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Application of Markov Model in Continuous Time for the Analysis and Prediction of Weekly Rainfall Pattern

L. Adamu[1], U.Y. Abubakar[1], D.Hakimi[1] and A.S. Gana[2]

[1]Department of Mathematics, Federal University of Technology Minna, Nigeria
[2]Department of Crop Production, Federal University of Technology Minna, Nigeria.

Abstract

A stochastic process whose future state depends on the present state and not on the past state is a Markov chain. This principle was used to formulate a four state stochastic model in continuous time. The model is developed to analyze and predict weekly rainfall pattern of Makurdi in continuous time to enable the inhabitants to plan for the uncertainty of rainfall. The parameters of the model were estimated from weekly rainfall amount of Makurdi, Benue state, Nigeria for a period of eleven years(2005-2015). It was observed that if it is no rainfall state in a given week, it would take at most 49%,27% and 16% of the time to make a transition to low rainfall, moderate rainfall, and high rainfall in the far future. Thus given the rainfall in a week, it is possible to determine quantitatively the probability of finding weekly rainfall in other states in the following week and in the long run.The transition probability matrix, also reveals that, a week of high rainfall cannot be followed by another week of high rainfall, a week of high rainfall cannot be followed by a week of no rainfall and a week of moderate rainfall cannot precede a week of high rainfall. These results are important information to the residents of Makurdi and environmental management scientists to plan for the uncertainty of rainfall, for effective and viable production.

Keywords: Markov chain, Transition probability, Weekly Rainfall Amount, Makurdi.

1.0 Introduction

Rainfall exhibits a strong variability in time and space across the globe. It is well established that rainfall is changing on both the global and the regional scale due to global warming [1]. Rainfall is the principal phenomenon driving many hydrological extremes such as floods, droughts, landslides, debris and mud-flows; its analysis and modeling are typical problems in applied hydrometeorology [2]. Hence, its stochastic modeling is necessary for the prevention of natural disaster. Understanding the rainfall distribution is equally necessary for future planning. This is applicable in areas like agriculture, industry, insurance, hydrological studies and the entire planning of a country economy. Consequently, information on rainfall probabilities is vital for the design of water supply management, supplementary irrigation schemes and the evaluation of alternative cropping system for effective soil water management plans [2]. Such information can also be beneficial in determining the best adapted plant species and the optimum time of seedling to re-establish vegetation on deteriorated rangelands. The aim of this research is to analysed and predict weekly rainfall pattern of Markurdi in continuous time. The results from the model could be useful to the residents of Makurdi for effective planning and viable production.

Overwhelming researchers within Nigeria and in around the world, have proposed several methods in attempt to provide information that will enable humanity to make best use of this random phenomenon, either for agricultural purposes or other purposes fundamental importance to life, such researchers include. Reported in[3]isthe study of the sequence of daily rainfall occurrence. It was found that the daily rainfall occurrence for the Tel Aviv data was successfully fitted with the first-order Markov chain model. Reported in [4] is an application of first- and second-order Markov chain models to dry and wet periods of annual stream flow series to reproduce the stochastic structure of hydrological droughts.

Presented in [5] is a three-state Markov chain to examine the pattern and distribution of daily rainfall in Uyo metropolis of Nigeria using 15 years (1995-2009) rainfall data. The results from their model was an important information to the residents of Uyo.Reported in [6]is an evaluation of effect of climate change on daily rainfall using first-order Markov chain model.

Corresponding author: L. Adamu, E-mail: lawal.adamu@futminna.edu.ng, Tel.: +2348036011769

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presented in [7] is the investigation of the variations of annual rainfall in Tirunelveli district, India based on stochastic method. The method was applied to observational data (1951-2010) and an ensemble of regional climate simulations for Central Europe (1971-2000, 2021-2050) for two type of compound extremes, heavy precipitation and cold in winter and hot and dry days in 10 summer. The research identify three regions in Europe, which are probably susceptible to a future change in the succession or dynamics of heavy precipitation and cold in winter, which are a region south western France, northern Germany and in Russia around Moscow. They concluded that, the change in the succession of hot and dry days in summer will probably affect regions in Spain and Bulgaria. The susceptibility to a dynamic change of hot and dry extremes in the Russian region will probably decrease. A developmental method of stochastic generator of monthly rainfall series had been presented in [8]. The work was based on the modeling of the occurrence and the quantity of rain in a separate way. The occurrence was treated in two stages. The first step considered the Markov chain according to the occurrence of annual statements (dry, average and wet). The second step used the monthly rankings. The amount of rain was calculated based on historical series according to the monthly ranking and the annual statement was noted. The method was applied to rainfall data recorded at five rainfall stations in semi-arid region of Central Tunisia. The generated series was then validated..

2.0 Materials and Methods

2.1 Study Area

The study area of this research is Makurdi, Makurdi is the capital city of Benue state of Nigeria. The city is located in central Nigeria along the Benue River , it is located on (latitude 7.7^{0}N, longitude 8.5^0). The data used in this research work were obtained from the archive of Nigerian Meteorological Agency, Maitama, Abuja. It is the daily rainfall record of Markudi, Benue state for the period of 11 years (2005 to 2015).

2.2 The Model Equations

Following [9], we consider a stochastic process that is discrete state and continuous time. A continuous time stochastic process $\{X(t)\}$ is an infinite family of random variables indexed by the continuous real variable that is for any fixed t, X(t) is a random variable and the collection of all these (for all t) is the stochastic process we ordinarily think of t as time, so we may expect $X(t_1)$, the random variable at time t_1, to be dependent on $X(t_o)$, where $t_0 < t_1$, but not upon $X(t_2)$, where $t_2 > t_1$. The value of $X(t_1)$ is the state of the process at time t_1.

let $p_{ij}(t)$, denotes transition probability function which is discrete state and continuous time , where

$$p_{ij}(t) = pr\{X(t) = j / X(0) = i\} \tag{1}$$

Now, a fair amount can be known about these $p_{ij}(t)$ functions just as a consequence of the fact that they are, for all t, probabilities. For example, they are non-negative bounded functions because a probability must lie between 0 and 1. The values of the functions at $t = 0$ can be deduced because $p_{ij}(0) = pr\{X(0) = j / X(0) = i\}, \quad clearly \quad for \quad i \neq j$

$$p_{ij}(0) = 0 \quad and \quad for \quad i = j \quad p_{jj}(0) = 1$$

If we fix i and vary j over all states, the sum of the $p_{ij}(t)$ must equal 1 (for all t)

Now, under the assumption that the $p_{ij}(t)$ are continuous functions then we may express p_{ij} for small δt by the use of Maclaurin's series, thus

$$p_{ij}(\delta t) = p_{ij}(0) + p'_{ij}(0)\delta t + o(\delta t^2) \tag{2}$$

Where $o(\delta t^2)$ represent all terms of the order of the $(\delta t)^2$ or higher. If we consider this expression for $i \neq j$, and let $q_{ij} = p'_{ij}(0)$ we obtain

$$p_{ij}(\delta t) = q_{ij}\delta t + o(\delta t^2) \tag{3}$$

This is a linear approximation to $p_{ij}(t)$ which is good approximation as long as δt is small. q_{ij} is called the transition rate from $i \quad to \quad j$

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For $i = j$, the Maclaurin's series expansion yields

$$p_{jj}(\delta t) = 1 + p'_{jj}(0)\delta t + o(\delta t^2)$$

let $q_{jj} = p'_{jj}(0)$ (4)

We get the linear approximation

$$p_{jj}(\delta t) = 1 + q_{jj}\delta t + o(\delta t^2)$$

If we now, consider the forward Chapman-Kolmogorov equation (5)
Which is a equation for studying stationary Markov processes thus:

$$p_{ij}(t+\delta t) = \sum_k p_{ik}(t) p_{kj}(\delta t)$$

for small δt, and substitute our linear approximation, we get

$$p_{ij}(t+\delta t) = p_{ij}(t)[1 + q_{jj}\delta t + o(\delta t^2)] + \sum_{k \neq j} p_{ik}(t)\left[q_{kj}\delta t + o(\delta t^2)\right]$$

$$\frac{p_{ij}(t+\delta t) - p_{ij}(t)}{\delta t} = p_{ij}(t) q_{jj} + \frac{p_{ij}(t) o(\delta t^2)}{\delta t} + \sum_{k \neq j}\left[p_{ik}(t) q_{kj} + \frac{p_{ik}(t) o(\delta t^2)}{\delta t}\right]$$

$$= \sum_k p_{ik}(t) q_{kj} + \sum_k \frac{p_{ik}(t) o(\delta t^2)}{\delta t}$$

Taking limit as $\delta t \to 0$

$$\frac{dp_{ij}(t)}{dt} = \sum_k p_{ik}(t) q_{kj} \quad (6)$$

In matrix form we have $\frac{dp(t)}{dt} = p(t)Q$ (7)

Where $\frac{dp(t)}{dt}$ is the matrix whose $(i,j)^{th}$ element is $\frac{dp_{ij}(t)}{dt}$,

$p(t)$ is the matrix whose $(i,j)^{th}$ element is $p_{ij}(t)$, and Q is the matrix whose $(i,j)^{th}$ element is q_{ij}

The elements of Q may be further related by extending the properties of P(t)
In particular, since for each i

$$\sum_j p_{ij}(t) = 1$$

$$\frac{d}{dt}\left[\sum_j p_{ij}(t)\right]_{t=0} = \left[\frac{d}{dt}(1)\right]_{t=0}$$

$$\sum_j \frac{d}{dt} p_{ij}(t)\Big|_{t=0} = 0$$

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$$\sum_j q_{ij} = 0$$

In words, each row of Q must sum to zero. Since off-diagonal element in non negative, the diagonal element q_{ii}, must be equal in magnitude and opposite in sign to the sum of others in the same rows. That is

$$q_{ii} = -\sum_{i \neq j} q_{ij} \quad (8)$$

To obtain the solution to Eq.7, the initial condition $P_i(0)$, $i = 1,2,3,4;$ must be specified

Taking the Laplace transform of Eq.7, we obtained

$$P(s) = P(0)(SI - Q)^{-1} \quad (9)$$

Thus p(t) is obtained as the inverse transform of P(s) [10]

2.3 Application

This model consider the weekly rainfall amount during the raining season in Makurdi. In this model, we wish to analyse and predict weekly rainfall pattern in continuous time scale. Thus weekly rainfall amount is modelled in continuous time and the basic unit of time in this model is a week.

Now, suppose that the amount of weekly rainfall in Makurdi in a week during the raining season is considered as a random variable X, the collection of these random variables over the weeks constitutes a stochastic process

$$X_n \quad , \quad n = 0,1,2,3..................$$

It is assumed that this stochastic process satisfies Markov properties reported in [11]. Let the weekly rainfall be modelled by four states, Markov model.

State1: No rainfall
State2: Low rainfall
State3: Moderate rainfall
State4: High rainfall

2.4 Transition Probability Matrix

$$p = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & 0 \\ 0 & P_{42} & P_{43} & 0 \end{bmatrix} \quad (10)$$

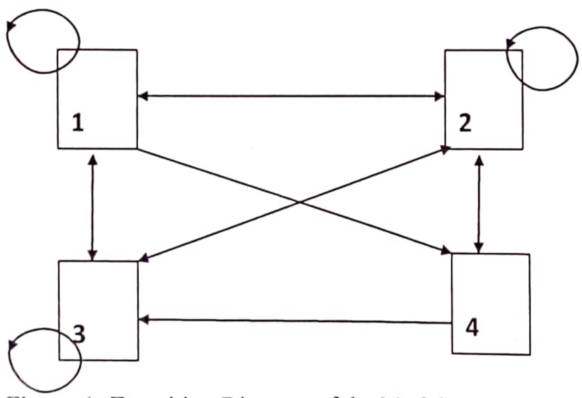

Figure 1: Transition Diagram of the Model

Table1: A summary of Weekly Rainfall Amount in Makurdi between 2005-2015 and states distribution

Weekly rainfall in mm	Frequency	State
0	68	No Rainfall
Rainfall amount <61	254	Low Rainfall
61-120	51	Moderate Rainfall
Rainfall amount >120	9	High Rainfall

From the Table 1, we obtained the transition count matrix below

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$$M=\begin{bmatrix}29 & 25 & 2 & 2\\ 26 & 161 & 47 & 7\\ 4 & 4 & 10 & 0\\ 0 & 6 & 2 & 0\end{bmatrix}$$

Normalizing this matrix using Eq.8,

$$A=\begin{bmatrix}-29 & 25 & 2 & 2\\ 26 & -80 & 47 & 7\\ 4 & 4 & -8 & 0\\ 0 & 6 & 2 & -8\end{bmatrix} \quad (11)$$

Thus, the matrix A can be interpreted as the reciprocal of the *mean times* of the negative exponentially distributed random variable having the cumulative distribution (12)

$1-\ell^{-\lambda t}$ and mean value $\frac{1}{\lambda}$.

The above matrix indicates that if the rainfall is no rainfall state, the time it takes to make a transition to low rainfall state is exponentially distributed with mean 25 weeks. That is, if the rainfall is in No rainfall state, it has a probability $\frac{1}{25}\delta t$ of making transition to low rainfall state, a probability $\frac{1}{2}\delta t$ of making transition to moderate rainfall state and a probability $\frac{1}{2}\delta t$ of making transition to high rainfall state in the time interval $(t,\ \ t+\delta t)$. Also if the rainfall is in low rainfall state it has a probability $\frac{1}{26}\delta t$ of making transition to no rainfall state, a probability $\frac{1}{47}\delta t$ of making transition to moderate rainfall state and a probability $\frac{1}{7}\delta t$ of making transition to high rainfall state in the time interval $(t,\ \ t+\delta t)$. Similarly, if the rainfall is in moderate rainfall state it has a probability $\frac{1}{4}\delta t$ of making transition to both no rainfall state and low rainfall state and a probability of 0 to high rainfall state, in the time interval $(t,\ \ t+\delta t)$. Similar interpretation is given to being in high rainfall state making transition to no rainfall state, low rainfall, and moderate rainfall with probabilities, 0, $\frac{1}{6}\delta t$, and $\frac{1}{2}\delta t$ respectively in the time interval $(t,\ \ t+\delta t)$.

The Matrix A can be expressed as the expected value of the exponential distribution thus

$$Q=\begin{bmatrix}-1.04 & 0.04 & 0.5 & 0.5\\ 0.04 & -0.2 & 0.02 & 0.14\\ 0.25 & 0.25 & -0.5 & 0\\ 0 & 0.17 & 0.5 & -0.67\end{bmatrix} \quad (13)$$

Now, to obtained solution for Eq.9, we have that

$$IS-Q=\begin{bmatrix}s & 0 & 0 & 0\\ 0 & s & 0 & 0\\ 0 & 0 & s & 0\\ 0 & 0 & 0 & s\end{bmatrix}-\begin{bmatrix}-2.50 & 0.5 & 1 & 1\\ 0 & -1.17 & 0.17 & 1\\ 0.33 & 0.25 & -0.58 & 0\\ 1 & 0.5 & 0 & -1.5\end{bmatrix} \quad (14)$$

$$=\begin{bmatrix}s+1.04 & -0.04 & -0.5 & -0.5\\ 0.04 & s+0.2 & -0.02 & -0.14\\ -0.25 & -0.25 & s+0.5 & 0\\ 0 & -0.17 & -0.5 & s+0.67\end{bmatrix} \quad (15)$$

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We suppose that the initial state of the process is $P(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$

Thus

$$P(s) = P(0)(SI - Q)^{-1} \tag{16}$$

$$P(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s+1.04 & -0.04 & -0.5 & 0.5 \\ -0.04 & s+0.2 & -0.02 & -0.14 \\ -0.25 & -0.25 & s+0.5 & 0 \\ 0 & -0.17 & 0.5 & s+0.67 \end{bmatrix}^{-1}$$

Solving Eq.16, using maple software we have the following equations to compute

$$P_{12}(t) = 0.488 - 0.679\ell^{-0.401t} + 1.06\times10^{-25}(1.80\times10^{24}\cos(0.153t) - 2.48\times10^{24}\sin(0.153t))\ell^{-1.004t}$$

$$P_{13}(t) = 0.266 + 0.545\ell^{-0.401t} - 2.129\times10^{-25}\ell^{-1004t}(3.806\times10^{24}\cos(0.153t) + 2.922\times10^{24}\sin(0.153t))$$

$$P_{14}(t) = 0.164 - 0.036\ell^{-0.401t} + 2.662\times10^{-24}\ell^{-1.006t}(-4.79\times10^{22}\cos(0.153t) + 8.77\times10^{23}\sin(0.153t))$$

$P_{12}(t)$ is the conditional probability that the weekly rainfall will be in low rainfall state at time t given that the weekly rainfall was in no rainfall state at time zero

$P_{13}(t)$ is the conditional probability that the weekly rainfall will be in moderate rainfall state at time t given that the weekly rainfall was in no rainfall state at time zero

$P_{14}(t)$ is the conditional probability that the weekly rainfall will be in high rainfall state at time t given that the weekly rainfall was in no rainfall state at time zero

$P_{11}(t)$ is the conditional probability that the weekly rainfall will be in no rainfall state at time t given that the weekly rainfall was in no rainfall state at time zero. This the compliment of $P_{12}(t)$, $P_{13}(t)$ and $P_{14}(t)$ this can be calculated at any point in time using $\sum_j p_{ij} = 1,\ ij = 1,2,3,4$, to obtain $P_{11}(t)$,one may use $P_{11}(t) = 1 - (p_{12}(t) + P_{13}(t) + P_{14}(t))$

The values of the equations evaluated for t = 0 to 25 are tabulated in Table2 and are illustrated graphically in Fig 2

Table 2: The Transition Probabilities

t	$P_{12}(t)$	$P_{13}(t)$	$P_{14}(t)$
0	0.000000	0.000000	0.000000
1	0.087732	0.302213	0.223710
2	0.197351	0.380988	0.225555
3	0.286862	0.379895	0.198041
4	0.351632	0.356771	0.178654
5	0.396334	0.332300	0.168832
6	0.426582	0.312415	0.164674
7	0.446882	0.297773	0.163215
8	0.460466	0.287480	0.162874
9	0.469547	0.280411	0.162928
10	0.475619	0.275616	0.163083
11	0.479679	0.272385	0.163235
12	0.482395	0.270215	0.163356
13	0.484212	0.268761	0.163444
14	0.485428	0.267786	0.163506
15	0.486786	0.267134	0.163548
16	0.487395	0.266697	0.163577
17	0.487559	0.266405	0.163596
18	0.487668	0.266209	0.163609
19	0.487741	0.266078	0.163618
20	0.487789	0.265991	0.163624
21	0.487823	0.265932	0.163627
22	0.487845	0.265893	0.163630
23	0.487859	0.265866	0.163632
24	0.487845	0.265849	0.163633
25	0.487859	0.265837	0.163634

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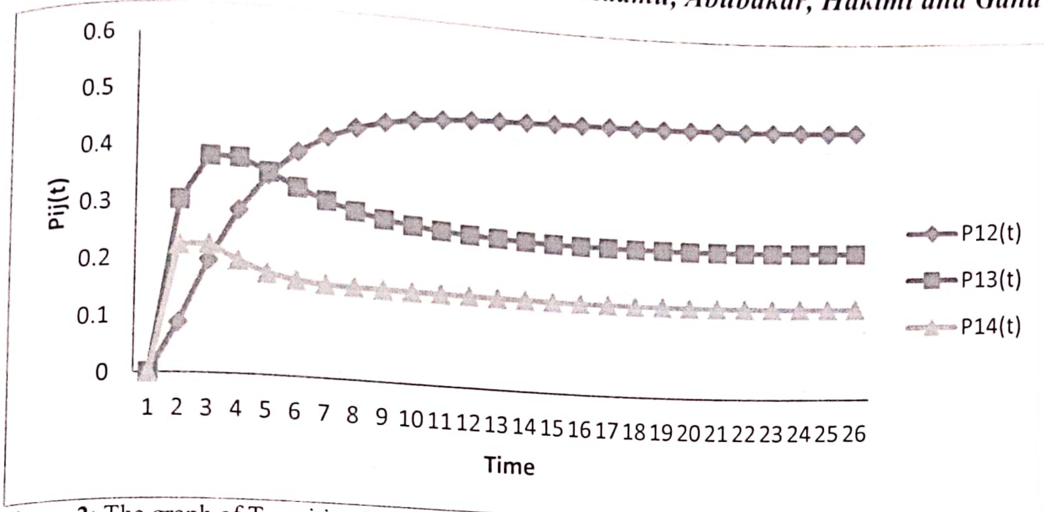

Figure 2: The graph of Transition Probabilities

We observe that the limit of each function as t goes to infinity or fairly large is immediately apparent, both in the functions themselves and in the graphs of the functions. The convergence is smooth and monotonic, as opposed to discontinuous, oscillating or both.

3.0 Discussion of Results

The result is presented in Table 2 and illustrated with graph in figure 2. The model enables us to determine the values of $P_{12}(t)$, $P_{13}(t)$, $P_{14}(t)$ respectively at any time t. It is observed from the table that P_{12} rose steadily and stabillzes to 0.49 as t tends to infinity, also P_{13} rose , later dropped and stabilized to 0.27 as t tends to infinity, similarly P_{14} rose and drop sharply and later stabilized to 0.16 as t tends to infinity. These are the equilibrium transition probabilities. For instance, if the process is No rainfall state in a given week, it would take at most 59% ,27% and16 % of the time to make transition to Low rainfall state, moderate rainfall, and high rainfall state respectively in the long run. Thus given the rainfall in a week it is possible to determine quantitatively the probability of finding rainfall in other states in the following week(s) and in the long run. From Eq.10, it can be observed that P_{41}, P_{34}, $P_{44}=0$, these mean that, there are not transition between these states. For example for the $P_{44}=0$, it means that, it is not possible for a week of high rainfall in Makurdi to be followed by another week of high rainfall , and also for the P_{41}, it is not possible to for a week of high rainfall to be followed by a week of no rainfall. Similarly, for P_{34}, a week of moderate rainfall cannot precede a week high Rainfall. The results from this model is an important information for the residents of Makurdi effective planning and production.

4.0 Conclusion

A stochastic model to analyze and predict weekly rainfall pattern of Makurdi in continuous time has been presented. The stochastic model was formulated based on the principle of Markov. It was observed that if it is norainfall state in a given week, it would take at most 49%,27% and 16% of the time to make a transition to low-rainfall , moderate rainfall, and high rainfall in the far future.. The model also reveals that; a week of high rainfall cannot be followed by another week of high rainfall, a week of high rainfall cannot be followed by a week of no rainfall and a week of moderate rainfall cannot precede a week high. The results from the model are an important information that could assists the residents to better understand the dynamics of weekly rainfall in Markurdi which may be helpful for effective planning and viable production.

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