi}{\partial y}=b\frac{\partial}{\partial y}\left(\exp(\lambda\theta)\frac{\partial\phi}{\partial y}\right)+M \qquad (6)$$

$$\frac{\partial\theta}{\partial t}+a\frac{\partial\theta}{\partial y}=\frac{\partial^2\theta}{\partial y^2}+f*\exp\left(\frac{\theta}{1+\varepsilon\theta}\right) \qquad (7)$$

where

$$a=\frac{v_o t_o}{h},\quad b=\frac{t_o\mu_o}{h},\quad M=\frac{-t_o}{\rho V_o}\frac{\partial p}{\partial x},\quad d=\frac{kt_o}{\rho},$$

$$\lambda=\alpha\left(\frac{RT_o^2}{E}\right),\quad f=\frac{t_o.E.Q\exp\left(-\frac{E}{RTo}\right)}{\rho.C_pRT_o^2}$$

Initial conditions are:

$$\theta(y,0)=0, \phi(y,0)=0 \qquad (8)$$

Boundary conditions are:

$$\theta(-1,t)=0, \phi(-1,t)=0, \theta(1,t)=0, \phi(1,0)=0 \qquad (9)$$

IV. STEADY STATE

We assume that the fluid properties and the variables of this flow are independent of time, i.e $\frac{d}{dt}=0$, then we have

$$a\frac{d\phi}{dy}=b\frac{d}{dy}\left(\exp(\lambda\theta)\frac{d\phi}{dy}\right)+M \qquad (10)$$

and

$$a\frac{d\theta}{dy}=d\frac{d^2\theta}{dy^2}+f\exp\left(\frac{\theta}{1+\varepsilon\theta}\right) \qquad (11)$$

as $\varepsilon\to 0$, $\frac{\theta}{1+\varepsilon\theta}=\theta$,

let $a=0$, $b=1$, $\varepsilon=0$, $M=1$, $d=1$, $f=\delta$ then

Equations (9) and (10) becomes,

$$\frac{d}{dy}\left(\exp(\lambda\theta)\frac{d\phi}{dy}\right)+1=0 \qquad (12)$$

$$\phi(-1)=\phi(1)=0 \qquad (13)$$

$$\frac{d^2}{dy^2}+\delta\exp(\theta)=0 \qquad (14)$$

$$\theta(-1)=\theta(1)=0 \qquad (15)$$

From equation (13), we have Buckmaster and Ludford [28],

$$\theta = 2\ln\left\{\exp\left(\frac{\theta_{max}}{2}\right)\sec hcy\right\} \quad (16)$$

$$\exp(\theta) = \exp(\theta_{max})\sec h^2 cy \quad (17)$$

and

$$c^2 = \frac{1}{2}\delta\exp(\theta_{max}) \quad (18)$$

taking $\delta = 0.4$, $\theta_{max} = 0.22/3.28$

A. CASE I, $\lambda = 1$

Equation (12) becomes

$$\exp(\theta_{max})\frac{d}{dy}\left(\sec h^2 cy\frac{d\phi}{dy}\right) = -1 \quad (19)$$

differentiate;

$$\phi'' - 2\tanh cy\phi'^2 cy\exp(-\theta_{max}) \quad (20)$$

and

$$\phi(-1) = \phi(1) = 0 \quad (21)$$

we resolve the above into system of equation

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y \\ \phi \\ \phi' \end{pmatrix} \quad (22)$$

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 1 \\ \phi' \\ \phi'' \end{pmatrix} = \begin{pmatrix} 1 \\ y_3 \\ 2y_3\tanh cy_1(\cosh^2 cy_1)\exp(-\theta_{max}) \end{pmatrix} \quad (23)$$

Satisfying,

$$\begin{pmatrix} y_1(-1) \\ y_2(-1) \\ y_3(-1) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ \beta \end{pmatrix} \quad (24)$$

where β is guessed such that $y_2(1) = 0$ and $\phi(-1) = 0$.

B. CASE II, $\lambda = \frac{1}{2}$

Equation (12) becomes

$$\exp\left(\frac{\theta_{max}}{2}\right)\frac{d}{dy}\left(\sec hcy\frac{d\phi}{dy}\right) = -1 \quad (25)$$

differentiate,

$$\phi'' - \tanh cy\phi' = -\cosh cy.\exp\left(-\frac{\theta_{max}}{2}\right) \quad (26)$$

and

$$\phi(-1) = \phi(1) = 0 \quad (27)$$

Also, we resolve the above into system of equation

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y \\ \phi \\ \phi' \end{pmatrix} \quad (28)$$

and

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 1 \\ \phi' \\ \phi'' \end{pmatrix} = \begin{pmatrix} 1 \\ y_3 \\ y_3\tanh(cy_1)\phi' - \cosh(cy_1).\exp\left(-\frac{\theta_{max}}{2}\right) \end{pmatrix} \quad (29)$$

satisfying

$$\begin{pmatrix} y_1(-1) \\ y_2(-1) \\ y_3(-1) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ \beta_2 \end{pmatrix} \quad (30)$$

where β_2 is guessed such that $y_2(1) = 0$, $\phi(-1) = 0$.

C. Theorem 1:

The equations (9) and (10) which satisfy the boundary conditions (14) and (15) has a unique solution (ϕ , θ) for each $\phi(0)$ [28]

Proof:

$$\frac{d\phi}{dy} = \frac{d}{dy}\left(\exp(\theta)\frac{d\phi}{dy}\right) + 1 \quad (31)$$

$$\phi(-1) = \phi(1) = 0 \quad (32)$$

Let

$$\begin{pmatrix} y \\ \phi \\ \phi' \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \quad (33)$$

and

$$\phi_3 = \exp(\theta)\frac{d\theta}{dy}\phi_3 + 1 + \exp(\theta)\phi_3', \quad (34)$$

then,

$$\begin{pmatrix} \phi_1' \\ \phi_2' \\ \phi_3' \end{pmatrix} = \begin{pmatrix} 1 \\ \phi_3 \\ \frac{\left(\phi_3 - 1 - \phi_3 \exp(\theta)\frac{d\theta}{dy}\right)}{\exp(\theta)} \end{pmatrix} \quad (35)$$

where

$$\exp(\theta) = \left\{\exp\left(\frac{\theta_{\max}}{2}\right)\sec hcy\right\}^2 \quad (36)$$

$\phi_1(0) = 0$, $\phi_2(0) = 0$, $\phi_1(0) = \alpha$ prescribed to satisfy $\phi(1) = 1$

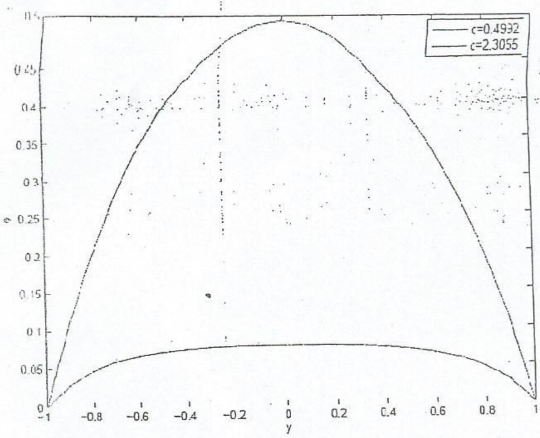
$$\phi_1' = f_1(y, \phi_1, \phi_2, \phi_3)$$
$$\phi_2' = f_2(y, \phi_1, \phi_2, \phi_3)$$
$$\phi_3' = f_3(y, \phi_1, \phi_2, \phi_3)$$

Then f_i, $i = 1, 2, 3$ are lipshits continuous.

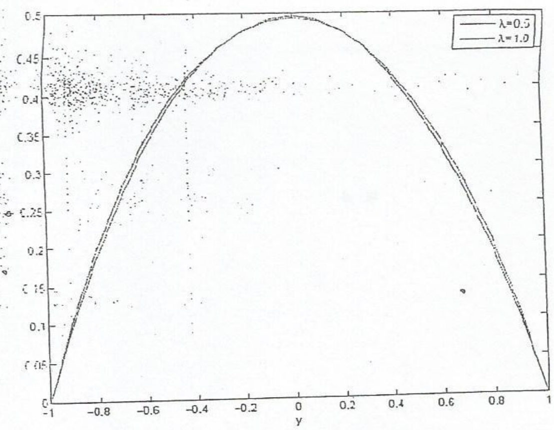
Hence by existence theorem the solution is unique.

V. CONCLUSIONS

It has been shown [29] that the temperature θ has two solutions. We investigated the behaviour of the velocity when viscosity, μ depends exponentially on temperature, θ. The existence of two velocity solutions for temperature depend viscous flow is just discovered here. From our results, it shows that the smaller value of maximum temperature ($(\theta_{\max})$ corresponds to the higher value of velocity. The graphs show that both velocity and temperature reaches the maximum at the middle of the channel $(y = 0)$.



(A)



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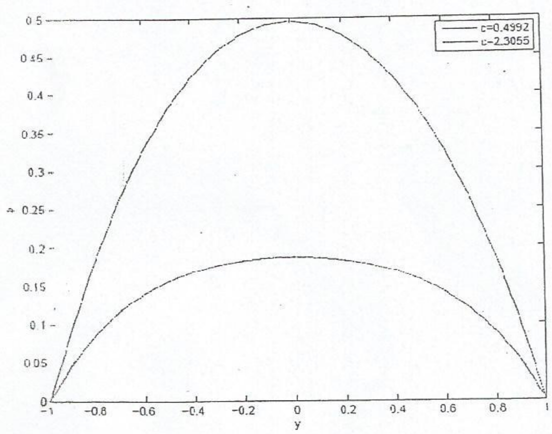

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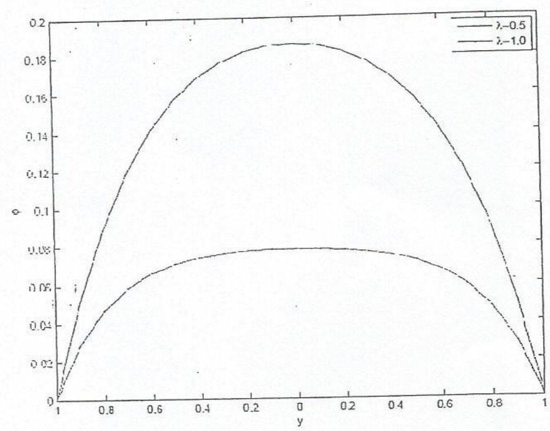

(D)

Figure 1: The velocity profiles (a) $\lambda = 1.0$, $\theta_{max} = 0.22$, c = 0.4992(- curve) and $\theta_{max} = 3.28$, c = 2.3055(.- curve); (b)) $\lambda = 0.5$, $\theta_{max} = 0.22$, c = 0:4992(-curve), and $\theta_{max} = 3.28$, c = 2.3055 (.- curve), (c) $\theta_{max} = 3.28$, c = 2.3055, $\lambda = 0.5$(- curve) and λ= 1:0(.- curve), (d) θ_{max} = 0:22 c = 0:4992, λ = 0.5(.- curve)

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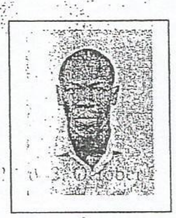
BIOGRAPHIES



Odutayo R. Rufai was born in Ago-Iwoye, Nigeria in 1977. Currently pursuing his PhD degree in Space Plasma Physics at University of the Western Cape, South-Africa. He received B.Sc (Hons) degree in Mathematics from Lagos State University, Nigeria in 2000, M.Sc in Mathematics (Computational Fluid Mechanics) in 2007 at Olabisi Onabanjo University, Ago-Iwoye, Nigeria. He is a lecturer at Department of Mathematics, Lagos state University, Nigeria. His research area are Space Plasma, Computational fluid Mechanics and Modelling.

Ademola M. Rabiu (1969, Nigeria), (M.Sc 1999 from Obafemi Awolowo University). Current address: Lecturer in the Department of Chemical Engineering, Cape Peninsula University of Technology, Cape Town, South Africa. He is a member of America Chemical Society, Society of Petroleum Engineers, South African Institution of Chemical Engineers, Nigerian Society of Engineers and Nigerian Society of Chemical Engineers. Fields of interest: Process Modelling and Simulation, Computational Fluid Dynamics, Numerical Analysis, Computer Oriented Algorithms.

Ismail B. Adefeso was born in Ile-Ife, Nigeria on July 2 1975. He had National Diploma in Electrical & Electronic Engineering. He obtained from Obafemi Awolowo University, Nigeria, BSc and MSc in Chemical Engineering in 2003 and 2010 respectively. He is a member of Nigerian Society of Chemical Engineers and American Chemical Society. He is presently pursuing his doctoral Degree in Chemical Engineering, Cape Peninsula University of Technology, Cape Town. His research interest is Mathematical modelling of Fuel Cells and Process optimization.

Kazeem O. Sanusi was born in Osogbo, Nigeria, in 1972. He received B.Eng in Materials and Metallurgical Engineering from Federal University of Technology Akure, Nigeria in 1999, MSc degree in Industrial and Production Engineering, University of Ibadan, Nigeria in 2003 and D.Tech degree in Mechanical Engineering from Cape Peninsula University of Technology, Cape Town, South Africa in 2011. He is currently a research fellow in the Department of Industrial Engineering, University of Stellenbosch, South Africa. His research interests are modelling and simulation of materials behaviours.

Sarafa O. Azeez was born in Nigeria, in 1968. He received B.Sc in 1994 and MSc in 1998 from Obafemi Awolowo University, Nigeria

and University of Lagos, Akoka, Nigeria respectively. Currently on leave for his doctoral program with University of Cape Town, South Africa specializing in process synthesis and integration, process computation and optimization. He lectures at the Federal University of Technology, Minna, Nigeria.