Numerical Assessment of Some Almost Runge-Kutta and Runge-Kutta Methods for First-Order Differential Equation

Khadeejah James Audu[1], Muideen Taiwo Kharashi[2], Yusuph Amuda Yahaya[3], James Nkereuwem Essien[4], Abraham Ajeolu Oluwasegun[5]

[1,2,4,5]Department of Mathematics, Federal University of Technology, Minna, Niger State, Nigeria.
[3] Department of Mathematics, North-Eastern University, Gombe, Nigeria
Corresponding email: k.james@futminna.edu.ng

Abstract

Numerical methods play a critical role in solving first-order Ordinary Differential Equations (ODEs), with their efficiency and accuracy being key considerations. This study conducts a detailed comparative analysis of four numerical schemes: the Almost Runge-Kutta fourth-order scheme (ARK4), the Almost Runge-Kutta third-order fourth-stage scheme (ARK34), the classical Runge-Kutta fourth-order scheme (RK4), and the Runge-Kutta fourth-order fifth-stage scheme (RK45). The methods are evaluated based on their computational accuracy, error behavior, and efficiency. Numerical experiments reveal that all methods deliver highly accurate solutions, with ARK4 emerging as the most effective due to its lower computational complexity. ARK4 demonstrates superior performance in achieving minimal absolute error with reduced computational effort, making it a suitable choice for solving first-order ODEs. This study highlights ARK4 as a viable alternative to conventional Runge-Kutta methods for practical applications.

Keywords: Numerical analysis, runge-kutta method, first-order ODEs, almost runge-kutta method, computational efficiency

1. Introduction

First-order differential equations are essential in describing a multitude of dynamic systems across various scientific and engineering disciplines. In physics, they model essential phenomena such as radioactive decay, Newton's law of cooling, and electrical circuits governed by Ohm's and Kirchhoff's laws. Engineering applications often rely on first-order differential equations to design and analyze systems, including control systems, thermal dynamics in buildings and fluid flow in pipelines. In finance, these equations are critical for modeling the time evolution of economic variables, such as interest rates and stock prices, which are pivotal for investment strategies and risk management. (Arora and Joshi, 2020; Audu *et al.* 2023; Benjamin and Okisamen, 2020; Audu *et al*. 2024).

The precise and efficient resolution of these equations is vital for progress in these areas. Traditional analytical approaches frequently prove inadequate because of the complexity and nonlinearity present in many real-world challenges, requiring the application of numerical methods. Among these, the Runge-Kutta methods are notable for their balance of simplicity, effectiveness, and precision. The fourth-order, four-stage Runge-Kutta method (RK4) and fourth-order, five-stage Runge-Kutta method (RK45) are particularly renowned for their ability to provide

high-precision solutions with relatively modest computational effort (Esekhaigbe and Aitusi, 2022; Workineh *et al.*, 2024; Paudel and Bhatta, 2023; Hetmaniok and Pleszczyński, 2022; Sundaram, 2022). Researchers in their works (Shior *et al.*, 2024; Kalogiratou *et al.*, 2020) provided essential insights into their accuracy, stability, and computational efficiency. However, ongoing research aims to enhance these methods further, leading to the development of variants such as the Almost Runge-Kutta (ARK) method. ARK method was developed by Rattenbury (2005) and other researchers (Abraham, 2010; Alimi, 2014; Ishaq, 2017; Adeboye *et al*., 2013; Ndanusa and Audu, 2016a) have improved on the method in establishing the third-order, four- stage (ARK34) method and fourth-order Almost Runge-Kutta method (ARK4) method. These ARK methods seek to improve upon the classical approaches by offering better stability and accuracy under certain conditions, making them a compelling subject for comparative analysis. Research on numerical methods for solving first-order differential equations lacks comprehensive comparative studies between Almost Runge-Kutta methods (ARK34 and ARK4) and Runge-Kutta methods (RK4 and RK45). A detailed investigation into their accuracy, stability and efficiency could offer valuable insights into their respective strengths and applications.
This paper presents a direct comparison of the selected methods by highlighting their errors and proximity of the approximate solution to the exact solution when used to solve equation of the form

$$\frac{dt}{du} = t'(u) = f(t,u) \tag{1}$$

The study fills a key gap in numerical analysis by comparing the selected methods. Its findings offer practical guidance for selecting optimal numerical techniques, improving simulations, and optimizing computational resources for solving complex first-order differential equations.

2. Materials and Methods

In this section, we analyzed the four methods chosen for the numerical assessment on first-order differential equations.

2.1 The Runge-Kutta Fourth-Order Method

Given a first-order differential equation in the form of (1), the RK4 method for solving ODE can be expressed in the following equation

$$t_{b+1} = t_b + \frac{s}{6}\left(\underline{k}_1 + 2\underline{k}_2 + 2\underline{k}_3 + \underline{k}_4\right) \tag{2}$$

where the stages $\left(\underline{k}_1, \underline{k}_2, \underline{k}_3 \text{ and } \underline{k}_4\right)$ are represented in (3)

$$\begin{aligned} \underline{k}_1 &= f\left(t_b, u_b\right) \\ \underline{k}_2 &= f\left(t_b + \frac{s}{2}, u_b + \frac{\underline{k}_1}{2}\right) \\ \underline{k}_3 &= f\left(t_b + \frac{s}{2}, u_b + \frac{\underline{k}_2}{2}\right) \qquad b = 0, 1, 2, 3, 4, \cdots, M \\ \underline{k}_4 &= f\left(t_b + s, u_b + \underline{k}_3\right) \end{aligned} \tag{3}$$

with an initial estimates (u_0 and t_0), step interval (s) and a total number of steps (M **),** the popular RK4M can be employed to compute solutions of any linear or non-linear ODEs utilising the following steps:

Step 1: Express the function $f(t,u)$ in a way such that $f(t,u) \in [q, f]$, select the step interval $\left(s = \frac{f-q}{M}\right)$ and initialize $t = t_0, u = u_0$.

Step 2: Iterate through M steps for each step $b = 0, 1, 2, 3, 4, \cdots, M$ and repeatedly perform the tasks:

(i) Compute the intermediate values $\underline{k}_1, \underline{k}_2, \underline{k}_3$ *and* $\underline{k}_4$ as represented in equation (3).

(ii) Update the solution $t: t_{b+1} = t_b + \frac{s}{6}\left(\underline{k}_1 + 2\underline{k}_2 + 2\underline{k}_3 + \underline{k}_4\right)$.

(iii) Update the independent variable $u: u_{b+1} = u_0 + bS$.

Step 3: Return the final solution after M iterations or output approximate solutions t_b for u_b.

Step 4: Terminate the procedure if $t_b \geq f$ provided that $\left\|t_{b+1} - t_b\right\| < \varepsilon$.

2.2 The Runge-Kutta Five Stage Fourth-Order Method

The explicit Runge-Kutta method of fourth-order with five stages, denoted by $k_1, ..., k_5$ for a single step, enables the solution of the equation using the general Runge-Kutta principles stated in works of authors (Fawzi *et al*., 2024). The RK45 method considered by Benjamin and Okisamen (2020) for obtaining solutions regarding first-order ODE can be expressed in the relation

$$t_{b+1} = t_b + \frac{s}{6}\left(\underline{k}_1 + 4\underline{k}_3 + \underline{k}_5\right) \tag{4}$$

where the stages $\left(\underline{k}_1, \underline{k}_3 \; and \; \underline{k}_5\right)$ are denoted in (5)

$$
\begin{aligned}
\underline{k}_1 &= f\left(t_b\right) \\
\underline{k}_2 &= f\left(t_b + \frac{s}{4}\underline{k}_1\right) \\
\underline{k}_3 &= f\left(t_b + s\left[\frac{1}{4}\underline{k}_1 + \frac{1}{4}\underline{k}_2\right]\right) \qquad\qquad b = 0, 1, 2, 3, 4, \cdots, M \\
\underline{k}_4 &= f\left(t_b + s\left[\frac{17}{8}\underline{k}_1 + \frac{4}{8}\underline{k}_2 - \frac{15}{8}\underline{k}_3\right]\right) \\
\underline{k}_5 &= f\left(t_b + s\left[-\frac{7}{4}\underline{k}_1 + \frac{9}{4}\underline{k}_2 + \frac{3}{4}\underline{k}_3 - \frac{1}{4}\underline{k}_4\right]\right)
\end{aligned}
\tag{5}
$$

Focusing on the RK45 formula in equations 4 and 5, if given an initial guess (u_0 and t_0) with step interval (s) for a total number of steps (M **),** then the numerical RK45 method can be applied in solving any linear or non-linear ODEs using the following procedure as follows:

Step 1: Express the function $f\left(t,u\right)$ in a way such that $f\left(t,u\right) \in \left[q, f\right]$, select the step interval $\left(s = \frac{f-q}{M}\right)$ and initialize $t = t_0, u = u_0$.

Step 2: Iterate through M steps for each step $b = 0, 1, 2, 3, 4, \cdots, M$ and repeat the following:

(i) Compute the intermediate values $\underline{k}_1, \underline{k}_2, \underline{k}_3, \underline{k}_4 \; and \; \underline{k}_5$ as stated in equation (5).

(ii) Update the solution t: $t_{b+1} = t_b + \frac{s}{6}\left(\underline{k}_1 + 4\underline{k}_3 + \underline{k}_5\right)$.

(iii) Update the independent variable $u : u_{b+1} = u_0 + bS$.

Step 3: Return the final solution after M iterations, that is return t_b

Step 4: Terminate the procedure if $t_b \geq f$ provided that $\left\| t_{b+1} - t_b \right\| < \varepsilon$.

2.3 The Almost Runge-Kutta Method

2.3.1 The Third-Order, Four-Stage ARK Method

The general third-order four stages scheme for ARK method is governed by (6)

$$\left[\begin{array}{c|c} M & \underline{U} \\ \hline N & P \end{array}\right] = \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 1 & \underline{c}_1 & \frac{1}{2}\underline{c}_1^2 \\ m_{21} & 0 & 0 & 0 & 1 & \underline{c}_2 - m_{21} & \frac{1}{2}\underline{c}_2^2 - m_{21}\underline{c}_1 \\ m_{31} & m_{32} & 0 & 0 & 1 & \underline{c}_3 - m_{31} - m_{32} & \frac{1}{2}\underline{c}_3^2 - m_{31}\underline{c}_1 - m_{32}\underline{c}_2 \\ n_1 & n_2 & n_3 & 0 & 1 & n_0 & 0 \\ \hline n_1 & n_2 & n_3 & 0 & 1 & n_0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & 0 & \lambda_0 & 0 \end{array}\right] \tag{6}$$

The four matrices M, N, U and P are utilized to construct the ARK methods where $M = [m_{ij}]$ s,s, $N = [n_{ij}]$ r,s, $\underline{U} = [\underline{u}_{ij}]$ s,r and $P = [p_{ij}]$ r,r . These methods transfer values in between steps for different stages. *r* represents the transferred quantity (valued) and *s* denotes the desired stage. The vector of abscissa is denoted as

$$\underline{c} = \begin{bmatrix} \underline{c}_1 \\ \underline{c}_2 \\ \underline{c}_3 \\ 1 \end{bmatrix}, \quad n = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ 0 \end{bmatrix}, \quad \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix}$$

Many authors have developed various ARK34M suitable for solving initial value problems related to first-order (Ndanusa and Audu, 2016; Butcher, 1997; Rattenbury, 2005; Ochoche, 2011). However, this study will consider the variant formulated by Ndanusa and Audu (2016b). A representation of the selected ARK34 method with $\underline{c}^T = [0.25, 0.5, 1.0, 1.0]$ is

$$\left[\begin{array}{c|c} M & \underline{U} \\ \hline N & P \end{array}\right] = \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 1 & 0.50 & 0.03125 \\ 0.89 & 0 & 0 & 0 & 1 & -0.39 & -0.0972 \\ -0.67 & 0.50 & 0 & 0 & 1 & 1.167 & 0.4167 \\ 0 & 0.67 & 0.17 & 0 & 1 & 0.17 & 0 \\ \hline 0 & 0.67 & 0.17 & 0 & 1 & 0.17 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -0.89 & 0 & -0.67 & 2 & 0 & -0.444 & 0 \end{array}\right] \tag{7}$$

2.3.2 The Fourth-Order, Four-Stage ARK Method

A general fourth-order, four-stage concerning ARK method is equally governed by the representation in equation (5). This study is interested in employing the ARK4 method constructed by Ndanusa and Audu (2016a) for the solutions of first-order ODE. The selected method with

$\underline{c}^T = [0.4412, 0.5, 1.0, 1.0]$, $\lambda_4 = 4$ is presented as

$$\left[\begin{array}{c|c} M & \underline{U} \\ \hline N & P \end{array}\right] = \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 1 & 0.4412 & 0.0097 \\ 0.1510 & 0 & 0 & 0 & 1 & 0.3495 & 0.0586 \\ -0.6021 & 2 & 0 & 0 & 1 & -0.3979 & -0.2344 \\ 0 & 0.6667 & 0.1667 & 0 & 1 & 0.1667 & 0 \\ \hline 0 & 0.6667 & 0.1667 & 0 & 1 & 0.1667 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -12.844 & 10.667 & -2.6667 & 4 & 0 & -0.844 & 0 \end{array}\right] \tag{8}$$

3. Numerical Investigations

This section aims to compare the performance of the four numerical methods discussed in the previous section for solving ordinary differential equations. The computations were conducted using Maple 2021 software and Python 2021 software to obtain the desired results for the problems.

Problem 1: We solve the ODE using the 4 methods.

$u' = t + u \qquad s = 0.1$, within the interval $0 \le t \le 1$

Exact solution can be obtained from the relation $u_E(t) = 2e^t - t - 1$

Problem 2: The ODE is solved using ARK34, ARK4, RK4 and RK45.

$$u' = \frac{u}{4}\left(1 - \frac{u}{20}\right), \quad s = 0.1 \qquad 0 \le t \le 1,$$

The exact solution is $u_E(t) = \dfrac{20}{1 + 19e^{\frac{-t}{4}}}$

Problem 3: We resolve the following ODE by methods of ARK34, ARK4, RK4 and RK45.

$$u' = \frac{u+t}{u-t}, \quad u_0 = 1, \quad t_0 = 0, \quad s = 0.1, \quad 0 \le t \le 1$$

Exact solution = $u_E(t) = t + \sqrt{1 + 2t^2}$

The computational and graphical results of problems 1 to 3 are tabulated in the following Tables and Figures.

Table 3.1: Numerical Output of RK4 and RK45 for Problem 1

t	Exact	RK4	RK4 Error	RK45	RK45 Error
0.0	1.0000000000	1.0000000000	0.0000000000	1.0000000000	0.0000000000
0.1	1.1103418362	1.1103416667	0.0000001695	1.1103417643	0.0000000718
0.2	1.2428055163	1.2428051417	0.0000003746	1.2428053576	0.0000001588
0.3	1.3997176152	1.3997169941	0.0000006210	1.3997173520	0.0000002632
0.4	1.5836493953	1.5836484802	0.0000009151	1.5836490074	0.0000003878
0.5	1.7974425414	1.7974412772	0.0000012642	1.7974420056	0.0000005358
0.6	2.0442376008	2.0442359242	0.0000016766	2.0442368902	0.0000007105
0.7	2.3275054149	2.3275032532	0.0000021617	2.3275044988	0.0000009162
0.8	2.6510818570	2.6510791266	0.0000027304	2.6510806998	0.0000011572
0.9	3.0192062223	3.0192028276	0.0000033948	3.0192047836	0.0000014387
1.0	3.4365636569	3.4365594883	0.0000041686	3.4365618902	0.0000017667

Table 3.2: Numerical Output of ARK4 and ARK34 for Problem 1

t	Exact	ARK4	ARK4 Error	ARK34	ARK34 Error
0.0	1.0000000000	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00
0.1	1.1103418362	1.1103416667	7.1899999954E-08	1.1103384722	3.3639999999E-06
0.2	1.2428055163	1.2428051417	2.6670000008E-07	1.2427988608	6.6555000000E-06
0.3	1.3997176152	1.3997169941	5.0180000000E-07	1.3997069895	1.0625700000E-05
0.4	1.5836493953	1.5836484801	7.8329999997E-07	1.5836340388	1.5356500000E-05
0.5	1.7974425414	1.7974412771	1.1185000000E-06	1.7974215765	2.0964900000E-05
0.6	2.0442376008	2.0442359241	1.5155999997E-06	2.0442100176	2.7583200000E-05
0.7	2.3275054149	2.3275032531	1.9838000003E-06	2.3274700532	3.5361700000E-05

0.8	2.6510818570	2.6510791265	2.5338000000E-06	2.6510373858	4.4471200000E-05
0.9	3.0192062223	3.0192028275	3.1774000000E-06	3.0191511166	5.5105700000E-05
1.0	3.4365636569	3.4365594882	3.9284000004E-06	3.4364961717	6.7485200000E-05

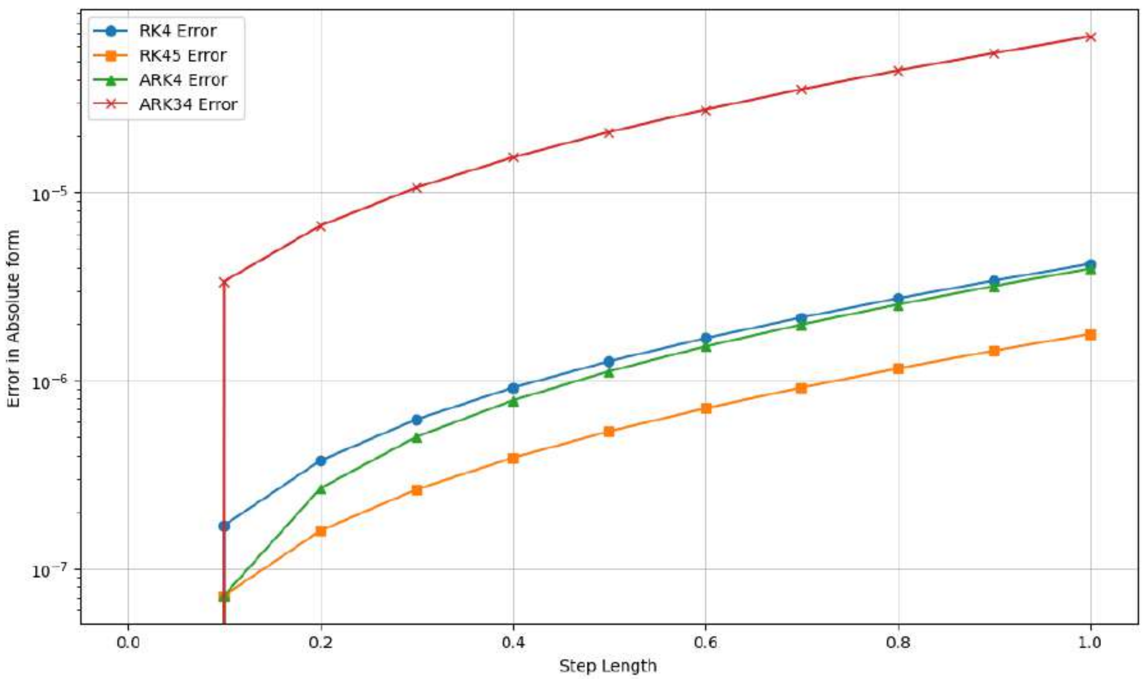

Figure 1. Error graph for problem 1

The provided graph illustrates the calculated errors for ARK 34, ARK 4, RK 4, and RK45 in relation to problem 1.

Table 3.3: Numerical Output of RK4 and RK45 for Problem 2

t	Exact	RK4	RK4 Error	RK45	RK45 Error
0.0	1.0000000000	1.0000000000	0.0000000000	1.0000000000	0.0000000000
0.1	1.0240189624	1.0240189623	0.0000000001	1.0240189623	0.0000000001
0.2	1.0485829964	1.0485829963	0.0000000001	1.0485829963	0.0000000001
0.3	1.0737029288	1.0737029287	0.0000000001	1.0737029287	0.0000000001
0.4	1.0993897267	1.0993897265	0.0000000002	1.0993897265	0.0000000002

0.5	1.1256544953	1.1256544950	0.0000000003	1.1256544950	0.0000000003
0.6	1.1525084759	1.1525084755	0.0000000004	1.1525084755	0.0000000004
0.7	1.1799630432	1.1799630427	0.0000000005	1.1799630428	0.0000000004
0.8	1.2080297028	1.2080297022	0.0000000006	1.2080297023	0.0000000005
0.9	1.2367200876	1.2367200870	0.0000000006	1.2367200871	0.0000000005
1.0	1.2660459552	1.2660459545	0.0000000007	1.2660459546	0.0000000006

Table 3.4: Numerical Output of ARK4 and ARK34 for Problem 2

t	Exact	ARK4	ARK4 Error	ARK34	ARK34 Error
0.0	1.0000000000	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00
0.1	1.0240189623	1.0240189623	0.0000000000E+00	1.0240189584	3.9000001010E-09
0.2	1.0485829964	1.0485829963	1.0000000830E-10	1.0485829943	2.0999999520E-09
0.3	1.0737029289	1.0737029287	2.0000001650E-10	1.0737029286	3.0000002480E-10
0.4	1.0993897267	1.0993897266	1.0000000830E-10	1.0993897285	1.8000001490E-09
0.5	1.1256544953	1.1256544951	2.0000001650E-10	1.1256544992	3.8999998790E-09
0.6	1.1525084759	1.1525084756	3.0000002480E-10	1.1525084820	6.1000000610E-09
0.7	1.1799630432	1.1799630429	2.9999980280E-10	1.1799630516	8.4000000290E-09
0.8	1.2080297028	1.2080297024	4.0000003310E-10	1.2080297136	1.0800000010E-08
0.9	1.2367200876	1.2367200871	5.0000004140E-10	1.2367201010	1.3400000000E-08
1.0	1.2660459552	1.2660459547	5.0000004140E-10	1.2660459713	1.6100000000E-08

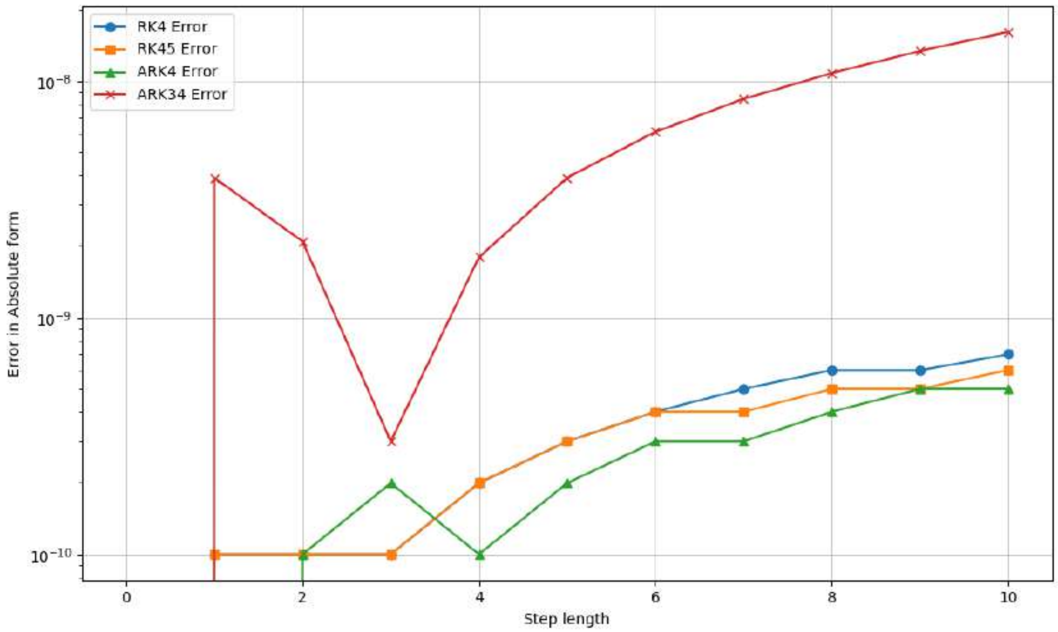

Figure 1. Error graph for problem 2

The provided graph displays the computed errors for ARK 34, ARK 4, RK 4, and RK45 in relation to Problem 2.

Table 3.5: Numerical Output of RK4 and RK45 for Problem 3

t	Exact	RK4	RK4 Error	RK45	RK45 Error
0.0	1.0000000000	1.0000000000	0.0000000000	1.0000000000	0.0000000000
0.1	1.1099504938	1.1099505755	0.0000000817	1.1066539351	0.0032965588
0.2	1.2392304845	1.2392307736	0.0000002891	1.2329046017	0.0063258829
0.3	1.3862780491	1.3862785808	0.0000005317	1.3774301541	0.0088478950
0.4	1.5489125293	1.5489132614	0.0000007321	1.5381571140	0.0107554153
0.5	1.5489125293	1.7247457310	0.0000008596	1.7126819218	0.0120629496
0.6	1.9114877049	1.9114886243	0.0000009194	1.8986292382	0.0128584666
0.7	2.1071247279	2.1071256588	0.0000009309	2.0938695410	0.0132551870
0.8	2.3099668871	2.3099677997	0.0000009126	2.2966059595	0.0133609276
0.9	2.5186414056	2.5186422842	0.0000008786	2.5053765792	0.0132648264
1.0	2.7320508076	2.7320516450	0.0000008374	2.7190159000	0.0130349076

Table 3.6: Numerical Output of ARK4 and ARK34 for Problem 3

t	Exact	ARK4	ARK4 Error	ARK34	ARK34 Error
0.0	1.0000000000	1.0000000000	0.0000000000E+00	1.0000000000	0.0000000000E+00
0.1	1.1099504938	1.1099505178	2.3999999987E-08	1.1099504239	6.9900000010E-08
0.2	1.2392304845	1.2392307341	2.4960000000E-07	1.2392278329	2.6516000000E-06
0.3	1.3862780491	1.3862786072	5.5809999999E-07	1.3862720147	6.0343999999E-06
0.4	1.5489125293	1.5489133617	8.3240000004E-07	1.5489029640	9.5652999998E-06
0.5	1.7247448714	1.7247458896	1.0182000001E-06	1.7247322162	1.2655200000E-05
0.6	1.9114877049	1.9114888194	1.1144999998E-06	1.9114727000	1.5004900000E-05
0.7	2.1071247279	2.1071258718	1.1439000001E-06	2.1071081495	1.6578400000E-05
0.8	2.3099668871	2.3099680179	1.1308000003E-06	2.3099494034	1.7483700000E-05
0.9	2.5186414056	2.5186424999	1.0942999999E-06	2.5186235316	1.7874000000E-05
1.0	2.7320508076	2.7320518540	1.0464000000E-06	2.7320329124	1.7895200000E-05

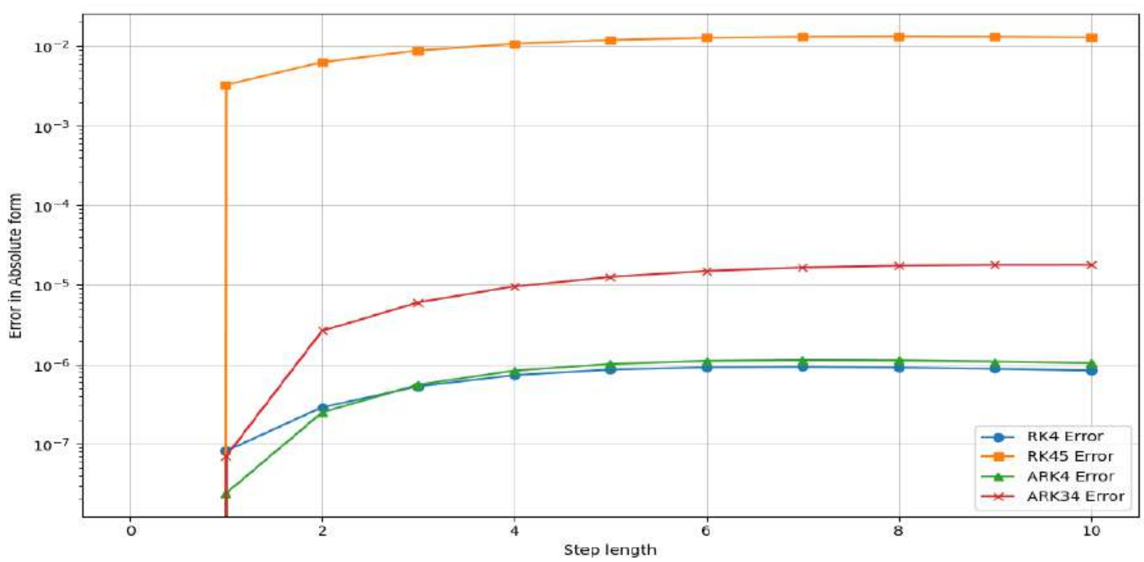

Figure 1. Error graph for problem 3

The provided graph presents the computed errors of ARK34, ARK4, RK4 and RK45 in relation to problem 3.

4. Discussion of Results

The results obtained are presented in Tables 1-6. The approximate numerical solutions were calculated using step size 0.1 and the results were compared with the exact solution. Further analysis on the error was carried out using graphical representation. The results provide insights into the efficiency of these numerical techniques.

Comparison of Numerical Approximations (Tables 1-6)

i. The tables indicate that using a consistent step size of 0.1, the RK4, RK45, ARK4, and ARK34 methods yield numerical solutions for all three problems that closely match the analytical results.
ii. The results from the RK4, RK45, ARK4, and ARK34 methods exhibit some variation, indicating that these approaches offer comparable levels of approximation to the exact solution.
iii. The findings highlight the efficiency of the four numerical methods in addressing the specified problems.

Error Comparison (Figures 1-3)

i. The graphical error representations yield a deeper insight into the effectiveness of each technique.
ii. It is evident that the ARK4 method consistently produces more accurate results and exhibits lower error values compared to the other methods.
iii. The error curves for the ARK4 method shown in Figures 1-3 demonstrate a trend toward zero, indicating convergence to the exact solution as long as the step size is kept constant.

Discoveries

i. The comparative assessment clearly demonstrates that the ARK4 method surpasses the other three approaches in solving Problems 1-3.
ii. The ARK4 technique stands out for its high accuracy and efficiency, consistently converging to the analytical solution with minimal error.

5. Conclusion

This study compared four numerical methods applied to solve first-order ordinary differential equations. Extensive numerical exploration yielded compelling results, demonstrating the exceptional accuracy of all four methodologies. An analysis of the result tables and graphical figures indicated that all methods converged. Error comparisons revealed that the ARK4 method provides more accurate results than the RK4, RK45, and ARK34 methods. The comparative assessment highlights the relative advantages and disadvantages of the RKMs and ARKMs concerning accuracy and computational efficiency. The findings of this research have practical implications for both researchers and practitioners in the field of numerical methods for differential equations. They enhance the understanding of the trade-offs involved in selecting an appropriate numerical method for specific problem types and guide the application of the ARK4 method in real-world scenarios. Future research would focus on conducting a comparative study of ARK4 with various numerical approaches for higher-order differential equations, such as second-order ODEs.

References

Aida, M. (2022). Fourth-Order Runge-Kutta Method for Solving Applications of System of First-Order Ordinary Differential Equations. *Enhanced Knowledge in Sciences and Technology*, *2*(1), 517-526. https://doi.org/10.30880/ekst.2022.02.01.056

Adeboye, K. R., Ochoche, A. & Olumide A. K. (2013). On Error Estimation in Almost Runge-Kutta (ARK) Methods. *Journal of Applied Mathematics & Bioinformatics, 3*(2)*,* 2013, 125-135.

Arora, G. & Joshi, V. (2020). Developments in Runge–Kutta Method to Solve Ordinary Differential Equations, *Recent Advances in Mathematics for Engineering*, 1-11.

Audu, K. J., Taiwo, A. R. & Soliu, A. A. (2023). Assessment of Numerical Performance of Some Runge-Kutta Methods and New Iteration Method on First Order Differential Problems. *Dutse Journal of Pure and Applied Sciences*, *9*(4a), 58-70. https://dx.doi.org/10.4314/dujopas.v9i4a.5

Audu, K. J., Udoh, V. J. & Garba, J. (2025). An Investigation Comparison of Higher Order Runge Kutta techniques for resolving first order differential equations," *Turkish Journal of Science and Technology*, *20*(2), 1-15. https://doi.org/10.55525/tjst.1433935

Abraham, O. & Adeboye, K. R. (2009). *On the derivation of third-order Almost Runge-Kutta (ARK) methods with four stages* ($s = 4, p = 3$). Paper presented at the *28th Annual Conference of Nigerian Mathematical Society,* University of Ilorin, Ilorin, Nigeria.

Abraham, O. (2010). *Development of some new classes of explicit Almost Runge-Kutta methods for non-stiff differential equations*. Unpublished doctoral dissertation. Federal University of Technology, Minna, Nigeria.

Alimi, O. K. (2014). *On the performance of Richardson extrapolation technique in estimating local truncation errors for explicit Almost Runge-Kutta methods.* Unpublished Master's thesis, Federal University of Technology, Minna, Nigeria.

Butcher, J. C. (1997). An introduction to Almost Runge-Kutta methods. *Applied Numerical Mathematics*, 24, 331-342. https://doi.org/10.1016/S0168-9274(97)00030-5

Benjamin, O. C. & Okisamen, A. B. (2020). On the Derivation and Implementation of a Fifth-Stage Fourth-Order Formula for Solving Initial Value Problems in Ordinary Differential Equations, *Journal of Mathematics*, *16*(4), 29-39. https://doi.org/10.9790/5728-1604042939

Esekhaigbe, A. C., & Aitusi, D. (2022). On the Consistency and Convergence Analysis of a Derived Explicit Fourth-Stage Fourth-Order Runge-Kutta Method. *Journal of Advance Research in Mathematics and Statistics, 9*(1), 29-35. https://doi.org/10.61841/9hxxvy46

Fawzi, F. A., Mechee, M. S. & AbdAllah, S. M. (2024). Development and Analysis of Embedded Explicit Runge-Kutta Methods for Directly Solving Special Fifth-Order Quasi-Linear Ordinary Differential Equations. *Mathematical Modelling of Engineering Problems*, *11*(8), 2136-2142. https://doi.org/10.18280/mmep.110815

Hetmaniok, E. & Pleszczyński, M. (2022). Comparison of the selected methods used for solving the ordinary differential equations and their systems, *Mathematics*, *10*(3), 06 **https://doi.org/10.3390/math10030306**

Ishaq, Z. A. (2017). Some Explicit Almost Runge – Kutta Methods of Orders Four and Five. Unpublished BTech thesis, Federal University of Technology, Minna, Nigeria.

Ndanusa, A., & Audu, K. J. (2016a). Design and Analysis of Some Third-Order Explicit Almost Runge-Kutta Methods. *Journal of Applied Mathematics, 2016,* 6, 13-21. https://doi.org/10.4236/am.2016.71002

Ndanusa, A. & Audu, K. J. (2016b). On Fourth and Fifth Order Explicit Almost Runge-Kutta Methods, *International Journal of Scientific and Innovative Mathematics Research*, *4*(1), 88-96. https://doi.org/10.4236/am.2016.71002

Paudel, D. R. & Bhatta, M. R. (2023). Comparative Study of Euler's Method and Runge-Kutta Method to Solve an Ordinary Differential Equation through a Computational Approach, *Academic Journal of Mathematics Education*, *6*(1), 81-85. https://doi.org/10.3126/ajme.v6i1.63802

Shior, M. M., Agbata, B. C., Gbor, G. D., Ezugorie, I. U., Topman, N. N. (2024) Solution of First Order Ordinary Differential Equations Using Fourth Order Runge-Kutta Method with MATLAB, *International Journal of Mathematics and Statistics Studies*, *12* (1), 54-63. https://doi.org/10.37745/ijmss.13

Sundaram, A. (2022). Application of runge – kutta method to population equations, *International Journal for Research in Applied Science and Engineering Technology*, 10(4), 719–724. https://doi.org/10.22214/ijraset.2022.41358

Workineh,Y., Mekonnen, H. & Belew, B. (2024). Numerical methods for solving second-order initial value problems of ordinary differential equations with Euler and Runge-Kutta fourth-order methods. *Frontier Applications in Mathematics and Statistics*. 10, 1360628. https://doi.org/10.3389/fams.2024.1360628