EXTENDED ACCELERATED OVER RELAXATION (EAOR) METHOD FOR SOLUTION OF A LARGE AND SPARSE LINEAR SYSTEM

AUDU, K. J., YAHAYA, Y. A., ADEBOYE, K. R., & ABUBAKAR, U. Y.
Department of Mathematics, Federal University of Technology, Minna, Nigeria
E-mail: k.james@futminna.edu.ng, yusuphyahaya@futminna.edu.ng, yedeaadeboye@gmail.com, usman.abu@futminna.edu.ng

Abstract
In this research, we introduce a stationary iterative method called Extended Accelerated Over Relaxation (EAOR) method for solving linear systems. The method, an extension of the Accelerated Over Relaxation (AOR) method, was *derived through the interpolation procedure with respect to the sub-matrices in application of a general linear stationary schemes. We studied the convergence properties of the method for special matrices such as L-, H- and irreducible diagonally dominant matrices and proposed some convergence theorems. Some numerical tests were carried out to test the efficiency of the proposed method with existing methods in terms of number of iterations, spectral radius and computational time. The results revealed the superiority of the proposed EAOR method over the AOR method in terms of convergence rate.*

Keywords: Convergence, EAOR method, Iterative method, linear equations, spectral radius

Introduction
Many physical problems encountered in engineering and sciences are modelled into Partial Differential Equations (PDEs). Discretization of such PDEs are carried out through some procedures like finite element volume, finite volume method and finite difference method which usually results into system of linear equations (Saad, 2003). Techniques employed for their solutions are direct methods and iterative methods. Iterative methods are often employed in solving large and sparse linear systems (Wu and Liu, 2014). The efficiency of any iterative method depends on its convergence. As well, it is necessary to investigate how fast the method converges this goes a long way to reduce cost, save storage and time.

Over the years, iterative methods starting from the Jacobi to Gauss-Seidel methods which are non-parameterized have been introduced. Then, the SOR was introduced with one parameter which has proved to outperform the methods of Jacobi and Gauss Seidel. The classical AOR method is a two- parameter iterative method that results into better convergence and greater efficiency than the Successive Over Relaxation (SOR) method in certain cases. The AOR method has been modified into different versions by the works of Wu and Liu (2014), Youssef and Farid (2015) and Vatti *et al*. (2019). Studies on stationary iterative methods have shown the need for an efficient iterative method that would guarantee an improvement to the efficiency and convergence rate of existing methods. Recently, Vatti *et al*. (2020) introduced a new version of the classical AOR method for solving consistently ordered linear systems.

In this research, we attempt to propose an efficient iterative method that would solve H, L and weak irreducible diagonally dominant linear systems, where such proposed method is expected to have increased rate of convergence.
We consider a large and sparse linear system of the type

$$Az = b \tag{1}$$

where the coefficient matrix $A = [a_{ij}]$ is a square nonsingular matrix, z is the unknown vector to be determined and b is the known vector. Several different iterative methods can be employed to solve equation (1) by the usual splitting of matrix A into

$$A = D - L_A - U_A \tag{2}$$

where L_A is strictly lower part of A, U_A is strictly upper part of A and D is the diagonal part of A repectively. Furthermore, a regular decomposition of A into

$$A = P - Q \tag{3}$$

is necessary to solve $Az = b$ in an iterative manner, where matrix P is a square non-singular matrix and Q is a square matrix. With the above decomposition in (3), the standard linear iterative methods for computing linear systems can be represented as:

$$z^{(i+1)} = P^{-1}Qz^{(i)} + Q^{-1}b, \qquad i = 0, 1,2, \cdots \tag{4}$$

Matrix $E = P^{-1}Q$ denotes the iteration matrix of (4). It is important to note that (4) converges to the true solution for any arbitrary initial guess $z^{(0)}$ provided $\rho(E) < 1$, where $\rho(E)$ designates the spectral radius of matrix E.

Hadjidimos (1978) invented the AOR method, a two-parameter *generalization of the Jacobi, Gauss-Seidel and SOR methods.* By employing the splittings in (2), (3) and the iterative scheme (4), the AOR method is governed by the relation

$$z^{(i+1)} = E_{\omega,r}z^{(i)} + (I - \omega L)^{-1}r\dot{b}, \qquad i = 0, 1,2, \cdots \tag{5a}$$

Where

$$E_{\omega,r} = (I - \omega L)^{-1}[(1 - r)I + (r - \omega)L + rU\,] \tag{5b}$$

Methodology

Derivation of the EAOR Method

Considering a general linear stationary iterative method of the form;

$$(\beta_1 D + \beta_2 L_A)z^{(i+1)} = (\beta_3 D + \beta_4 L_A + \beta_5 U_A)z^{(i)} + \beta_6 b, \tag{6}$$

Where $\beta_i, i = 1(1)$ and $(\beta_1 \neq 0)$. We divide through by β_i and let $\beta_i' = \frac{\beta_i}{\beta_1}$, $i = 1(1)$ to obtain

$$(D + \beta_2' L_A)z^{(i+1)} = (\beta_3' D + \beta_4' L_A + \beta_5' U_A)z^{(i)} + \beta_6' b\,, \quad i = 0,1, 2, \cdots, \tag{7}$$

Such that $\frac{\beta_1}{\beta_1} = 1$ and (6) can be written as

$$[D + \beta_2' L_A - \beta_3' D - \beta_4' L_A - \beta_5' U_A]z = \beta_6' b \tag{8}$$

Sufficient condition for the iterative method (7) to be consistent with $Az = b$ are

$$(1 - \beta_3')D + (\beta_2' - \beta_4')L_A - \beta_5' U_A = \beta_6' A\,, \;\; \beta_6' \neq 0 \tag{9}$$

Comparing (2) and (9), we obtain the following relationships

$$\beta_2' - \beta_4' = -\beta_6', \quad 1 - \beta_3' = \beta_6', \quad \text{ and } -\beta_5' = -\beta_6' \tag{10}$$

The choice we make for the two free variables in (10) are;

$$\begin{gathered} \beta_2' = -v - r \\ \beta_6' = \omega \end{gathered} \tag{11}$$

Which gives the following solution of the linear equations in (9)

$$\beta_2' = -v - r, \quad \beta_3' = 1 - \omega, \quad \beta_4' = \omega - v - r, \quad \beta_5' = \omega, \;\; \beta_6' = \omega \tag{12}$$

Where $\omega \neq 0,\ v \neq 0,\ \text{and}\ r \neq 0$ are fixed parameters. Consequently substitution of (12) into (6) gives the proposed method

$$[D - vL_A - rL_A]z^{(i+1)} = [(1 - \omega)D + [\,\omega - v - r]L_A + \omega U_A]z^{(i)} + \omega b \tag{13}$$

Letting $D^{-1}L_A = L,\ D^{-1}U_A = U,\ D^{-1}D = I$ and $D^{-1}b = \bar{b}$, then we have

$$[I - vL - rL]z^{(i+1)} = [(1 - \omega)I + [\omega - v - r]L + \omega U]z^{(i)} + \omega\bar{b}, \qquad i = 0,1, 2, \cdots, \tag{14}$$

Method (13) or (14) is the proposed Extended Accelerated Over-Relaxation method called EAOR. It can also be written as

$$z^{(i+1)} = E_{v,\omega,r}z^{(i)} + [I - vL - rL]^{-1}\omega\bar{b} \tag{15}$$

Where $E_{v,\omega,r}$ is the iteration matrix given as

$$E_{v,\omega,r} = (I - vL - rL)^{-1}[(1 - \omega)I + [\omega - v - r]L + \omega U] \tag{16}$$

The spectral radius of the EAOR method is the highest eigenvalue of its iteration matrix denoted as $\rho(E_{v,\omega,r})$. For specific values of the parameters, the EAOR iterative method reduces to the following methods:
$\omega = \omega,\ r = r,\ v = 0;$ AOR method
$\omega = 0,\ r = 1,\ v = 0;$ Jacobi method
$\omega = \omega,\ r = \omega,\ v = 0;$ SOR method
$\omega = 1,\ r = 1,\ v = 0;$ Gauss-Seidel method

Convergence of the EAOR Method

Definitions and Lemmas

Definition 1: A square matrix A is an $L-$matrix if $a_{ii} > 0$ and $a_{ij} \leq 0$, $i \neq j$ for all $i, j = 1, 2, \cdots, N$.
Definition 2: A matrix $A = [a_{ij}]$ is irreducible if and only if its directed graph $G(A)$ is strongly connected.
Definition 3: A square irreducible matrix A is said to be weakly diagonally dominant if

$$|a_{ii}| \geq \sum_{j=1, j\neq i}^{n} |a_{ij}|, \qquad i = 1, 2, \cdots, n$$

Definition 4: Let a matrix $A = (a_{i,j})$ be a nonsingular matrix and define its comparison matrix as $C\langle A\rangle = \begin{cases} -|a_{i,j}|, & \text{when } i \neq j \\ |a_{i,i}|, & \text{when } i = j \end{cases}$, then A is an $H-$matrix if $C\langle A\rangle$ is a $M-$matrix.

Lemma 1 (Varga, (2000))
Let $A \geq 0$ be an irreducible square matrix. Then

i. A has a positive real eigenvalue equal to its spectral radius.
ii. To the spectral radius of A denoted as $\rho(A)$, there corresponds an eigenvector $z > 0$.
iii. $\rho(A)$ increases when any entry of A increases.
iv. $\rho(A)$ is a simple eigenvalue of A.

Lemma 2 (Young, (2014))
Suppose A and B are two matrices of compatible size with $\rho(A)$ representing the spectral radius of A and $\rho(B)$ representing the spectral radius of B, if $|A| \leq B$ implies that $\rho(A) \leq \rho(B)$, where $|A|$ represents the moduli of the corresponding elements of A.

$L-$matrices

Theorem 1: If matrix A is an $L-$matrix, for all v, r and ω such that $0 < v + r < \omega < 1$, then the EAOR iterative method converges for $\rho(E_{0,0,1}) < 1$.
This theorem will be established using similar ideas from works of Youssef and Farid (2015) and Hadjidimos (1978).
Proof: Suppose that $\lambda = \rho(E_{v,r,\omega}) \geq 1$, with the assumption in mind, we can easily obtain

$$[I - (v + r)L]^{-1} = I + (v + r)L + (v + r)^2L^2 + \cdots + (v + r)^{N-1}L^{N-1} \geq 0 \qquad (17)$$

Therefore, for the matrix of iteration, it gives

$$\begin{aligned} E_{v,r,\omega} &= [I - vL - rL]^{-1}[(1 - \omega)I + [\omega - (v + r)]L + \omega U] \\ &= I + (v + r)L + (v + r)^2L^2 + \cdots + (v + r)^{N-1}L^{N-1} \\ &\quad \times [(1 - \omega)I + (\omega - v - r)L + \omega U\ ] \\ &= (1 - \omega)I + (v + r)(1 - \omega)L + (v + r)^2(1 - \omega)L^2 + (\omega - v - r)L + (v + r)(\omega - v - r)L^2 \\ &\quad + (v + r)^2(\omega - v - r)L^3 + \omega U + \omega(v + r)LU + \omega(v + r)^2L^2U + \cdots \\ &= (1 - \omega)I + (v + r)(1 - \omega)L + \omega U + G \geq 0 \end{aligned} \qquad (18)$$

Where G represents non-negative terms, implying that $E_{v,r,\omega}$ is a nonnegative matrix. Since λ is an eigenvalue of $E_{v,r,\omega}$, if $z \neq 0$ is the eigenvector corresponding to λ, then it implies that $E_{v,r,\omega}\, z = \lambda z$ which yields the following equations below;

$$[I - vL - rL]^{-1}[(1-\omega)I + [\omega - (v+r)]L + \omega U]z = \dot{\lambda} z$$

$$\left[\frac{\omega + r(\dot{\lambda}-1) + v(\dot{\lambda}-1)}{\omega} L + U\right] z = \left(\frac{\dot{\lambda}+\omega-1}{\omega}\right) Iz \qquad (19)$$

Equation (19) indicates that $\frac{\dot{\lambda}+\omega-1}{\omega}$ is an eigenvalue of $\frac{\omega+r(\dot{\lambda}-1)+v(\dot{\lambda}-1)}{\omega}L+U$, hence (19) yields

$$\frac{\dot{\lambda}+\omega-1}{\omega} \;\le\; \rho\left(\frac{\omega + r(\dot{\lambda}-1) + v(\dot{\lambda}-1)}{\omega} L + U\right) \qquad (20)$$

It is easily seen that $\frac{\omega+r(\dot{\lambda}-1)+v(\dot{\lambda}-1)}{\omega} \ge 1$ so that

$$0 \le \frac{\omega + r(\dot{\lambda}-1) + v(\dot{\lambda}-1)}{\omega} L + U \;\le \frac{\omega + r(\dot{\lambda}-1) + v(\dot{\lambda}-\ 1)}{\omega}(L+U)$$

$$= \frac{\omega + r(\dot{\lambda}-1) + v(\dot{\lambda}-1)}{\omega} E_{0,0,1} \qquad (21)$$

Where $E_{0,0,1}$ is the Jacobi matrix and combining equations (20) with (21), yields

$$\dot{\lambda}+\omega-1 \;\le \omega + r(\dot{\lambda}-1) + v(\dot{\lambda}-1)\rho(E_{0,0,1}) \qquad (22)$$

Simple manipulation on (22) results into

$$\rho(E_{0,0,1}) \ge \frac{\dot{\lambda}+\omega-1}{\omega + r(\dot{\lambda}-1) + v(\dot{\lambda}-1)} \ge 1 \qquad (23)$$

And thus we obtain

$$\rho(E_{0,0,1}) \ge 1 \qquad (24)$$

Now, suppose $\rho(E_{0,0,1}) < 1$, then it becomes

$$\frac{\dot{\lambda}+\omega-1}{\omega + r(\dot{\lambda}-1) + v(\dot{\lambda}-1)} < 1 \qquad (25)$$

Which implies $\dot{\lambda} < 1$, so that if $\rho(E_{0,0,1}) < 1$ then the EAOR method equally converges {that is $\rho(E_{v,r,\omega}) < 1$} since the spectral radius of the Jacobi matrix {$\rho(E_{0,0,1})$} is incorporated inside the EAOR iterative method. Hence, the theorem is completed and proved.

Irreducible matrices with weak diagonal dominance

Theorem 2: If A is an irreducible matrix with weak diagonal dominance, then the EAOR iterative method converges for all $0 < v + r < 1$ and $0 < \omega \le 1$.

We shall employ the ideas of authors like Wu and Liu (2014) and Youssef and Farid (2015) to prove the theorem.

Proof: Let A be an irreducible matrix, for some eigenvalue $\dot{\lambda}$ of $E_{v,r,\omega}$, we assumed that $|\dot{\lambda}| \ge 1$. For this $\dot{\lambda}$, the relationship in (26) holds

$$\det(E_{v,r,\omega} - \dot{\lambda} I) = 0 \qquad (26)$$

Simple transformation was performed on (26) to obtain

$$\det(\mathrm{R}) = 0 \qquad (27)$$

Where R is given as

$$\mathrm{R} = I - \frac{\omega + r(\dot{\lambda}-1) + v(\dot{\lambda}-1)}{\dot{\lambda}+\omega-1} L - \frac{\omega}{\dot{\lambda}+\omega-1} U \qquad (28)$$

The moduli of the coefficients of L and U in (28) are less than one. To prove this, it is sufficient and necessary to prove that

$$\left.\begin{array}{c} |\dot{\lambda}+\omega-1| \ge |\omega + r(\dot{\lambda}-1) + v(\dot{\lambda}-1)| \\ and \\ |\dot{\lambda}+\omega-1| \ge |\omega| \end{array}\right\} \qquad (29)$$

suppose $\dot{\lambda}^{-1} = qe^{i\theta}$, where q and θ are real with $0 < q \le 1$, then the first inequality in (29) is equivalent to;

$$[1-(v+r)^2] + [1-(v+r)^2]q^2 - [1-(v+r)^2]2q\cos\theta + [1-(v+r)]2q\omega\cos\theta$$
$$-[1-(v+r)]2q^2\omega \ge 0 \qquad (30)$$

Which holds for $v + r = 1$, otherwise it is equivalent to;

$$(1 + v + r) + (1 + v + r)q^2 - [(1 + v + r) - \omega]2q\cos\theta - 2qr^2\omega \geq 0 \qquad (31)$$

Since the expression in the brackets above is nonnegative, (31) holds for all θ if and only if it holds for $\cos\theta = 1$, hence (31) is equivalent to

$$(1 - q)[(1 + v + r)(1 - q)] + 2q\omega \geq 0 \qquad (32)$$

Which is true, similarly, the second inequality in (29) is equivalently

$$1 + q^2 - 2q(1 - \omega)\cos\theta - 2\omega q^2 \geq 0 \qquad (33)$$

Which for same reason, must be satisfied for $\cos\theta = 1$, then equation (33) is equivalent to

$$(1 - q)[1 - q + 2\omega q] \geq 0 \qquad (34)$$

Which holds for $q = 1$, thus we have

$$\left.\begin{array}{c} \left|\dfrac{\omega + r(\dot{\lambda} - 1) + v(\dot{\lambda} - 1)}{\dot{\lambda} + \omega - 1}\right| \leq 1 \\ \text{and} \\ \left|\dfrac{\omega}{\dot{\lambda} + \omega - 1}\right| \leq 1 \end{array}\right\} \qquad (35)$$

Given that A is irreducible with a weak diagonal dominance, it signifies that $D^{-1}A = I - L - U$ equally have the same properties too. This is also true for R, considering that 1 is greater than the modulus of the coefficients of L and U and they differs from zero. Hence it implies that $\det(\mathrm{R}) \neq 0$ which contradicts (27) and consequently (26). Hence $\rho(E_{v,r,\omega}) < 1$ impying that the EAOR method converges and this completes the proof.

H –matrices

Theorem 3: If the coefficient matrix A is an H –matrix with the domain $0 < v + r < \omega < 1$, then the EAOR method converges to the true solution for any initial guess $z^{(0)}$.

The proof of theorem 3 will be established using similar ideas from works of Yun (2008) and Wu and Liu (2014)

Proof: Let A be an $H-$ matrix with splitting of A into $I - L - U$ and the regular splitting of A into $\omega A = P - Q$ by the EAOR method having choices $P = (I - vL - rL)$ and $Q = (1 - \omega)I + (\omega - v - r)L + \omega U$, then the comparison matrix $C\langle A\rangle = I - |L| - |U|$, with choices of $P_{C\langle A\rangle}$ and $Q_{C\langle A\rangle}$ is obtained as

$$\left.\begin{array}{r} P_{C\langle A\rangle} = (I - (v + r)|L|) \\ Q_{C\langle A\rangle} = \big((1 - \omega)|I| + (\omega - v - r)|L| + \omega|U|\big) \end{array}\right\} \qquad (36)$$

Suppose $P_{C\langle A\rangle} = (I - (v + r)|L|)$, then one can obtain

$$\begin{aligned} |(I - (v + r)L)^{-1}| &= |I + (v + r)L + (v + r)^2L^2 + (v + r)^3L^3 + \cdots + (v + r)^{N-1}L^{N-1}| \\ &\leq (I + (v + r)|L| + (v + r)^2|L|^2 + (v + r)^3|L|^3 + \cdots + (v + r)^{N-1}|L|^{N-1}) \\ &\leq (I - (v + r)|L|)^{-1} = {P_{C\langle A\rangle}}^{-1} \end{aligned} \qquad (37)$$

Let $Q_{C\langle A\rangle} = (1 - \omega)|I| + (\omega - v - r)|L| + \omega|U|$ and taking the modulus of the matrix Q to obtain

$$\begin{aligned} |1 - \omega)I + (\omega - v - r)L + \omega U| &= |(1 - \omega)I + (\omega - v - r)L + \omega U| \\ &= \big((1 - \omega)I + (\omega - v - r)|L| + \omega|U|\big) \\ &\leq (1 - \omega)I + (\omega - v - r)|L| + \omega|U| = Q_{C\langle A\rangle} \end{aligned} \qquad (38)$$

This implies that $|Q| \leq Q_{C\langle A\rangle}$, and similarly, the modulus of matrix $E_{v,r,\omega}$ gives

$$\begin{aligned} |E_{v,r,\omega}| &= \left|(I - (v + r)L)^{-1} \times \big((1 - \omega)I + (\omega - v - r)L + \omega U\big)\right| \\ &= \left|\begin{pmatrix} I + (v + r)L + (v + r)^2L^2 + (v + r)^3L^3 + \cdots + (v + r)^{N-1}L^{N-1} \\ \times [(1 - \omega)I + [\omega - (v + r)]L + \omega U\ ] \end{pmatrix}\right| \\ &\leq (1 - \omega)I + (1 - \omega)(v + r)|L| + (1 - \omega)(v + r)^2|L|^2 + (\omega - v - r)|L| \\ &\quad + (v + r)(\omega - v - r)|L|^2 + (v + r)^2(\omega - v - r)|L|^3 + \omega + \omega(v + r)|L||U| \\ &\quad + \omega(v + r)^2|L|^2|U| + \cdots \\ &\leq [\mathrm{I} - (v + r)|L|]^{-1} \times \big((1 - \omega)I + (\omega - v - r)|L| + \omega|U|\big) = {P_{C\langle A\rangle}}^{-1}Q_{C\langle A\rangle} \end{aligned} \qquad (40)$$

Obviously, it is shown that $|E_{v,r,\omega}| \leq P_{C\langle A\rangle}{}^{-1}Q_{C\langle A\rangle}$, since it holds, then applying lemma 2 to it implies that

$$\rho(E_{v,r,\omega}) \leq \rho(P_{C\langle A\rangle}{}^{-1}Q_{C\langle A\rangle}) \quad (41)$$

And $\rho(P_{C\langle A\rangle}{}^{-1}Q_{C\langle A\rangle}) < 1$ if and only if $\omega[I - |B|]$ is a monotone matrix, that is

$$\begin{aligned} P_{C\langle A\rangle} - Q_{C\langle A\rangle} &= I - (v + r)|L| - [(1 - \omega)I + (\omega - v - r)|L| + \omega|U|] \\ &= \omega[I - |B|] \end{aligned} \quad (42)$$

Given that A is an H matrix then $\omega[I - |B|]$ with $\omega > 0$ is a monotone matrix, therefore the EAOR iterative method converges for $H-$ matrices and the theorem is completed.

Numerical Tests

We computed the spectral radius of the proposed EAOR method and compared the result with spectral radii of AOR and some variants of AOR methods. We further obtain their convergence results. All computations were performed using Maple 2017 software and the stopping criteria utilized is $|z^{(i+1)} - z^{(i)}| < 1.0 \times 10^{-10}$, where $z^{(i+1)}$ is the approximate solution at $(i + 1)th$ iteration and $z^{(i)}$ is the approximate solution at ith iteration. The results are presented in Tables 1 and 2 with the following notations;

$K_{AOR} = (I - \omega L)^{-1}[(1 - r)I + (r - \omega)L + rU\,]$: *Iteration matrix of the classical AOR method*

$K_{EAOR} = (I - vL - rL)^{-1}[(1 - \omega)I + (\omega - v - r)L + \omega U]$: *Iteration matrix of EAOR method*

Example 1: Solve the linear equations generated from discretization of the Laplace equation $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$ on a unit square found in Vatti (2016), whose coefficient matrix is an Irreducible matrix with weak diagonal dominance matrix, expressed as $Az = b$ by AOR and EAOR methods.

$$\begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \\ z_9 \end{pmatrix} = \begin{pmatrix} 1500 \\ 1000 \\ 1000 \\ 2000 \\ 0 \\ 0 \\ 1500 \\ 1000 \\ 1000 \end{pmatrix} \quad (43)$$

Example 2: Solve the system of linear equations from Mohammed and Rivaie (2017), whose coefficient matrix is an L matrix in the form $Az = b$ by methods of the classical AOR and EAOR.

$$\begin{pmatrix} 7 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 7 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 7 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 7 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 7 & -1 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & 7 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 7 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & 0 & -1 & 7 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 7 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 7 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \\ z_9 \\ z_{10} \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} \quad (44)$$

Table 1: Results of Spectral Radius for Example 1

ω	r	$\rho(K_{AOR})$	v	$\rho(K_{EAOR})$
0.1	0.04	0.9702892488	0.05	0.9697412031
0.2	0.08	0.9397055636	0.10	0.9373783471
0.3	0.12	0.9082009036	0.15	0.9026234150
0.4	0.16	0.8757229293	0.20	0.8651200559
0.5	0.20	0.8422144385	0.25	0.8244179744
0.6	0.24	0.8076127016	0.30	0.7799331029
0.7	0.28	0.7718486727	0.35	0.7308822855
0.8	0.32	0.7348460470	0.40	0.6761686319
0.9	0.36	0.6965201257	0.45	0.6141612268

Table 2: Results of Spectral Radius for Example 2

ω	r	$\rho(K_{AOR})$	v	$\rho(K_{EAOR})$
0.1	0.04	0.9695659299	0.05	0.9690220033
0.2	0.08	0.9382651880	0.10	0.9359663473
0.3	0.12	0.9060538059	0.15	0.9005738634
0.4	0.16	0.8728842724	0.20	0.8625309526
0.5	0.20	0.8387051160	0.25	0.8214516534
0.6	0.24	0.8034604202	0.30	0.7768509636
0.7	0.28	0.7670892570	0.35	0.7281033573
0.8	0.32	0.7295250213	0.40	0.6743746009
0.9	0.36	0.6906946434	0.45	0.6145013535

Table 3: Convergence Result for Example 1

ITERATIVE METHODS	NO OF ITERATIONS	CPU TIME (seconds)
AOR	81	0.296
EAOR	56	0.266

Table 4: Convergence Result for Example 2

ITERATIVE METHODS	NO OF ITERATIONS	CPU TIME (seconds)
AOR	60	0.109
EAOR	43	0.031

Discussion of Results

Tables 1 and 2 shows the performance of the proposed EAOR method and the classical AOR method in terms of their spectral radii with varied values of the parameters (r, v and ω). It reveals that the spectral radii of both methods are less than 1 and this tells us that the methods are convergent, but how fast will they converge to the true solution depends on the closeness of their spectral radii to zero. A point to note here is that the closer the spectral radius is to zero, the faster the convergence. We observed that the spectral radius of the EAOR method is smaller than that of the AOR method, that is to say $\rho\rho(K_{EAOR}) < (K_{AOR}) < 1$. This clearly indicates that the EAOR method will converge faster to the true solution than the method of AOR since it has a lower spectral radius.

From Tables 3 and 4, it is observed that the EAOR method converges at 56th and 43rd iterations in comparison to 81st and 60th iterations of the AOR method for examples 1 and 2 respectively. The convergence result indicates that the EAOR method require less number of iterations and computational time to achieve convergence than AOR method. As discussed in Sebro (2018), numerical methods that register small numbers of iterations requires less computer storage to store its data. This means that the EAOR method requires less computer storage compared to AOR method.

Conclusion

In this study, an improved iterative method has been proposed, namely Extended Accelerated Over Relaxation method. We investigated the convergence of the method for some special matrices theoretically and validated the results through some numerical experiments. The results proved that the method is convergent for irreducible weak diagonally dominant, L- and H- matrices. Also, it has been shown that the proposed EAOR method is more efficient than AOR in terms of computational time, storage capacity and spectral radius. Determination of optimum value for the parameters that would maximize the convergence rate of the method will be considered in future studies.

References

Hadjidimos, A. (1978). Accelerated over relaxation method. *Mathematics of computation,* (32), 149 -157.

Mohammed, F. D., & Rivaie, M. (2017). Jacobi-Davisson, Gauss-Seidel and Successive Over-Relaxation for solving systems of linear equations. *Applied Mathematics and Computational Intelligence*, 6, 41-52.

Saad, Y. (2003). *Iterative methods for sparse linear systems.* Society for industrial and applied Mathematics, USA.

Sebro, A. (2018). Extrapolated refinement of generalized Gauss-Seidel method for solving systems of linear equation. Masters Thesis, Jimma University, Ethiopia. Published.

Varga, R. S. (2000). *Matrix iterative analysis*. Berlin, Germany: Springer.

Vatti, V. B. K. (2016). *Numerical analysis iterative methods.* I. K International Publishing House, Pvt, Limited, New Delhi, India.

Vatti, V. B. K., Rao, G. C., & Pai, S. S. (2019). Parametric Accelerated Over-Relaxation (PAOR) method. *Numerical optimization in Engineering and Sciences, select Proceedings in NOIEAS*, 979:283-288.

Vatti, V. B. K., Rao, G. C., & Pai, S. S. (2020). Reaccelerated Over-Relaxation (ROR) method. *Bulettin of the International Mathematical Virtual Institute.* 10(2): 315-324.

Wu, S., & Liu, Y. (2014). A New Version of Accelerated Overrelaxation Iterative method. *Journal of Applied Mathematics,* (725360): 1-6.

Young, D. M. (2014). *Iterative solutions of large linear systems*, Elsevier.

Youssef, I. K., & Farid, M. M. (2015). On the Accelerated Overrelaxation Method. *Pure and Applied Mathematics Journal,* 4, 26 – 31.

Yun, J. H. (2008). Convergence of SSOR multisplitting methods for and H −matrix. *Journal of Computational and Applied Mathematics,* Elsevier, 217, 252-258.